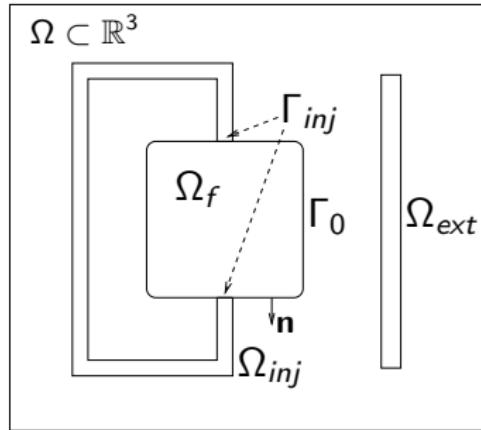


A Stokes-MHD problem

M. Discacciati and R. Griesse

RICAM

We consider the following setting:



and we take an electric current

$$\mathbf{J} = \begin{cases} \mathbf{J}_f & \text{in } \Omega_f \quad (\text{unknown}) \\ \mathbf{J}_{inj} & \text{in } \Omega_{inj} \quad (\text{given}) \\ \mathbf{J}_{ext} & \text{in } \Omega_{ext} \quad (\text{given}) \\ \mathbf{0} & \text{elsewhere} \end{cases}$$

This current generates a magnetic field \mathbf{B} that is the solution of the curl-div problem:

$$\nabla \times (\mu^{-1} \mathbf{B}) = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } \mathbb{R}^3,$$

or, equivalently, setting $\mathbf{B} = \nabla \times \mathbf{A}$,

$$\text{find } \mathbf{A} \quad \text{s.t.} \quad \nabla \times (\mu^{-1} \nabla \times \mathbf{A}) = \mathbf{J} \quad \text{in } \mathbb{R}^3.$$

In the fluid domain Ω_f , we consider the Stokes equations with Lorentz force: find (\mathbf{u}, p) s.t.

$$\begin{aligned}-\eta \Delta \mathbf{u} + \nabla p - \mathbf{J}_f \times (\nabla \times \mathbf{A}) &= \mathbf{f} \quad \text{in } \Omega_f \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega_f,\end{aligned}$$

with Dirichlet boundary conditions $\mathbf{u} = \mathbf{g}$ on $\partial\Omega_f$.

Moreover, we consider also the electric current-potential problem:
find (\mathbf{J}_f, ϕ) s.t.

$$\begin{aligned}\sigma^{-1} \mathbf{J}_f + \nabla \phi - \mathbf{u} \times (\nabla \times \mathbf{A}) &= 0 \quad \text{in } \Omega_f \\ \nabla \cdot \mathbf{J}_f &= 0 \quad \text{in } \Omega_f,\end{aligned}$$

with boundary conditions:

$$\mathbf{J}_f \cdot \mathbf{n} = \mathbf{J}_{inj} \cdot \mathbf{n} \quad \text{on } \Gamma_{inj}, \quad \mathbf{J}_f \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_0.$$

The coupled MHD problem that we consider reads:

find $(\mathbf{A}, \mathbf{u}, p, \mathbf{J}_f, \phi)$ s.t.

$$\begin{aligned}\nabla \times (\mu^{-1} \nabla \times \mathbf{A}) &= \mathbf{J} && \text{in } \Omega \subset \mathbb{R}^3 \\ -\eta \Delta \mathbf{u} + \nabla p - \mathbf{J}_f \times (\nabla \times \mathbf{A}) &= \mathbf{f} && \text{in } \Omega_f \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega_f \\ \sigma^{-1} \mathbf{J}_f + \nabla \phi - \mathbf{u} \times (\nabla \times \mathbf{A}) &= 0 && \text{in } \Omega_f \\ \nabla \cdot \mathbf{J}_f &= 0 && \text{in } \Omega_f,\end{aligned}$$

with the essential boundary conditions:

$$\begin{aligned}\mathbf{n} \times \mathbf{A} &= 0 \text{ on } \partial\Omega, & \mathbf{u} &= \mathbf{g} \text{ on } \partial\Omega_f, \\ \mathbf{J}_f \cdot \mathbf{n} &= \mathbf{J}_{inj} \cdot \mathbf{n} \text{ on } \Gamma_{inj}, & \mathbf{J}_f \cdot \mathbf{n} &= 0 \text{ on } \Gamma_0.\end{aligned}$$

Let \mathcal{T}_h be a regular triangulation of $\overline{\Omega}$ made up of tetrahedra. We assume that the triangulations induced on the subdomains $\Omega \setminus \overline{\Omega}_f$ and Ω_f are compatible on $\partial\Omega_f$.

We consider:

1. Nédélec elements for approximating \mathbf{A}

$$Y_n^A = \{\mathbf{a}_h \in H_{\text{curl}}^0(\Omega) \mid \mathbf{a}_{h|K} \in R_n \quad \forall K \in \mathcal{T}_h\}, \quad n > 1;$$

2. Raviart-Thomas elements for approximating (\mathbf{J}_f, ϕ) :

$$Y_m^J = \{\mathbf{y}_h \in H_{\text{div}}^0(\Omega_f) \mid \mathbf{y}_{h|K} \in \mathbb{D}_m \quad \forall K \in \mathcal{T}_h\}, \quad m \geq 1,$$

$$W_{m-1}^\phi = \{\psi_h \in L_0^2(\Omega_f) \mid \psi_{h|K} \in \mathbb{P}_{m-1} \quad \forall K \in \mathcal{T}_h\};$$

3. Taylor-Hood elements for approximating (\mathbf{u}, p) :

$$W_r^u = \{\mathbf{v}_h \in C^0(\overline{\Omega}_f) \mid \mathbf{v}_{h|K} \in \mathbb{P}_r \quad \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega_f), \quad r \geq 2,$$

$$W_{r-1}^p = \{q_h \in C^0(\overline{\Omega}_f) \mid q_{h|K} \in \mathbb{P}_{r-1} \quad \forall K \in \mathcal{T}_h\} \cap L_0^2(\Omega_f).$$

The discrete weak form of the coupled problem reads:

find $\mathbf{A}_h \in Y_n^A$, $\mathbf{u}_h \in W_r^u$, $p_h \in W_{r-1}^p$, $\mathbf{J}_{f,h} \in Y_m^J$, $\phi_h \in W_{m-1}^\phi$ s.t.

$$\int_{\Omega} \mu^{-1} (\nabla \times \mathbf{A}_h) \cdot (\nabla \times \mathbf{W}_h) - \int_{\Omega} \mathbf{J}_h \cdot \mathbf{W}_h = 0$$

$$\int_{\Omega_f} \eta \, \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h - \int_{\Omega_f} p_h \, \nabla \cdot \mathbf{v}_h - \int_{\Omega_f} \mathbf{J}_{f,h} \times (\nabla \times \mathbf{A}_h) \cdot \mathbf{v}_h = 0$$

$$\int_{\Omega_f} q_h \, \nabla \cdot \mathbf{u}_h = 0$$

$$\int_{\Omega_f} \sigma^{-1} \mathbf{J}_{f,h} \cdot \mathbf{K}_h - \int_{\Omega_f} \phi_h \, \nabla \cdot \mathbf{K}_h + \int_{\Omega_f} \mathbf{K}_h \times (\nabla \times \mathbf{A}_h) \cdot \mathbf{u}_h = 0$$

$$\int_{\Omega_f} \varphi_h \, \nabla \cdot \mathbf{J}_{f,h} = 0$$

for all $\mathbf{W}_h \in Y_n^A$, $\mathbf{v}_h \in W_r^u$, $q_h \in W_{r-1}^p$, $\mathbf{K}_h \in Y_m^J$, $\varphi_h \in W_{m-1}^\phi$.

If we denote by $y = (\mathbf{A}_h, \mathbf{u}_h, p_h, \mathbf{J}_{f,h}, \phi_h)$ the state variable and by u an appropriate set of control variables, we can rewrite the former system in the compact form $e(y, u) = 0$.

Then, the linearization wrt the state variables $e_y(y, u)\delta y$ reads

$$\begin{aligned}
& \int_{\Omega} \mu^{-1} (\nabla \times \delta \mathbf{A}_h) \cdot (\nabla \times \mathbf{W}_h) - \int_{\Omega} \delta \mathbf{J}_h \cdot \mathbf{W}_h \\
& \int_{\Omega_f} \eta \nabla \delta \mathbf{u}_h \cdot \nabla \mathbf{v}_h - \int_{\Omega_f} \delta p_h \nabla \cdot \mathbf{v}_h - \int_{\Omega_f} \delta \mathbf{J}_{f,h} \times (\nabla \times \mathbf{A}_h) \cdot \mathbf{v}_h - \int_{\Omega_f} \mathbf{J}_{f,h} \times (\nabla \times \delta \mathbf{A}_h) \cdot \mathbf{v}_h \\
& \int_{\Omega_f} q_h \nabla \cdot \delta \mathbf{u}_h \\
& \int_{\Omega_f} \sigma^{-1} \delta \mathbf{J}_{f,h} \cdot \mathbf{K}_h - \int_{\Omega_f} \delta \phi_h \nabla \cdot \mathbf{K}_h + \int_{\Omega_f} \mathbf{K}_h \times (\nabla \times \mathbf{A}_h) \cdot \delta \mathbf{u}_h + \int_{\Omega_f} \mathbf{K}_h \times (\nabla \times \delta \mathbf{A}_h) \cdot \mathbf{u}_h \\
& \int_{\Omega_f} \varphi_h \nabla \cdot \delta \mathbf{J}_{f,h}
\end{aligned}$$

We denote

$$M_{ij} = \left(\int_{\Omega} \mu^{-1} (\nabla \times \delta \mathbf{A}_h) \cdot (\nabla \times \mathbf{W}_h) \right)_{ij} \quad F_{ij} = \left(\int_{\Omega} \delta \mathbf{J}_h \cdot \mathbf{W}_h \right)_{ij}$$

$$A_{ij} = \left(\int_{\Omega_f} \eta \nabla \delta \mathbf{u}_h \cdot \nabla \mathbf{v}_h \right)_{ij} \quad B_{ji} = \left(- \int_{\Omega_f} q_h \nabla \cdot \mathbf{u}_h \right)_{ij}$$

$$D_{ij} = \left(\int_{\Omega_f} \sigma^{-1} \delta \mathbf{J}_{f,h} \cdot \mathbf{K}_h \right)_{ij} \quad E_{ji} = \left(- \int_{\Omega_f} \varphi_h \nabla \cdot \delta \mathbf{J}_{f,h} \right)_{ij}$$

$$C(\mathbf{A}_h)_{ij} = - \int_{\Omega_f} \delta \mathbf{J}_h^i \times (\nabla \times \mathbf{A}_h) \cdot \mathbf{v}_h^j$$

$$G(\mathbf{J}_{f,h})_{ij} = - \int_{\Omega_f} \mathbf{J}_{f,h}^i \times (\nabla \times \delta \mathbf{A}_h) \cdot \mathbf{v}_h^j$$

$$H(\mathbf{u}_h)_{ij} = \int_{\Omega_f} \mathbf{K}_h^i \times (\nabla \times \delta \mathbf{A}_h^j) \cdot \mathbf{u}_h$$

Then, we have the matrix formulation

$$\begin{pmatrix} M & & -F \\ G(\mathbf{J}_{f,h})^T & A & B^T & C(\mathbf{A}_h)^T \\ & B & & \\ H(\mathbf{u}_h) & -C(\mathbf{A}_h) & D & E^T \end{pmatrix} \begin{pmatrix} \delta\mathbf{A}_h \\ \delta\mathbf{u}_h \\ \delta p_h \\ \delta\mathbf{J}_h \\ \delta\varphi_h \end{pmatrix}$$

In addition, the full KKT matrix, which is in general

$$\begin{pmatrix} L_{yy} & L_{yu} & e_y^\top \\ L_{uy} & L_{uu} & e_u^\top \\ e_y & e_u & \end{pmatrix},$$

is given by

$$\left(\begin{array}{cccc|ccc} -G(\lambda_y)^T & -G(\lambda_y) & -H(\lambda_u)^T & & M & G(\mathbf{J}_h) & H(\mathbf{u}_h)^T \\ & * & & & A & B^T & -C(\mathbf{A}_h)^T \\ -H(\lambda_u) & & & & B & C(\mathbf{A}_h) & D \\ \hline & & & & -F^T & & E^T \\ M & G(\mathbf{J}_h)^T & A & B^T & -F & C(\mathbf{A}_h)^T & \\ & & B & & & & \\ H(\mathbf{u}_h) & -C(\mathbf{A}_h) & & D & & E^T & \\ & & & E & & & \end{array} \right)$$

The “*” correspond to entries which depend on the specific type of control chosen and the objective functional in the optimization.