

Preconditioning of KKT Systems

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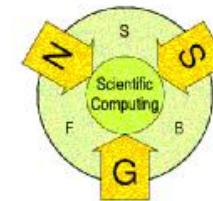
Project F1309 - Multilevel Solvers for Large Scale Discretized Optimization Problems

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An Optimal Control Problem

👉 objective:

$$\min J(y, u) = \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \|u\|_{L^2(\Omega)}^2$$

👉 state equation:

$$-\Delta y + cy = u \quad \text{in } \Omega$$

$$\frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma$$

👉 Ω bounded domain with $C^{0,1}$ -boundary Γ , $\nu > 0$, $y_d \in L^2(\Omega)$, $c \in L^\infty(\Omega)$



Optimality System

$$\begin{aligned}-\Delta y + y &= u \\ \frac{\partial y}{\partial n} &= 0 \\ -\Delta p + p &= -(y - y_d) \\ \frac{\partial p}{\partial n} &= 0 \\ \nu u - p &= 0\end{aligned}$$

- $y, p \in V = H^1(\Omega), V_h = \{v \in V \cap C(\bar{\Omega}) : v|_T \in P_1, \forall T \in \mathcal{T}\}$
- $u \in Q = L^2(\Omega), Q_h = \{q \in Q : q|_T \in P_1, \forall T \in \mathcal{T}\}$



Linear System

$$\begin{pmatrix} M_y & 0 & A \\ 0 & \nu M_u & -B^T \\ A & -B & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}$$

- 👉 $M_y := ((\Phi_i, \Phi_j))_{i,j=1}^{n_P}, \quad M_u := ((\phi_i, \phi_j))_{i,j=1}^{3n_T}, \quad c := (y_d, \Phi_i)_{i=1}^{n_P}$
- 👉 $A := ((\nabla \Phi_i, \nabla \Phi_j))_{i,j=1}^{n_P}, \quad B := ((\phi_i, \Phi_j))_{i=1, \dots, 3n_T; j=1, \dots, n_P}$
- 👉 $\Phi_i \in V_h \subset H^1(\Omega) \cap C(\bar{\Omega})$, piecewise linear
- 👉 $\phi_i \in Q_h \subset L^2(\Omega)$, piecewise linear



Eigenvalues

Theorem 1 (Rusten/Winther). Suppose that M_y and M_u are positive definite and that $(A \mid -B)$ has full rank. Let $\mu_1 \geq \dots \geq \mu_n$ be the combined eigenvalues of M_y and M_u , and let $\sigma_1 \geq \dots \geq \sigma_m$ be the singular values of $(A \mid -B)^T$. Then $\sigma(K) \subset I^- \cup I^+$, where

$$I^- = \left[\frac{1}{2} \left(\mu_n - \sqrt{\mu_n^2 + 4\sigma_1^2} \right), \frac{1}{2} \left(\mu_1 - \sqrt{\mu_1^2 + 4\sigma_m^2} \right) \right]$$

$$I^+ = \left[\mu_n, \frac{1}{2} \left(\mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2} \right) \right]$$



"ideal" Preconditioner

- neutralizing the mesh-dependence of the mass matrices by application of

$$\begin{pmatrix} M_y^{-1/2} & 0 & 0 \\ 0 & M_u^{-1/2} & 0 \\ 0 & 0 & I \end{pmatrix} \hookrightarrow \begin{pmatrix} I & 0 & M_y^{-1/2} A^T \\ 0 & I & M_u^{-1/2} B^T \\ A M_y^{-1/2} & B M_u^{-1/2} & 0 \end{pmatrix}$$

- further transformation by application of

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_y^{1/2} A \end{pmatrix} \hookrightarrow \begin{pmatrix} I & 0 & I \\ 0 & I & D^T \\ I & D & 0 \end{pmatrix}, \quad D = M_y^{1/2} A^{-1} B M_u^{-1/2}$$



Preconditioner

- the "ideal" preconditioner is given by

$$P_1 = \begin{pmatrix} M_y^{1/2} & 0 & 0 \\ 0 & M_u^{1/2} & 0 \\ 0 & 0 & AM_y^{-1/2} \end{pmatrix}$$

- assume, that preconditioners \tilde{M}_y, \tilde{M}_u are available, e.g. $\tilde{M}_y = \text{diag}(M_y)^{1/2}$, $\tilde{M}_u = \text{diag}(M_u)^{1/2}$ and an approximate inverse \tilde{A}^{-1} is known

$$P_1^{-1} = \begin{pmatrix} \tilde{M}_y^{-1} & 0 & 0 \\ 0 & \tilde{M}_u^{-1} & 0 \\ 0 & 0 & \tilde{M}_y^T \tilde{A}^{-1} \end{pmatrix}$$



Preconditioner

- the preconditioned KKT matrix is

$$P_1^{-1} K P_1^{-T} = \begin{pmatrix} \tilde{M}_y^{-1} M_y \tilde{M}_y^{-T} & 0 & \tilde{M}_y^{-1} A^T \tilde{A}^{-T} \tilde{M}_y \\ 0 & \tilde{M}_u^{-1} M_u \tilde{M}_u^{-T} & -\tilde{M}_u^{-1} B^T \tilde{A}^{-T} \tilde{M}_y \\ \tilde{M}_y^T \tilde{A}^{-1} A \tilde{M}_y^{-T} & -\tilde{M}_y^T \tilde{A}^{-1} B \tilde{M}_u^{-T} & 0 \end{pmatrix}$$

- and we expect that

$$P_1^{-1} K P_1^{-T} = \begin{pmatrix} \tilde{I} & 0 & \tilde{I} \\ 0 & \tilde{I} & -\tilde{M}_u^{-1} B^T \tilde{A}^{-1} \tilde{M}_y \\ \tilde{I} & \underbrace{-\tilde{M}_y^T \tilde{A}^{-1} B \tilde{M}_u^{-T}}_{\tilde{D}} & 0 \end{pmatrix}$$



$$\tilde{A} = A$$

- Let $B \in \mathbb{R}^{m \times n}$. The singular values σ_i of $(I \mid B)$ are given by $\sigma_i = \sqrt{1 + \tau_i^2(B)}$, where $\tau_i(B)$ are the singular values of B .
- singular values of $\tilde{M}_y^T A^{-1} B \tilde{M}_u^{-T}$:

h	σ_{\min}	σ_{\max}
$1/2$	$8.08 \cdot 10^{-3}$	1
$1/4$	$2.32 \cdot 10^{-3}$	1
$1/8$	$6 \cdot 10^{-4}$	1
$1/16$	$1.52 \cdot 10^{-4}$	1
$1/32$	$3.8 \cdot 10^{-5}$	1



Eigenvalues ($\tilde{A} = A$)

- 👉 eigenvalues of the preconditioned mass matrices: $\mu_1 = 2, \quad \mu_n = \frac{1}{2}$
- 👉 singular values of $(I \mid \tilde{M}_y^T A^{-1} B \tilde{M}_u^{-T})$: $\sigma_1 = 1, \quad \sigma_m = \sqrt{2}$
- 👉 Rusten/Winther: $I^- = [-1.186, -0.4142], \quad I^+ = [0.5, 2.732]$
- 👉 computed:

h	λ_1	λ_n	λ_{n+1}	λ_{n+m}
1/2	2.732	0.5	-0.482	-0.781
1/4	2.732	0.5	-0.433	-0.781
1/8	2.732	0.5	-0.421	-0.781
1/16	2.732	0.5	-0.417	-0.781
1/32	2.732	0.5	-0.415	-0.781



Numerical Results

- $\tilde{M}_y := (\text{diag}(M_y))^{1/2}$, $\tilde{M}_u := (\text{diag}(M_u))^{1/2}$, $\tilde{A} = A$
- solved by MINRES, SYMMLQ, tolerance: 10^{-4} and 10^{-8}

h	tol=10 ⁻⁴		tol=10 ⁻⁸	
	MINRES	SYMMLQ	MINRES	SYMMLQ
1/2	10	10	11	11
1/4	17	17	21	21
1/8	24	24	39	39
1/16	28	28	46	46
1/32	29	30	49	49
1/64	31	31	50	50
1/128	33	33	49	52
1/256	34	34	51	51



Numerical Results

- again: $\tilde{M}_y := (\text{diag}(M_y))^{1/2}$, $\tilde{M}_u := (\text{diag}(M_u))^{1/2}$
- \tilde{A}^{-1} : one V-cycle of Multigrid iteration with 1 Gauss-Seidel iteration for pre- and postsmothing

h	tol=10 ⁻⁴		tol=10 ⁻⁸	
	MINRES	SYMMLQ	MINRES	SYMMLQ
1/2	10	10	11	11
1/4	23	23	35	35
1/8	32	32	51	51
1/16	39	39	64	65
1/32	44	45	72	72
1/64	49	49	77	77
1/128	54	54	82	82
1/256	57	58	86	87



Numerical Results

- again: $\tilde{M}_y := (\text{diag}(M_y))^{1/2}$, $\tilde{M}_u := (\text{diag}(M_u))^{1/2}$
- \tilde{A}^{-1} : one V-cycle of Multigrid iteration with 3 Gauss-Seidel iterations for pre- and postsmothing

h	tol=10 ⁻⁴		tol=10 ⁻⁸	
	MINRES	SYMMLQ	MINRES	SYMMLQ
1/2	10	10	11	11
1/4	22	23	25	25
1/8	26	31	42	42
1/16	34	36	47	48
1/32	37	39	50	50
1/64	36	42	52	53
1/128	44	44	54	55
1/256	47	49	56	56

