Special Radon Semester, Linz, 2005.

WORK SUMMARY

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I. LECTURES

LECTURES ON A POSTERIORI ERROR CONTROL

History. Error indicators for finite element methods.

Functional a posteriori error estimates.

A posteriori estimates for the Stokes problem.

A posteriori estimates for the linear elasticity problem.

A posteriori estimates for mixed methods.

Evaluation of errors arising due to data indeterminacy.

A posteriori estimates for iteration methods.

Functional a posteriori estimates for variational inequalities.

Open version in Internet

Electronic version of the Lectures is exposed on the cite of RICAM

http://www.ricam.oeaw.ac.at/sscm/structure/lectures/repin/repin.html

II. Talks on Conferences and Workshops

Kick-off Meeting, October 3 - 4.

"A posteriori estimates and adaptivity in solid mechanics".

Computational Mechanics Challenges Day. October 21.

"Mathematical modeling of perfectly elasto-plastic problems".

International Workshop on "Direct and Inverse Field Computations in Mechanics" Workshop, Linz, November 7 - 11.

- A. Gaevskaya and S. Repin. "A posteriori error estimates for elliptic and parabolic problems".
- A. Gaevskaya, R. Hoppe and S. Repin. "Error Majorants for Distributed Optimal Control Problems with Control Constraints".

Miniworkshop "Error-Estimates". November 15.

"A posteriori estimates in non–energy quantities".

Miniworkshop DG Methods. November 29.

"A posteriori estimates for DG methods".

III. New Researches Started

Topic I.

A posteriori estimates for problems in solid mechanics with nonlinear boundary conditions.

Topic II.

A posteriori estimates for discontinuous Galerkin methods.

In many real–life problems "classical" boundary conditions cannot adequately describe the situation. For example

- **obstacles (unilateral boundary conditions),**
- **friction type boundary conditions,**
- **"soft" (or flexible) contacts (e.g. Winkler obstacles)**

are typical in the theory of solid bodies.

It these problems the "main" boundary condition is stated on the part Γ_0, i.e.

$$\mathbf{u}(\mathbf{x}) = \mathbf{u_0}(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{\Gamma_0}$$

and **on another part Γ_1 the conditions are more complicated.** On Γ_1 the body may be subject to the action of surface forces or contact an obstacle with friction or without, satisfy linear or nonlinear Winkler law, etc. In practice, one can meet a wide spectrum of conditions that have one common form:

$$-\sigma_{\mathbf{n}}(\mathbf{x}) \in \partial \mathbf{j}(\gamma \mathbf{u}(\mathbf{x})) \quad \mathbf{x} \in \mathbf{\Gamma_1}\,, \tag{1}$$

where γ is the trace, $\mathbf{j} : \mathbb{R}^{\mathbf{d}} \longrightarrow \mathbb{R}$ is a convex lower semicontinuous functional called the **boundary dissipative potential**.

Variational inequalities

Such models are reduced to stationary or evolutionary variational inequalities, e.g.,

$$\mathbf{a}(\mathbf{u}, \mathbf{v} - \mathbf{u}) - (\mathbf{f}, \mathbf{v} - \mathbf{u}) + \boldsymbol{\Upsilon}(\mathbf{v}) - \boldsymbol{\Upsilon}(\mathbf{u}) \geq \mathbf{0}, \qquad \forall \mathbf{v} \in \mathbf{K}.$$

How to solve these problems numerically on a given mesh is, in general, clear. Various numerical methods for variational inequalities has been developed in 70-90':

- Regularization techniques;
- Penalization techniques;
- Saddle-point algorithms.

Open problems

How to perform **reliable modeling** for such type problems, what includes

(a) **guaranteed error bounds,**

(b) **efficient error indicators,**

(c) **mesh adaptation strategies.**

These questions for **variational inequalities** have started receiving attention only recently. In many parts it is still an OPEN AREA.

Difficulties in applying standard FEM approaches to a posteriori error control

I. It is hardly possible to say that approximations computed are the Galerkin approximations.

II. Residuals may be defined in very weak (implicit) form, so that their direct analysis is faced with serious difficulties.

III. Solutions to variational inequalities have limited reqularity even for smooth data. Therefore, methods based on the effects as *supeconvergence* are unlikely to be successful.

It is natural to develop here
A POSTERIORI ERROR ESTIMATION METHODS OF THE FUNCTIONAL TYPE.

These methods are not restricted to Galerkin solutions and require no extra regularity of the exact solutions.

This work has been started during the Radon Semester jointly with Dr. J. Valdman. Results are exposed in the paper:

FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS.

We plan to publish it in the RICAM Preprint Series and further to be submitted in one of the international journals in Numerical Analysis.

Functional type a posteriori estimate for a scalar–valued boundary–value problem with nonlinear boundary conditions

In the problem considered

$$\mathbf{a}(\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_\Omega \mathbf{A}\nabla\mathbf{v}\cdot\nabla\mathbf{v}\,\mathbf{dx}; \qquad \Upsilon(\gamma\mathbf{v}) := \int\limits_{\Gamma_1} \mathbf{j}(\gamma\mathbf{v})\mathbf{ds},$$

where $\gamma : H^1 \rightarrow H^{1/2}$ is the trace operator and for the friction model

$$\mathbf{j}(\mathbf{z}) := \mu|\mathbf{z}|$$

Let **v** be an approximate solution. Then the following estimate holds.

A form of the a posteriori estimate

$$\frac{1}{2}\|\mathbf{v}-\mathbf{u}\|_{\mathbf{a}}^{2} \leq$$

$$\leq (1+\beta)\int_{\Omega}(\mathbf{A}\nabla\mathbf{v}\cdot\nabla\mathbf{v}+\mathbf{A}^{-1}\mathbf{y}^*\cdot\mathbf{y}^*-\nabla\mathbf{v}\cdot\mathbf{y}^*)\mathbf{dx}+$$

$$+\frac{1}{2}\left(1+\frac{1}{\beta}\right)\mathbf{C}_{\Omega,\Gamma_1}\left(\int_{\Omega}(\mathrm{div}\mathbf{y}^*+\mathbf{f})^2\mathbf{dx}+\right.$$

$$\left.+\Upsilon(\gamma\mathbf{v})+\Upsilon^*(-\delta_{\mathbf{n}}\mathbf{y}^*)+\langle\delta_{\mathbf{n}}\mathbf{y}^*,\gamma\mathbf{v}\rangle\right)$$

where $\delta_n : H(\Omega, \mathrm{div}) \to H^{-1/2}$ is the normal trace operator and Υ^* is the polar to Υ. For summable traces we simply have

$$\Upsilon^*(-\delta_{\mathbf{n}}\mathbf{y}^*) = \int\limits_{\Gamma_1}\mathbf{j}^*(-\delta_{\mathbf{n}})\mathbf{ds}$$

3 components of the error

We observe that for such problems the Error Majorant has 3 terms:

- The term that penalizes violations of $\mathbf{y}^* = \mathbf{A}\nabla\mathbf{v}$;
- The term that penalizes violations of $\mathbf{div}\mathbf{y}^* + \mathbf{f} = \mathbf{0}$;
- The term that penalizes violations of the nonlinear boundary condition $-\delta_{\mathbf{n}}\mathbf{y}^* = \partial\mathbf{j}(\gamma\mathbf{v})$.

Possible lines of future joint researches

First results concern a problem that is used in the theory of variational inequalities as a simple model of the friction-type boundary conditions. It suggests ways of the a posteriori error control and construction of adaptive methods in more complicated problems.

Challenging problems of high practical value arise if we take the basic operator as **the linear elasticity operator** or **the elasto-plasticity operator**.

Thus, natural tasks to solve next are as follows:

Open problems

Task I.
Reliable a posteriori error bounds of approximation errors and adaptive methods for ELASTICITY problems with nonlinear boundary conditions: unilateral problems, friction, Signorini conditions, Winkler contact conditions etc.
Task II.
Reliable a posteriori estimates and adaptive methods in ELASTO-PLASTICITY with linear and nonlinear boundary conditions.

A posteriori error control for DG methods

Another big challenging problem in modern numerical analysis is

Reliable error control and adaptive methods for nonconforming methods.

Again "classical" a posteriori methods would face serious difficulties by the same reasons as before.

Another research topic started during the Radon Semester

Reliable error control by functional type a posteriori error estimates for Discontinuous Galerkin approximations of elliptic type problems.

First part of the work associated with diffusion model was carried out during the Semester jointly with Prof. R. Lazarov and Dr. S. Tomar.

First results.

First results has been exposed on a MiniWorkshop on DG approximations and are presented in a paper

A POSTERIORI ERROR ESTIMATES FOR DISCONTINUOUS GALERKIN METHOD.

Also, we plan to publish the results in the RICAM Preprint Series and further submitted to an international journal on numerical analysis.

Our aim is to obtain **directly computable two–sided estimates** for the quantities (the so–called "brocken norms")

$$\|[\mathbf{u}-\mathbf{u_h}]\| \quad \text{and} \quad |||[\mathbf{u}-\mathbf{u_h}]|||\,,$$

where

$$\|[\mathbf{v}]\|^2 := \sum_{\mathbf{T_h}\in\mathcal{T}_\mathbf{h}} \int_{\mathbf{T_h}} \mathbf{A}\nabla\mathbf{v}\cdot\nabla\mathbf{v}\mathbf{dx},$$

$$|||[\mathbf{v}]|||^2 := \sum_{\mathbf{T_h}\in\mathcal{T}_\mathbf{h}} \int_{\mathbf{T_h}} \mathbf{A}\nabla\mathbf{v}\cdot\nabla\mathbf{v}\mathbf{dx} + \sum_{\mathbf{e}\in\mathcal{E}_\mathbf{h}} \int_{\mathbf{e}} \mathbf{\Phi}([\mathbf{v}])\mathbf{ds}.$$

A form of such an estimate

Let $\|\eta\|^2_{\bar{a}} := \int_\Omega \mathbf{A}^{-1}\eta \cdot \eta \, \mathbf{dx}$. Then

$$[\![\mathbf{u} - \mathbf{u_h}]\!] \leq [\![\mathbf{v} - \mathbf{u_h}]\!] + \|\mathbf{A}\nabla\mathbf{v} - \mathbf{y}\|_{\bar{a}} + \mathbf{C}_\Omega \|\mathrm{div}\mathbf{y} + \mathbf{f}\|,$$

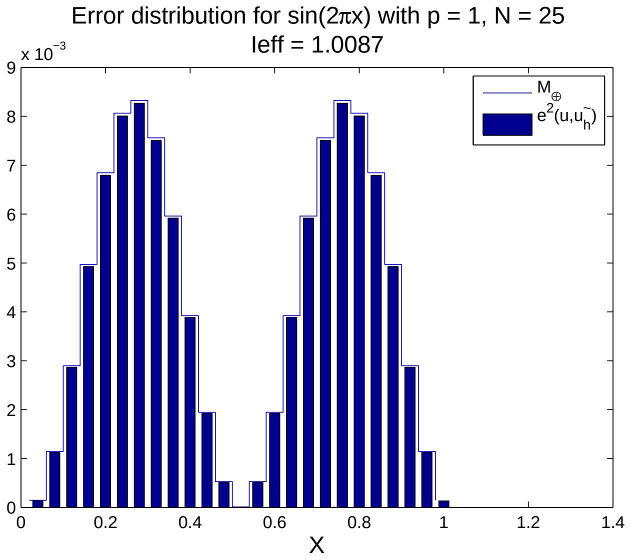
Here, $\mathbf{v}$ and $\mathbf{y}$ are "free" functions. Since for $\mathbf{v} = \mathbf{u}$ and $\mathbf{y} = \mathbf{A}\nabla\mathbf{u}$ the left hand side coincides with the right hand one we see that these estimates are also sharp and (as those for the conforming approximations) have no "gap". Once such functions are selected, the right hand side presents a natural decomposition of the overall error into three terms:

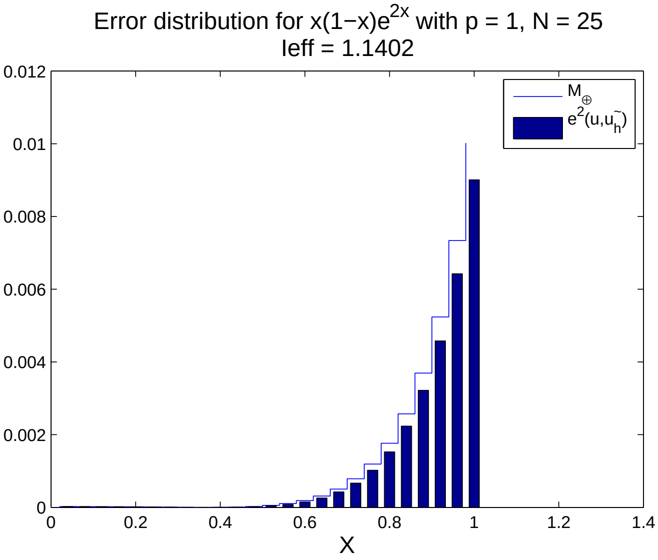
- **error due to nonconformity**,
- **error in the duality relation for fluxes**,
- **error in the equilibrium equation for fluxes**.

Numerical tests (performed by Dr. S. Tomar)

In the first series of tests model 1D problem were solved by DG approximations of various order. Since exact solutions are known we can compute the true errors and compare them with those computed by the new error estimation technology.

Below we present two results.


Error distribution for sin(2πx) with p = 1, N = 25
Ieff = 1.0087
x 10^−3
9
8
7
6
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
X
M⊕
e²(u,ũ_h)



Error distribution for x(1−x)e^2x with p = 1, N = 25
Ieff = 1.1402
0.012
0.01
0.008
0.006
0.004
0.002
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
X
M⊕
e^2(u,ũh)

In the both researches:

1. "A posteriori error control for nonlinear boundary conditions" and

2. "A posteriori error control for DG approximations".

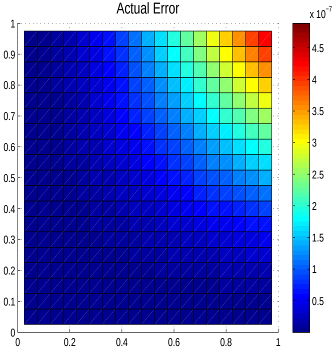
it was of cause necessary to first **understand and practically implement the functional error estimation technology to linear elliptic problems** in 2D.

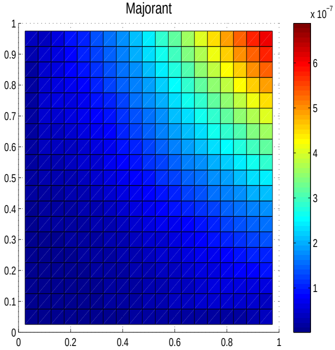
This work was very successfully done by Drs. **S. Tomar** and **J. Valdman** who wrote their own versions of the

Error Majorant Estimator

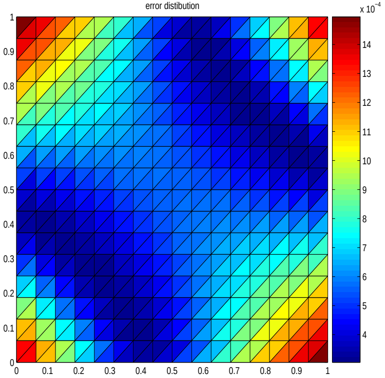
adopted for triangular elements (J. Valdman) and quadrilateral elements (S. Tomar).

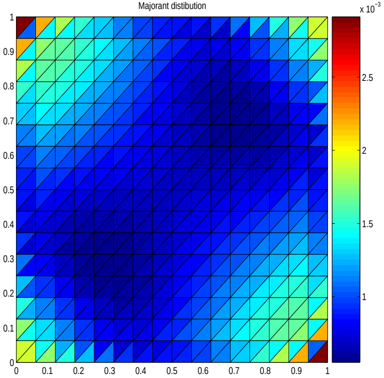
Quadrilateral elements (S. Tomar)



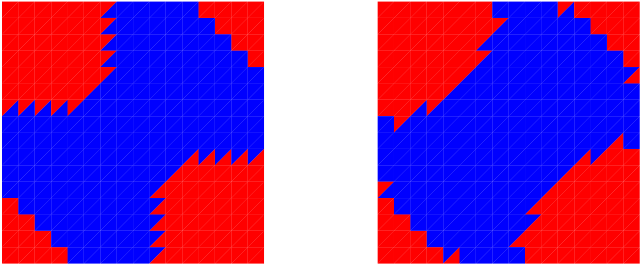


Triangular elements (J. Valdman).





Triangular elements (marking) (J. Valdman).



Method (a) - averaged stresses:

N	term left	term right	beta	majorant	error (squared)	I_{eff}
9	8.85353e-03	1.33109e-02	1.50	2.21644e-02	8.68056e-03	1.60
25	6.34626e-03	8.28169e-03	1.30	1.46279e-02	3.21791e-03	2.13
81	2.25775e-03	3.49366e-03	1.55	5.75142e-03	8.93391e-04	2.54
289	7.09748e-04	1.45984e-03	2.06	2.16959e-03	2.29411e-04	3.08
1089	2.23711e-04	6.34695e-04	2.84	8.58407e-04	5.77403e-05	3.86
4225	7.22314e-05	2.86546e-04	3.97	3.58778e-04	1.44595e-05	4.98
16641	2.38677e-05	1.33175e-04	5.58	1.57042e-04	3.61639e-06	6.59
66049	8.03261e-06	6.32145e-05	7.87	7.12471e-05	9.04193e-07	8.88

Method (c) - majorant optimization:

N	term left	term right	beta	majorant	error (squared)	I_{eff}
9	7.58101e-03	4.78880e-03	0.63	1.23698e-02	8.68056e-03	1.19
25	4.44820e-03	2.03327e-03	0.46	6.48147e-03	3.21791e-03	1.42
81	1.37711e-03	6.69829e-04	0.49	2.04694e-03	8.93391e-04	1.51
289	3.64732e-04	1.94158e-04	0.53	5.58890e-04	2.29411e-04	1.56
1089	9.27069e-05	5.24558e-05	0.57	1.45163e-04	5.77403e-05	1.59
4225	2.32999e-05	1.36760e-05	0.59	3.69759e-05	1.44595e-05	1.60
16641	5.83644e-06	3.49948e-06	0.60	9.33592e-06	3.61639e-06	1.61
66049	1.46035e-06	8.86344e-07	0.61	2.34669e-06	9.04193e-07	1.61

Third International Workshop on Reliable Methods for Mathematical Modeling. RM^3

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