Inverse problems and model identification in mechanics

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Progress Report

Overview

- Projected Newton for inverse problems with constraints
- Homogenization for piezoelectric stack actuator
- Further discussions and new contacts
- Activities

Constrained Inverse Problems

joint work with Andreas Neubauer

Example: Identify $a = a(x)$ in

$$\nabla(a\nabla u) = f \text{ in } \Omega \subset \mathbb{R}^n$$

from measurements of u, under the constraint

$$\underline{\gamma} \leq a(x) \leq \overline{\gamma} \quad \forall x \in \Omega$$

$\rightsquigarrow$ ellipticity and boundedness of $\nabla(a\nabla\cdot) : H_0^1(\Omega) \to H^{-1}(\Omega)$.

Formulation as Operator Equation in Hilbert Space

$$F(a) = y^\delta$$

$$\begin{array}{rcl} F : \mathcal{D}(F) \subset X & \rightarrow & Y \\ a & \mapsto & u \text{ solution to } \nabla(a\nabla u) = f \end{array}$$

L^∞ measurement error, Y Hilbert space $\rightarrow$ $Y = L^2(\Omega)$.

L^∞ bound constraints, X Hilbert space $\rightarrow$ $X = H^{\frac{n}{2}+\varepsilon}(\Omega)$, $\mathcal{D}(F) = B_\rho^X(a_0)$
$\rightsquigarrow$ oversmoothing of coefficient, ill-posedness is artificially increased!

alternative:
$X = L^2(\Omega)$, $\quad \mathcal{D}(F) = \{a \in L^2(\Omega) \mid \underline{\gamma} \leq a \leq \overline{\gamma} \quad \text{a.e. in } \Omega\}$
$\rightsquigarrow$ jumps in coefficients are possible

Regularization Methods for Constrained Inverse Problems (I)

Tikhonov-Philips:

$$\min_x \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2 \quad x \in \mathcal{D}(F)$$ [Neubauer 88], [Engl&Hanke&Neubauer96]

Landweber:

$$x_{k+1} = Proj_{\mathcal{D}(F)}\Big(x_k - F'(x_k)^*(F(x_k) - y^\delta)\Big)$$ [Eicke 92], linear case

proof only goes through for special cases of $\mathcal{D}(F)$!

Levenberg-Marquardt:

$$x_{k+1} = Proj_{\mathcal{D}(F)}\Big(x_k - (F'(x_k)^*F'(x_k)) + \alpha_k I)^{-1}(F(x_k) - y^\delta)\Big)$$

analogously to Landweber!

Iteratively regularized Gauss-Newton:

$$x_{k+1} = Proj_{\mathcal{D}(F)}\Big(x_0 - (F'(x_k)^*F'(x_k)) + \alpha_k I)^{-1}F'(x_k)^*\Big(F(x_k) - y^\delta - F'(x_k)(x_k - x_0)\Big)\Big)$$

Regularization Methods for Constrained Inverse Problems (II)

Iteratively regularized Gauss-Newton:

$$x_{k+1} = Proj_{\mathcal{D}(F)}\Big(x_0 - \underbrace{(F'(x_k)^*F'(x_k)) + \alpha_k I)^{-1}F'(x_k)^*}_{linear\ Tikhonov}\Big(F(x_k) - y^\delta - F'(x_k)(x_k - x_0)\Big)\Big)$$

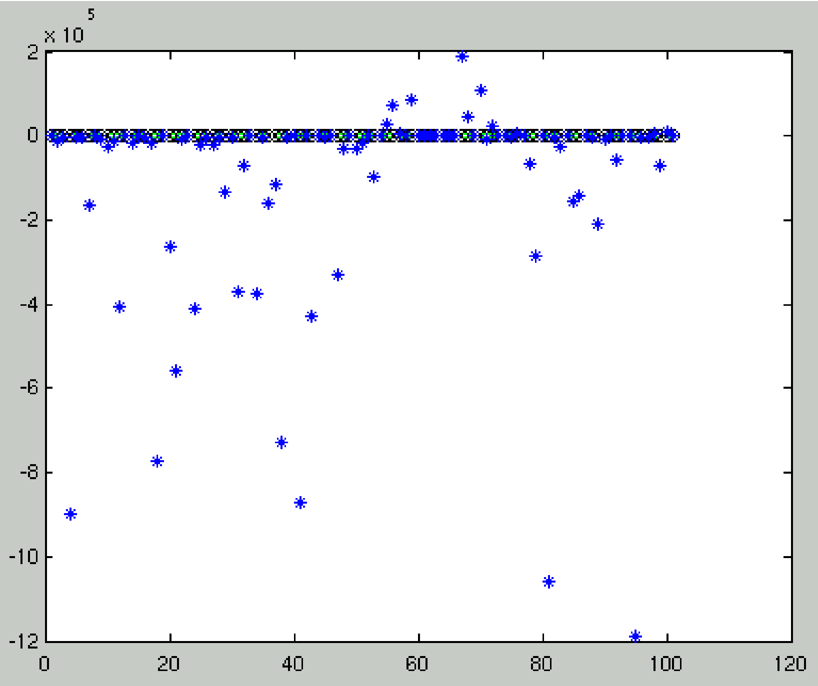
- convergence and optimal convergence rates
  also for linear Landweber or iterated Tikhonov in place of linear Tikhonov
- projection onto general convex closed domain $\mathcal{D}(F)$
- generalization to large class of linear regularization methods
  in place of linear Tikhonov (e.g., regularization by discretization, ν-methods)
  for sufficiently smooth solutions ($\mu \geq \frac{1}{2}$ in source condition)

Numerical Tests (I)

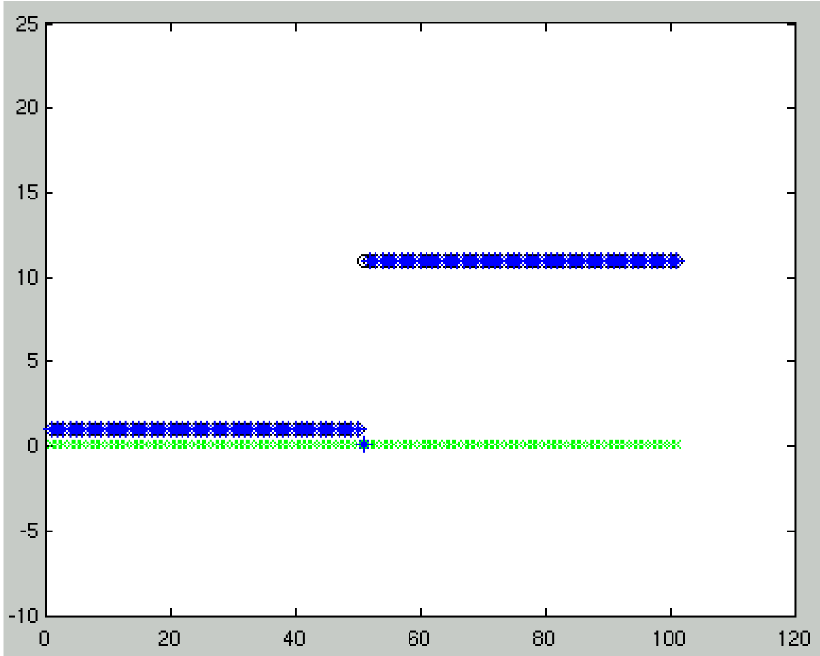
$$
\begin{aligned}
a(x) &= \begin{cases} 1 & x \in [0, \frac{1}{2}] \\ (1+c) & x \in (\frac{1}{2}, 1] \end{cases} \\
u(x) &= \sin(\pi\, x) \\
\underline{\gamma} &= 0.1 \quad \overline{\gamma} \;=\; 25 \\
c &= 10 \text{ (height of jump)} \\
a_0(x) &\equiv 0.1
\end{aligned}
$$

Numerical Tests (II)

Gauss-Newton without projection
after 150 iterations

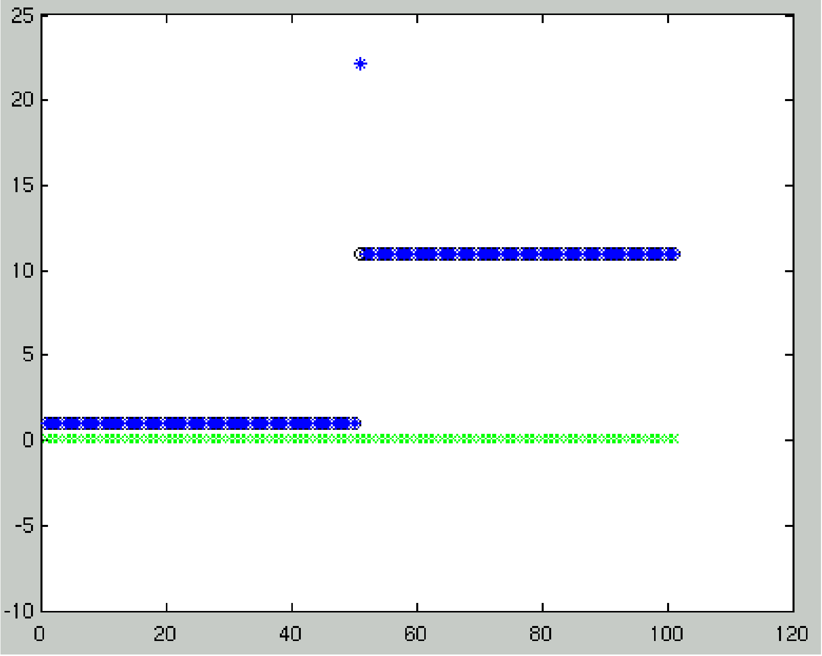


Gauss-Newton with projection
after 129 iterations

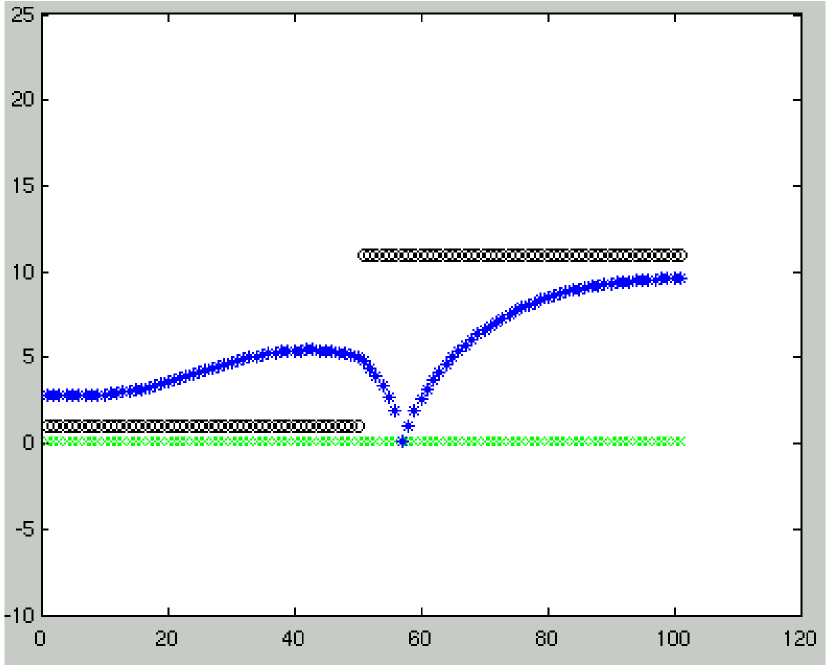


Numerical Tests (III)

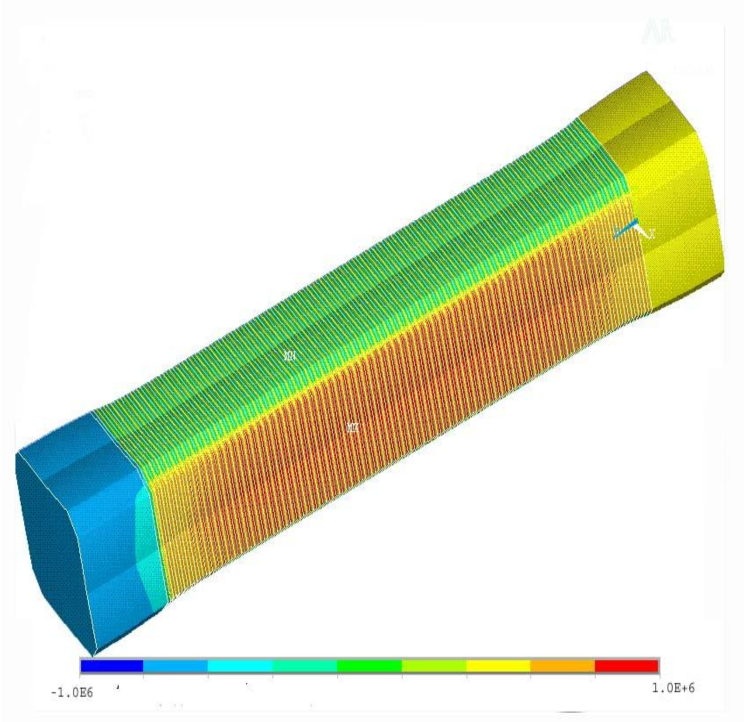
Levenberg-Marquardt with projection after 105 iterations



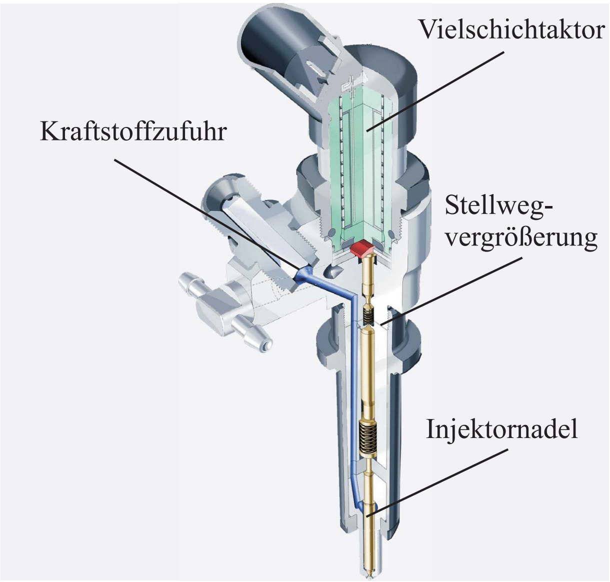
Landweber with projection after 1000 iterations



Piezoelectic Stack Actuator



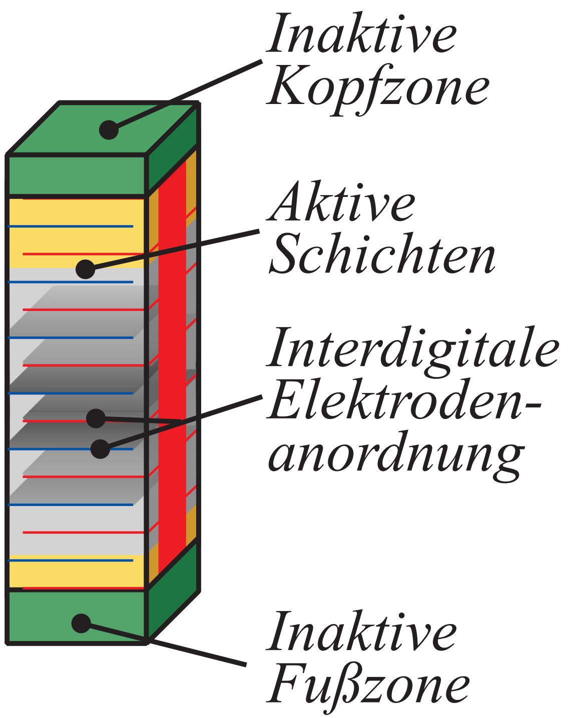
Injection valve Common-Rail-Dieselmotor



300-400 layers, $\sim$ 130.000 unknowns, CPU: 244.43 s

Homogenization for Piezoelectic Stack Actuator

joint work with Joachim Schöberl and Tom Lahmer



idea: compute solution as superposition of
global part + details on cells

challenge: *finite* periodic structure

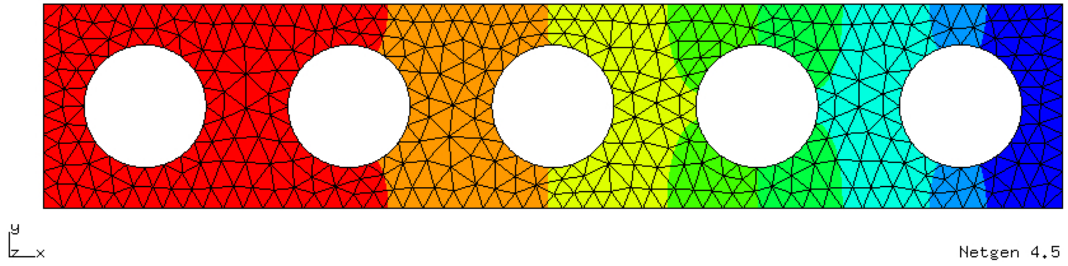
Homogenization

model problem

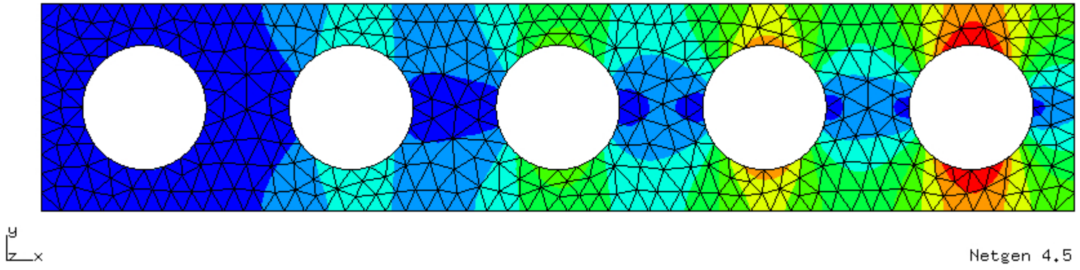
$$\begin{aligned} \Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma_D \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_N \end{aligned}$$

on p-periodic finite domain Ω.

solution:

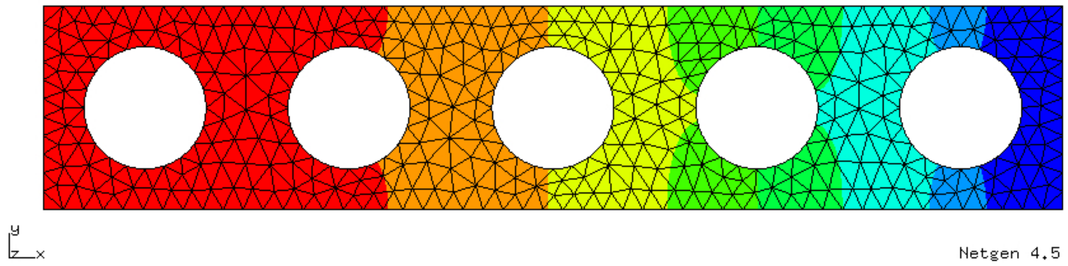


flux:



Homogenization Idea

$$\begin{aligned} \Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma_D \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_N \end{aligned}$$



Ansatz

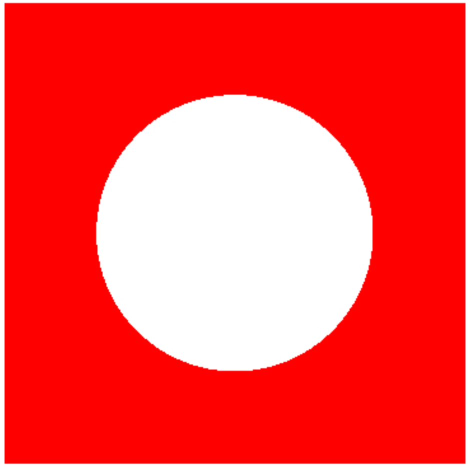
$$u(x,y) = \sum_{i=1}^{N} \underbrace{u_i(x,y)}_{global} \underbrace{\psi_i(\frac{x}{p}, y)}_{periodic}$$

cf. [Babuška&Morgan, Schwab&Matache&Rüegg] "gFEM", "micro shape functions"

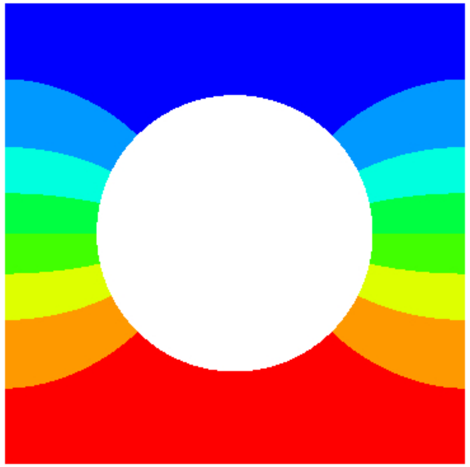
unit cell: we choose ψ_i as eigenfunctions with periodic Dirichlet BC
global: FEM with one element per cell (or per n cells) for easy assembling.

Eingenfunctions on the Unit Cell

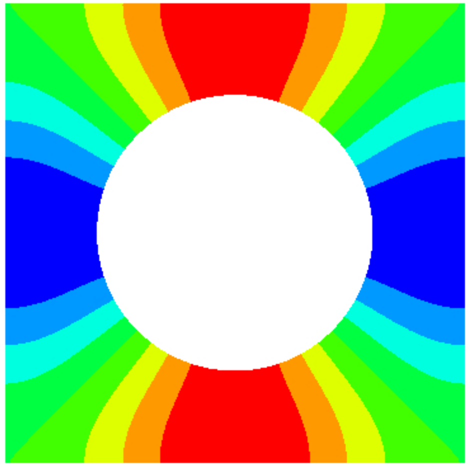
ef_1:



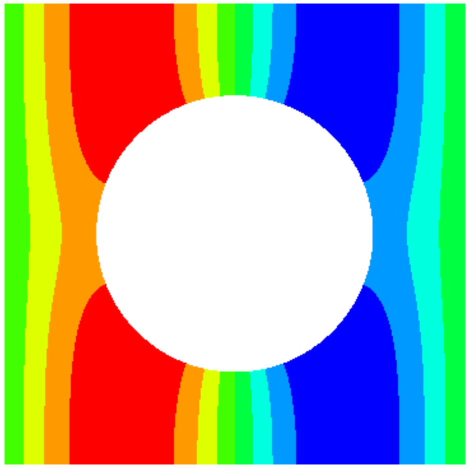
ef_2:



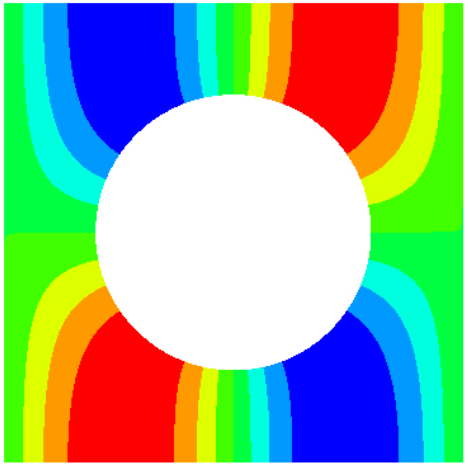
ef_3:



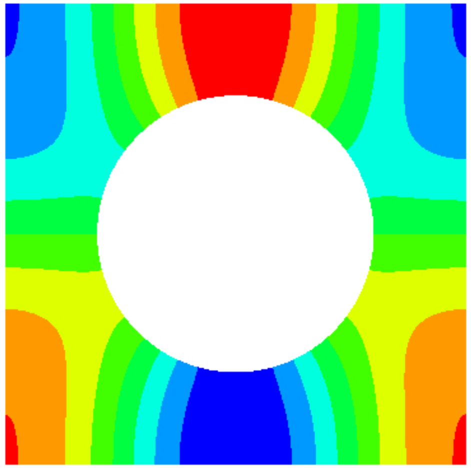
ef_4:



ef_5:



ef_6:



Numerical Tests

reference value: $f \cdot u = 38.190489$

$N_{ef} \backslash ord_{glob}$	1	2	3	4
1	29.58	29.77	34.09	34.78
2	29.6	29.77	35.32	35.58
3	29.7	30.33	36.31	37.29
4	32.5	35.11	37.69	38.18
5	34.5	37.54	37.88	38.19
6	34.5	37.54	37.96	38.20

$$DOFs = DOFs(ord_{glob}) * N_{ev}$$

ord_{glob}	1	2	3	4
$DOFs(ord_{glob})$	12	33	64	105

Discussions and New Contacts

- Samuel Armstutz: Topological asymptotics for piezoelectricity
- Marc Kamlah: Hysteresis modelling in piezoelectricity
- Volker Schulz: Optimum experiment design for piezoelectric material parameter id.
- Gen Nakamura: Identifiability of nonlinear coefficient curves
- Herbert Egger, Boris Vexler: Adaptivity in parameter identification

Further Activities

- Miniworkshop *Direct and Inverse Problems in Piezoelectricity*
  - 11 talks
  - 20-25 participants

- lecture *Parameter Identification in PDEs*
  - 14 lectures
  - 10-30 participants