SPECIAL RADON SEMESTER ON COMPUTATIONAL MECHANICS

Topic: Novel Discretization and Solver Techniques in Mechanics

ROBUST PARALLEL ALGEBRAIC MULTIGRID AND MULTILEVEL TECHNIQUES

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PCG methods

Consider the weak formulation of a given elliptic b.v.p. in the form

$$a(u, v) = \mathcal{F}(v), \qquad \forall v \in \mathcal{V},$$

and the related FEM problem

$$a_h(u_h, v_h) = \mathcal{F}_h(v_h), \qquad \forall v_h \in \mathcal{V}_h.$$

We are interested in the efficient solution of the resulting large-scale FEM linear systems

$$A\mathbf{u} = \mathbf{f}.$$

The construction of robust Preconditioned Conjugate Gradient (PCG) iterative solution methods is addressed to some special properties of the stiffness matrix A, among which are that:

- A **is symmetric and positive definite (SPD);**
- A **is large and even very large but sparse.**

Robust multilevel preconditioning

The preconditioner is optimal if

$$\left.\begin{array}{l} \varkappa(C^{-1}A) = O(1) \\ \mathcal{N}(C^{-1}\mathbf{v}) = O(N) \end{array}\right\} \rightarrow \mathcal{N}_{PCG}(A^{-1}\mathbf{f}) = O(N).$$

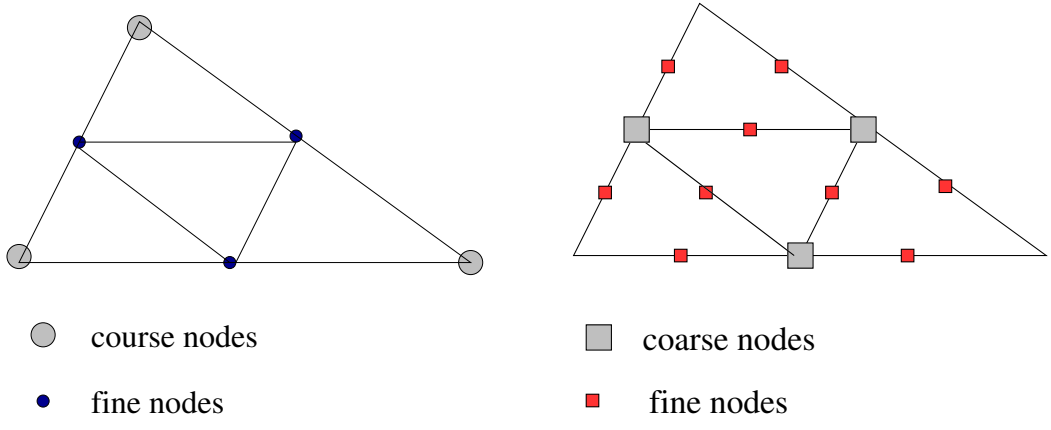
Various multilevel extensions of two-level FEM lead to PCG iterative methods of optimal computational complexity with respect to the size of the system.

- **The stiffness matrix becomes additionally ill-conditioned if the coefficients are anisotropic or, equivalently, when the mesh aspect ratio increases.**
- **The condition number $\varkappa(A)$ deteriorates also for parameter dependent problems like almost incompressible elasticity, thin plates or shells, Navier-Stokes equations, etc.**
- **For such ill-conditioned problems we need specially developed robust preconditioners.**

Lectures and seminars

1. On the role of the strengthened CBS inequality in the theory of algebraic multilevel iteration methods. (L1)
2. Optimal multilevel preconditioning of strongly anisotropic problems. Part I: Conforming FEM. (L2)
3. Locally optimized parameters of AMLI algorithms for conforming FEM problems. (S1)
4. Optimal multilevel preconditioning of strongly anisotropic problems. Part 2: Non-conforming FEM. (L3)
5. Robust AMLI preconditioning of FEM elasticity problems. (L4)
6. Algebraic multilevel preconditioning of finite element matrices using local Schur complements. (L5)
7. Multilevel methods with aggregation for nonconforming FEM problems. (S2)
8. Multilevel preconditioning of rotated bilinear non-conforming FEM problems. (RS)
9. Multilevel methods for DG system. (RS)

Lectures

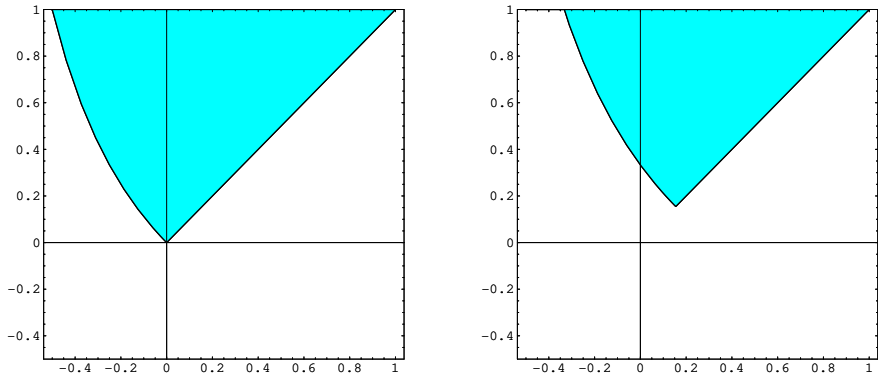


Courant conforming and Crouzeix-Raviart non-conforming linear FEs

- **Theory of the Algebraic MultiLevel Iteration (AMLI) methods.**
- **Local estimates of the constant in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality.**
- **Optimal algorithms for strongly anisotropic problems.**
- **Optimal algorithms almost incompressible elasticity problems.**

Seminars

Investigation of locally optimized AMLI parameters γ and $\kappa(C_{11}^{-1}A_{11}^{(k+1)})$.



Domains $\widetilde{D}(0)$ and $\widetilde{D}(\frac{\pi}{6})$ of the parameters (α, β).

Seminar contributions:

- **Conforming FEM: improved estimates of κ_A and κ_M**

  Ivan Georgiev, IPP, BAS, Sofia

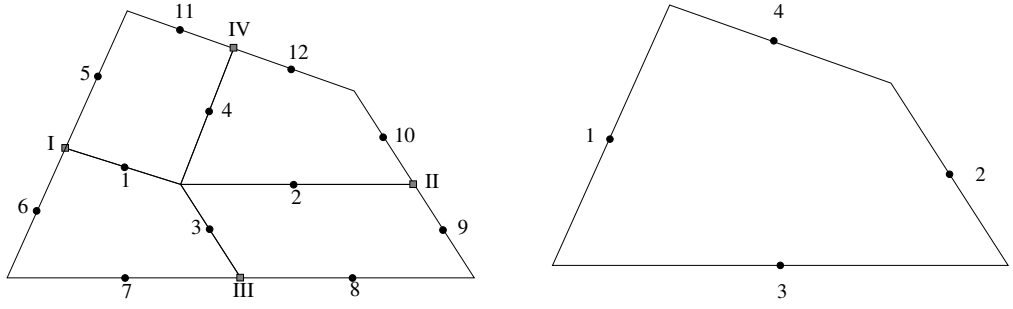
- **Non-conforming FEM: locally optimized parameters of the aggregation algorithm and improved estimates of γ_{DA}**

  Josef Synka, IIM, Joh. Kepler University, Linz

Joint research results

”Multilevel preconditioning of rotated bilinear non-conforming FEM problems”, RICAM Report

Ivan Georgiev, IPP, BAS, Sofia
Johannes Kraus, RICAM, AAS, Linz
Svetozar Margenov, IPP, BAS, Sofia



Uniform refinement of Rannacher-Turek rotated bilinear non-conforming FEs.

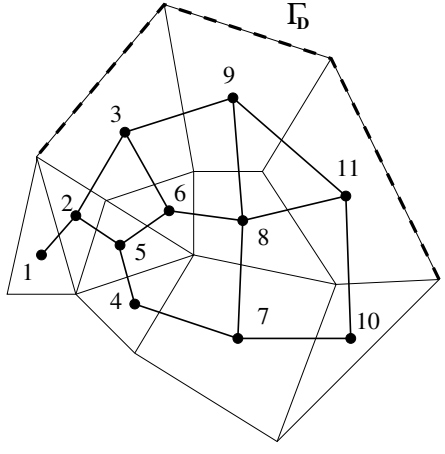
The obtained results include:

- **General setting of hierarchical two-level decompositions.**
- **New uniform estimates of the CBS constant.**
- **AMLI preconditioners of optimal complexity.**
- **Numerical tests for large-scale problems.**

”CBS constants for graph-Laplacians and application to multilevel methods for DG systems”, RICAM Report

Raytcho Lazarov, Texas A&M University, College Station

Svetozar Margenov, IPP, BAS, Sofia



Partition $\mathcal{T}$ and related graph-Laplacian

The obtained results include:

- General setting of two-grid method for DG systems.
- AMLI preconditioning of graph-Laplacians.
- Novel estimates of the CBS constant for graph-Laplacians.

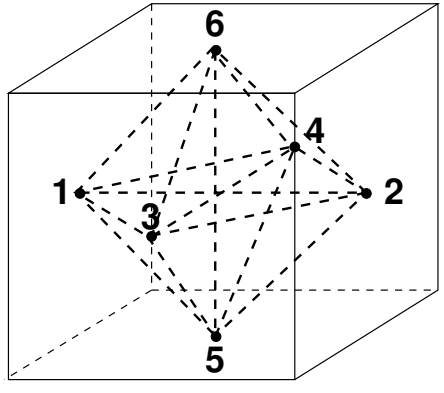
”Locally optimized MIC(0) preconditioning of rotated trilinear non-conforming FEM problems”, Report in progress

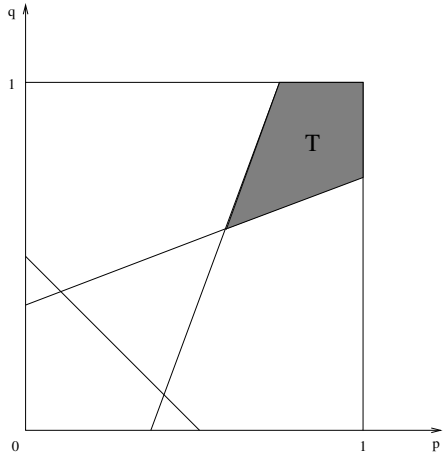
Ivan Georgiev, IPP, BAS, Sofia

Svetozar Margenov, IPP, BAS, Sofia

Johannes Kraus, RICAM, AAS, Linz

Josef Schicho, RICAM, AAS, Linz





Rannacher-Turek rotated trilinear non-conforming FE and related cases of anisotropy

Approximation of the anisotropic local stiffness matrix by M-matrix:

- Symbolic technique to find the locally optimal solution.
- Diagonal compensation.
- Approximation based on minimizing the Frobenius norm.
- Parametric splitting.

Problems for future research

- Theory of the multilevel methods:
  - Novel approaches for hierarchical decomposition of non-nested FE spaces.
  - Generalized techniques for estimates of the CBS constant.
  - Theory of the nonlinear multilevel methods.
- Multilevel methods for non-standard FEM systems:
  - Non-conforming FEs.
  - Mixed FEMs.
  - Discontinuous Galerkin approximations.
- Specific target research problems:
  - Robust methods for 3D elliptic problems.
  - Multilevel preconditioning of coupled problems.
  - Novel multilevel algorithms for graph-Laplacians.

Recent topics of joint interest

Short list of joint research problems:

- AMLI preconditioning of rotated trilinear (3-D) non-conforming FEM systems.
- Multilevel preconditioning of unstructured graph-Laplacians.
- Application of AMLI solvers for non-conforming FEs to Stokes problems.
- Novel multilevel algorithms for Nedelec FEs.
- Development of robust multilevel methods for higher order FEs.
- Application of symbolic techniques for optimization of AMLI parameters.

Research training materials:

- The lectures and seminar materials are available at the web-site. The seminar items are equipped with a particular list of open problems and exercises.
- The preparation of lecture notes based on the present course is under discussions.