A discontinuos Galerkin method with nonoverlapping domain decomposition for the Stokes and Navier-Stokes problems

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1. General introduction:

This Paper is devoted to the numerical analysis of a discontinuous Galerkin method with nonoverlapping domain decomposition, of order $k = 1, 2$ or 3 for solving the steady incompressible Stokes and Navier-Stokes systems of equations.

The authors analyze the discrete Stokes problem by establishing a uniform discrete inf-sup condition for the pressure, which is vital for proving the optimal estimation for the velocity and pressure.

The nonlinear convection term of the Navier-Stokes equation is discretized by adapting to totally discontinuous velocities the upwind scheme introduced by Lesaint and Raviart. The authors analyzed the nonlinearity by proving the uniform L^p estimates for the discrete velocity. This enables us to prove the existence of discrete solution and error estimates.

Outline of the presentation

2. Model Stokes and Navier-Stokes problems

Let Ω be a Lipschitz domain of R^2. Let $\mathbf{f} \in H^{-1}(\Omega)^2$ and $\mu > 0$.

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1}$$
$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \tag{2}$$
$$\mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega. \tag{3}$$

If additionally, $\int_\Omega p = 0$, there exist a unique solution $\mathbf{u} \in H_0^1(\Omega)^2$, $p \in L_0^2(\Omega)$ [12], where

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\},$$
$$L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}.$$

[12]. Girault and Raviart *Finite element methods for the steady Navier-Stokes problem in polyhedra*, Springer Series in Computational Mathematics 5 (1986).

2. Model Stokes and Navier-Stokes problems

However, in what follows, we shall need both the gradient of $\mathbf{u}$ and the pressure p have a trace on line segments. For this, it suffices for instance that the data $\mathbf{f}$ belong to $L^{4/3}(\Omega)^2$.

In [14], we have that if Ω is a Lipschitz polygon and $\mathbf{f} \in L^{4/3}(\Omega)^2$, then the solution $(\mathbf{u}, p)$ belongs to $W^{2,4/3}(\Omega)^2 \times W^{1,4/3}(\Omega)$ with continuous dependence on $\|\mathbf{f}\|_{L^{4/3}(\Omega)}$. Thus each component of the gradient of $\mathbf{u}$ has a trace on a line segment e, this trace belongs to $W^{1,4/3}(e) \hookrightarrow L^2(e)$. The same result holds for the trace of the pressure. Therefore, the trace is well defined and belongs to $L^2(e)$.

[14]. Grisvard *Elliptic problems in nonsmooth domains*, Pitman Monographs and Studies in Mathematics 24.

2. Model Stokes and Navier-Stokes problems

Stokes system is a linearized vision of the Navier-Stokes system

$$\begin{aligned} -\mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0}, \quad \text{on } \partial\Omega. \end{aligned} \tag{4}$$

where

$$\mathbf{u}\cdot\nabla\mathbf{u} = \sum_{i=1}^{2} u_i \frac{\partial\mathbf{u}}{\partial x_i}$$

is the convection term.

[12], (4),(2),(3) always has a solution (not necessary unique) $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$.

Since $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p < \infty$, we have $\mathbf{u}\cdot\nabla\mathbf{u}$ belongs to $L^{4/3}(\Omega)^2$. Therefore, if $\mathbf{f}$ belongs to $L^{4/3}(\Omega)^2$, every solution $(\mathbf{u}, p)$ of (4),(2),(3) belongs to $W^{2,4/3}(\Omega)^2 \times W^{1,4/3}(\Omega)$.

3. Notation and preliminaries

Assume Ω is a Lipschitz polygon partitioned into two sub-domains Ω_1 and Ω_2, both Lipschitz polygons, with interface γ, i.e., $\Omega = \Omega_1 \cup \gamma \cup \Omega_2$ (see Figure 1).

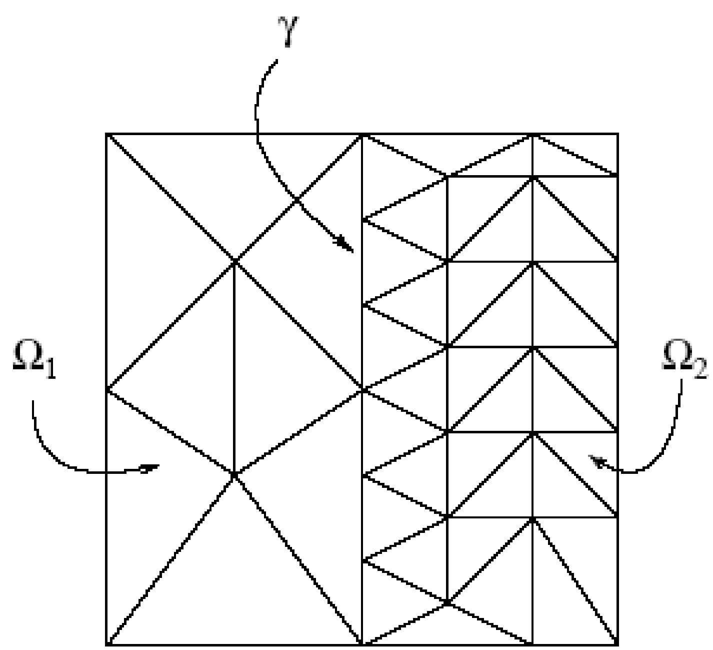

FIGURE 1. Example of two subdomains where γ_h^2 is a subgrid of γ_h^1.

3. Notation and preliminaries For $i = 1, 2$, define

- $\mathcal{E}_h^i$ be a regular family of triangulations of $\overline{\Omega_i}$, consisting of triangles of maximum diameter h;

- Γ_h^i be the set of all edges of $\mathcal{E}_h^i$ that do not lie on γ, set $\Gamma_h = \Gamma_h^1 \cup \Gamma_h^2$;

- γ_h^i be the set of edges of $\mathcal{E}_h^i$ that lie on γ.

At the interface γ, the two meshes $\mathcal{E}_h^i$ are related by two assumptions:

Hypothesis H1: Either γ_h^1 is a subgrid of γ_h^2 or γ_h^2 is a subgrid of γ_h^1;

Hypothesis H2: There exist two constants L_1 and L_2 independent of h such that for any pair of segments $e_1 \in \gamma_h^1$ end $e_2 \in \gamma_h^2$ such that $|e_1 \cap e_2| > 0$, we have

$$\frac{|e_1|}{|e_2|} \leq L_1 \quad \text{and} \quad \frac{|e_2|}{|e_1|} \leq L_2.$$

3. Notation and preliminaries

Some definition: for a domain $\mathcal{O} \subset R^2$,

$$
\begin{aligned}
W^{k,r}(\mathcal{O}) &= \{v \in L^r(\mathcal{O}) : \quad \forall |m| \le k, \quad \partial^m v \in L^r(\mathcal{O})\}, \\
X &= \{v \in L^2(\Omega) : \quad \forall E \in \mathcal{E}_h, \quad v|_E \in W^{2,4/3}(E)\}, \\
M &= \{v \in L^2_0(\Omega) : \quad \forall E \in \mathcal{E}_h, \quad v|_E \in W^{1,4/3}(E)\}. \\
H^k(\mathcal{O}) &= W^{k,2}(\mathcal{O}), \\
|\!|\!|\cdot|\!|\!| &= \sum_{E \in \mathcal{E}_h} \|\cdot\|^2_{k,E}, \text{ where } \|\cdot\|_{k,\mathcal{O}} \text{ Sobolev norm of } H^k(\mathcal{O}). \\
\mathbf{X} &= X^2 \\
\mathcal{D}(\mathcal{O}) &= \{\text{Infinitely differentiable function with compact support on } \mathcal{O}\}, \\
\mathcal{D}'(\mathcal{O}) &= \{\text{Distributions on } \mathcal{O}\}, \\
\mathbf{v} &= (v_i)_i \\
\nabla \mathbf{v} &= \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j} \\
[\![\phi]\!] &= (\phi|_{E^k})|_e - (\phi|_{E^l})|_e, \\
\{\phi\} &= \frac{1}{2}(\phi|_{E^k})|_e + \frac{1}{2}(\phi|_{E^l})|_e,
\end{aligned}
$$

3. Notation and preliminaries

Introduce the following bilinear forms on $\mathbf{X} \times \mathbf{X}$ and $\mathbf{X} \times M$

$$
\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \sum_{E \in \mathcal{E}_h} \int_E \nabla \mathbf{u} : \nabla \mathbf{v} - \sum_{e \in \Gamma_h \cup \gamma_h^1} \int_e \{\nabla \mathbf{u}\} \mathbf{n}_e \cdot [\![\mathbf{v}]\!] \\
&+ \epsilon^* \sum_{e \in \Gamma_h \cup \gamma_h^1} \int_e \{\nabla \mathbf{v}\} \mathbf{n}_e \cdot [\![\mathbf{u}]\!], \\
b(\mathbf{v}, p) &= -\sum_{E \in \mathcal{E}_h} \int_E p \nabla \cdot \mathbf{v} + \sum_{e \in \Gamma_h} \int_e \{p\} [\![\mathbf{v}]\!] \cdot \mathbf{n}_e + \sum_{e \in \gamma_h^1} \int_e p_\gamma [\![\mathbf{v}]\!] \cdot \mathbf{n}_1, \\
J_0(\mathbf{u}, \mathbf{v}) &= \sum_{e \in \Gamma_h \cup \gamma_h^1} \frac{\sigma_e}{|e|} \int_e [\![\mathbf{u}]\!] \cdot [\![\mathbf{v}]\!],
\end{aligned}
$$

where p_γ is the trace of p on the coarser mesh, i.e.

$$
p_\gamma = \begin{cases} p|_{\Omega_1}, & \text{if } \gamma_h^2 \text{ is a subgrid of } \gamma_h^1; \\ p|_{\Omega_2}, & \text{if } \gamma_h^1 \text{ is a subgrid of } \gamma_h^2. \end{cases}
$$

For ϵ^*, $\epsilon^* = 1$ is the nonsymmetric case (**Case A**); while $\epsilon^* = -1$ is the symmetric case (**Case B**).

3. Notation and preliminaries

Remark 3.1 The form b defined by (7) can also be written

$$b(\mathbf{v}, p) = -\sum_{E \in \mathcal{E}_h} \int_E p \nabla \cdot \mathbf{v} + \sum_{e \in \Gamma_h \cup \gamma_h^1} \int_e \{p\} [\![\mathbf{v}]\!] \cdot \mathbf{n}_e + \frac{\epsilon}{2} \sum_{e \in \gamma_h^1} \int_e [\![p]\!] [\![\mathbf{v}]\!] \cdot \mathbf{n}_1,$$

where

$$\epsilon = \begin{cases} 1, & \text{if } \gamma_h^2 \text{ is a subgrid of } \gamma_h^1; \\ -1, & \text{if } \gamma_h^1 \text{ is a subgrid of } \gamma_h^2. \end{cases}$$

Remark 3.2 Note that if $\mathbf{u}$ and $\mathbf{v}$ both belong to $H_0^1(\Omega)^2$, then formally

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \qquad \text{and} \qquad b(\mathbf{v}, p) = -\int_\Omega p \nabla \cdot \mathbf{v},$$

which are the standard bilinear forms associated with the Stokes problems.

3. Notation and preliminaries With these forms, we consider the following variational problem: Find $\mathbf{u} \in \mathbf{X}$ and $p \in M$, solution of

$$\mu(a(\mathbf{u},\mathbf{v}) + J_0(\mathbf{u},\mathbf{v})) + b(\mathbf{v},p) = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \qquad \forall \mathbf{v} \in \mathbf{X}, \tag{5}$$

$$b(\mathbf{u},q) = 0, \qquad \forall q \in M. \tag{6}$$

Remark 3.3 Note that all functions $\mathbf{v}$ in $\mathbf{X}$ satisfy

$$-\sum_{E\in\mathcal{E}_h} \int_E \nabla \cdot \mathbf{v} + \sum_{e\in\Gamma_h\cup\gamma_h^1} \int_e [\![\mathbf{v}]\!] \cdot \mathbf{n}_e = 0.$$

Therefore we can relax the zero mean-value constraint in (6); i.e., (6) is equivalent to

$$b(\mathbf{u},q) = 0, \quad \forall q \in Q = \{q \in L^2(\Omega) : \forall E \in \mathcal{E}_h, q|_E \in W^{1,4/3}(E)\}.$$

Lemma 3.4 If $\mathbf{f} \in L^{4/3}(\Omega)^2$. If $(\mathbf{u},p)$ is the solution of (1)-(3), then $(\mathbf{u},p)$ satisfies the variational problem (5), (6) and conversely.

Remark 3.5 Note that the jump term J_0 plays no part in the proof and therefore the statement of Lemma 3.4 is valid even if J_0 is suppressed from (5).

3. Notation and preliminaries

In order to approximate $\mathbf{u}$ and p, we introduce two finite-dimensional spaces $\mathbf{X}_h \subset \mathbf{X}$ and $M_h \subset M$, such that

$$\begin{aligned} X_h &= \{v_h \in X : \quad \forall E \in \mathcal{E}_h, \quad v_h \in \mathbb{P}_k(E)\}, \quad \mathbf{X}_h = X_h \times X_h, \\ M_h &= \{q_h \in M : \quad \forall E \in \mathcal{E}_h, \quad q_h \in \mathbb{P}_{k-1}(E)\}. \end{aligned}$$

With these spaces, the discrete scheme is: find $(\mathbf{U}, P) \in \mathbf{X}_h \times M_h$ such that

$$\forall \mathbf{v}_h \in \mathbf{X}_h, \quad \mu(a(\mathbf{U}, \mathbf{v}_h) + J_0(\mathbf{U}, \mathbf{v}_h)) + b(\mathbf{v}_h, P) = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h, \tag{7}$$

$$\forall q_h \in M_h, \quad b(\mathbf{U}, q_h) = 0. \tag{8}$$

We denote by $\mathbf{V}_h$ the kernel of b in $\mathbf{X}_h$:

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h; \quad \forall q_h \in M_h, \quad b(\mathbf{v}_h, q_h) = 0\}.$$

3. Notation and preliminaries

Finally, we recall the approximation properties of $\mathbf{X}_h$ and M_h.

Recall that the meshes $\mathcal{E}_h^i$ are regular. For $k \geq 1$, it is easy to construct an operator $r_h \in \mathcal{L}(L_0^2(\Omega); M_h)$, such that, for any $E \in \mathcal{E}_h$,

$$\forall q \in \mathbb{P}_{k-1}(E) \qquad \int_E q(r_h(p) - p) = 0, \tag{9}$$

and for any real number $s \in [0, k]$,

$$\forall q \in H^s(\Omega) \cap L_0^2(\Omega), \qquad \|q - r_h(q)\|_{0,E} \leq c h_E^s |q|_{s,E}. \tag{10}$$

For each $k = 1, 2, 3$, there exists an operator $\mathbf{R}_h^i \in \mathcal{L}(H^1(\Omega_i)^2; \mathbf{X}_h(\Omega_i))$, where $\mathbf{X}_h(\Omega_i)$ denotes the space $\mathbf{X}_h$ restricted to Ω_i, such that for any $E \in \mathcal{E}_h$,

3. Notation and preliminaries

$$\forall \mathbf{v} \in H^1(\Omega_i)^2, \quad \forall q_h \in \mathbb{P}_{k-1}(E), \quad \int_E q_h \nabla \cdot (\mathbf{R}_h^i(\mathbf{v}) - \mathbf{v}) = 0, \quad (11)$$

$$\forall v \in H^1(\Omega_i)^2, \quad \forall e \text{ of } \Gamma_h^i, \quad \forall \mathbf{q}_h \in \mathbb{P}_{k-1}(e)^2, \quad \int_e \mathbf{q}_h \cdot [\![\mathbf{R}_h^i(\mathbf{v})]\!] = 0, \quad (12)$$

$$\forall \mathbf{v} \in H_0^1(\Omega_i)^2, \quad \forall e \in \partial\Omega_i, \forall \mathbf{q}_h \in \mathbb{P}_{k-1}(e)^2, \quad \int_e \mathbf{q}_h \cdot \mathbf{R}_h^i(\mathbf{v}) = 0, \quad (13)$$

$$\forall s \in [1, k+1], \quad \forall \mathbf{v} \in H^s(\Omega_i)^2, \quad |\mathbf{v} - \mathbf{R}_h^i(\mathbf{v})|_{1,E} \le c h_E^{s-1} |\mathbf{v}|_{s,\Delta_E}, \quad (14)$$

where Δ_E is a suitable macro-element containing E. Also, for $m = 0$ or 1, for any $t \ge 2$, for $s \in [1, k+1]$,

$$\forall \mathbf{v} \in W^{s,t}(\Omega_i)^2, \quad |\mathbf{v} - \mathbf{R}_h^i(\mathbf{v})|_{W^{m,t}(E)} \le C h_E^{s-m} |\mathbf{v}|_{W^{s,t}(\Delta_E)}. \quad (15)$$

Furthermore, each triangle $E \in \mathcal{E}_h^i$ has at least one side e such that

$$\forall \mathbf{v} \in H^1(\Omega_i)^2, \quad \int_e (\mathbf{R}_h^i(\mathbf{v}) - \mathbf{v}) = \mathbf{0} \quad (16)$$

3. Notation and preliminaries

An easy consequence of (16) is the following lemma

Lemma 3.6 Assume that $\mathcal{E}_h^i$ is a regular family of trianglations. Then there exists a constant C independent of h, such that

$$\forall \mathbf{v} \in (H_0^1(\Omega_i))^2,$$
$$\left(\sum_{e \in \Gamma_h^i} \frac{1}{|e|} \| \left[\!\left[\mathbf{v} - \mathbf{R}_h^i(\mathbf{v}) \right]\!\right] \|_{0,e}^2 \right)^{\frac{1}{2}} \leq C \left\| \nabla (\mathbf{v} - \mathbf{R}_h^i(\mathbf{v})) \right\|_{0,\Omega_i}.$$

4. An inf-sup condition For proving the inf-sup condition, we define a norm on $\mathbf{X}$ which is more appropriate than the broken H^1 norm.

$$\forall \mathbf{v}_h \in \mathbf{X}_h, \quad [\![\mathbf{v}_h]\!] = \left(\sum_{E \in \mathcal{E}_h} \|\nabla \mathbf{v}\|^2_{0,E} + J_0(\mathbf{v}_h, \mathbf{v}_h) \right)^{1/2} .$$

Under the Hypotheses H1 and H2, we have the following preliminary result.

Lemma 4.1 Let the mesh $\mathcal{E}_h$ is regular and satisfy Hypothesis H2. Let $\mathbf{v} \in H^1(\Omega)^2$, let $k = 1, 2$ or 3 and let $\mathbf{R}_h(\mathbf{v})$ denote the operators $\mathbf{R}^1_h(\mathbf{v})$ in Ω_1 and $\mathbf{R}^2_h(\mathbf{v})$ in Ω_2 satisfying (11)-(16). Then there exists a constant C depending only on k, L_1, L_2 and the triangle-regular constant σ such that $\forall p_h \in M_h$,

$$\left| \sum_{e \in \Gamma_h \cup \gamma^1_h} \int_e \{p_h\} \, [\![\mathbf{R}_h(\mathbf{v}) - \mathbf{v}]\!] \cdot \mathbf{n}_e \right| \leq C \|p_h\|_{0,D^{12}} \, |\!|\!| \nabla (\mathbf{R}_h(\mathbf{v}) - \mathbf{v}) |\!|\!|_{0,D^{12}} ,$$

where $D^{12} = D_1 \cup D_2$ and D_i denotes the union of elements of $\mathcal{E}^i_h$ adjacent to γ.

Now we address the existence and uniqueness of discrete scheme (7), (8).

$$\begin{aligned} \forall \mathbf{v}_h \in \mathbf{X}_h, \quad \mu(a(\mathbf{U},\mathbf{v}_h) + J_0(\mathbf{U},\mathbf{v}_h)) + b(\mathbf{v}_h, P) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}_h, \\ \forall q_h \in M_h, \quad b(\mathbf{U}, q_h) &= 0. \end{aligned}$$

Consider $\mathbf{f} = \mathbf{0}$, let $\mathbf{v}_h = \mathbf{U}$. In the nonsymmetric case, the existence and uniqueness is obvious. For symmetric case, we make the following assumption:

Hypothesis H3:[27] There exists a constant $K > 0$, independent of h, such that

$$\forall \mathbf{v} \in \mathbf{X}_h, \quad a(\mathbf{v}_h,\mathbf{v}_h) + J_0(\mathbf{v}_h,\mathbf{v}_h) \geq K \, [\![\mathbf{v}_h]\!]^2 .$$

Then existence and uniqueness will be followed also for symmetric case.

[27]: Wheeler, *An elliptic collocation-finite element method with interior penalties*, SIAM J. Numer. Anal. 15, 152-161.

With the fact that γ_h^1 is a subgrid of γ_h^2 (or vice versal) we will have the inf-sup condition theorem:

<u>Theorem 4.2</u> Let the regular mesh $\mathcal{E}_h$ satisfy Hypotheses H1 and H2. Then there exists a constant $\beta^* > 0$, independent of h, such that

$$\inf_{p_h \in M_h} \sup_{\mathbf{v}_h \in \widetilde{\mathbf{X}}_h} \frac{b(\mathbf{v}_h, p_h)}{[\![\mathbf{v}_h]\!] \, \|p_h\|_0} \geq \beta^*,$$

where

$$\widetilde{\mathbf{X}}_h = \{\mathbf{v}_h \in \mathbf{X}_h : \forall e \in \Gamma_h, \quad \int_e \mathbf{q}_h \cdot [\![\mathbf{v}_h]\!] = 0, \quad \forall \mathbf{q}_h \in \mathbb{P}_{k-1}(e)^2\}.$$

Theorem 4.2 can be extended by induction to a fixed number of subdomains.

Then, the inf-sup condition allows us to construct a good approximation operator.

Corollary 4.3 Under the assumption of Theorem 4.2, there exists an approximation operator $\mathbf{P}_h \in \mathcal{L}(H_0^1(\Omega)^2; \widetilde{\mathbf{X}}_h)$ such that for any $s \in [1, k+1]$:

$$\forall \mathbf{v} \in H_0^1(\Omega)^2, \quad \forall q_h \in M_h, \quad b(\mathbf{P}_h(\mathbf{v}) - \mathbf{v}, q_h) = 0,$$

$$\forall \mathbf{v} \in (H^s(\Omega) \cap H_0^1(\Omega))^2, \quad [\![\mathbf{P}_h(\mathbf{v}) - \mathbf{v}]\!] \leq C h^{s-1} |\mathbf{v}|_{s,\Omega},$$

$$\forall \mathbf{v} \in H_0^1(\Omega)^2, \quad \forall e \in \Gamma_h, \quad \forall \mathbf{q} \in \mathbb{P}_{k-1}(e^2), \quad \int_e [\![\mathbf{P}_h(\mathbf{v}) - \mathbf{v}]\!] \cdot \mathbf{q} = 0.$$

5. Error estimation

Theorem 5.1 Let $k = 1, 2$ or 3 be the degree of the polynomials in the definition of $\mathbf{X}_h$ and assume that the solution $(\mathbf{u}, p)$ of problem (1)-(3) belongs to $H^{k+1}(\Omega)^2 \times H^k(\Omega)$. Then, if the regular triangulation satisfies Hypotheses H1 and H2, and if H3 holds, the solution $(\mathbf{U}, P)$ of (7),(8) satisfies the error estimate

$$[\![\mathbf{u} - \mathbf{U}]\!] \leq Ch^k(|\mathbf{u}|_{k+1} + \frac{1}{\mu}|p|_k),$$

where C is independent of h and μ.

Theorem 5.2 Under the assumption and notation of Theorem 5.1, we have

$$\|p - P\|_0 \leq Ch^k(\mu|\mathbf{u}|_{k+1} + |p|_k),$$

with a constant C independent of h and μ.

5. Error estimation

We now address the estimate for the velocity in the L^2 norm. The convergence is optimal for the symmetric case (**Case A**), but lose a power of h in nonsymmetric case (**Case B**).

Theorem 5.3 Assume that Ω is convex. Then, under the hypotheses of Theorem 5.1, there exist a constant C, independent of h and μ such that

$$\|\mathbf{u} - \mathbf{U}\|_0 \leq Ch^{k+1}(|\mathbf{u}|_{k+1} + \frac{1}{\mu}|p|_k), \qquad \epsilon^* = -1,$$
$$\|\mathbf{u} - \mathbf{U}\|_0 \leq Ch^{k}(|\mathbf{u}|_{k+1} + \frac{1}{\mu}|p|_k), \qquad \epsilon^* = +1.$$

6. Navier-Stokes problem

Recall the Navier-Stokes system (4),(2),(3)

$$\begin{aligned} -\mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0}, \quad \text{on } \partial\Omega. \end{aligned}$$

Discretize the nonlinear convection term $\mathbf{u}\cdot\nabla\mathbf{u}$ with an upwind discretization of $\mathbf{u}\cdot\nabla\mathbf{z}$: $\forall \mathbf{u},\mathbf{z},\theta \in \mathbf{X}$

$$\begin{aligned} c(\mathbf{u};\mathbf{z},\theta) &= \sum_{E\in\mathcal{E}_h}\left(\int_E (\mathbf{u}\cdot\nabla\mathbf{z})\cdot\theta + \int_{\partial E_-} |\{\mathbf{u}\}\cdot\mathbf{n}_E|(\mathbf{z}^{int}-\mathbf{z}^{ext})\cdot\theta^{int}\right) \\ &+ \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E (\nabla\cdot\mathbf{u})\mathbf{z}\cdot\theta - \frac{1}{2}\sum_{e\in\Gamma_h\cup\gamma_h^1}\int_e [\![\mathbf{u}]\!]\cdot\mathbf{n}_e\{\mathbf{z}\cdot\theta\}, \end{aligned}$$

where

$$\partial E_- = \{\mathbf{x}\in\partial E : \{\mathbf{u}\}\cdot\mathbf{n}_E < 0\},$$

and the superscript int (resp. ext) refers to the trace of the function on a side of E coming from the interior of E (resp. coming from the exterior of E on that side).

6. Navier-Stokes problem

Then we discretize Navier-Stokes system by: find $(\mathbf{U}, P) \in \mathbf{X}_h \times M_h$ such that

$$\begin{aligned} &\forall \mathbf{v}_h \in \mathbf{X}_h, \\ \mu(a(\mathbf{U}, \mathbf{v}_h) + J_0(\mathbf{U}, \mathbf{v}_h)) + c(\mathbf{U}; \mathbf{U}, \mathbf{v}_h) + b(\mathbf{v}_h, P) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h, \quad (17) \\ \forall q_h \in M_h, \quad b(\mathbf{U}, q_h) &= 0. \quad (18) \end{aligned}$$

It is easy to see that, when $\mathbf{u}, \mathbf{z}, \theta \in H_0^1(\Omega)^2$, c reduces to

$$c(\mathbf{u}; \mathbf{z}, \theta) = \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{z}) \cdot \theta + \frac{1}{2} \int_\Omega (\nabla \cdot \mathbf{u}) \mathbf{z} \cdot \theta.$$

Considering Lemma 3.4, we have that every solution of the Navier-Stokes problem (4),(2),(3) is also a solution of (17) (18) and conversely.

6. Navier-Stokes problem

Lemma 6.1 The trilinear form c defined satisfies the following "integration by parts" for all $\mathbf{u}, \mathbf{z}, \theta$ in $\mathbf{X}$

$$\begin{aligned} c(\mathbf{u}; \mathbf{z}, \theta) &= -\sum_{E \in \mathcal{E}_h} \left(\int_E (\mathbf{u} \cdot \nabla \theta) \cdot \mathbf{z} + \frac{1}{2} \int_E (\nabla \cdot \mathbf{u}) \mathbf{z} \cdot \theta \right) \\ &+ \frac{1}{2} \sum_{e \in \Gamma_h \cup \gamma_h^1} \int_e [\![\mathbf{u}]\!] \cdot \mathbf{n}_e \{\mathbf{z} \cdot \theta\} \\ &- \sum_{E \in \mathcal{E}_h} \int_{\partial E_-} |\{\mathbf{u}\} \cdot \mathbf{n}_E| \mathbf{z}^{ext} \cdot (\theta^{int} - \theta^{ext}) + \int_{\Gamma_+} |\mathbf{u} \cdot \mathbf{n}| \mathbf{z} \cdot \theta, \end{aligned}$$

where Γ_+ is the subset of $\partial\Omega$ where $\mathbf{u} \cdot \mathbf{n} > 0$. In particular, if $\mathbf{z} = \theta$, we obtain

$$c(\mathbf{u}; \mathbf{z}, \mathbf{z}) = \frac{1}{2} \sum_{E \in \mathcal{E}_h} \int_{\partial E_-} |\{\mathbf{u}\} \cdot \mathbf{n}_E| \|\mathbf{z}^{ext} - \mathbf{z}^{int}\|^2 + \frac{1}{2} \int_{\Gamma_+} |\mathbf{u} \cdot \mathbf{n}| \|\mathbf{z}\|^2 \geq 0.$$

6. Navier-Stokes problem

Lemma 6.2 Assume that the regular triangulation satisfies Hypothesis H2. Then for each real number $p \in [2, \infty)$ there exists a constant $C(p)$, independent of h, such that

$$\forall v \in H^1(\mathcal{E}_h), \quad \|v\|_{L^p(\Omega)} \le C(p) \, [\![v]\!] \, .$$

Here

$$H^1(\mathcal{E}_h) = \{v \in L^2(\Omega) : \forall E \in \mathcal{E}_h, \quad v|_E \in H^1(E)\}.$$

6. Navier-Stokes problem With Lemma 6.1 and Lemma 6.2, together with inf-sup condition Theorem 4.2, we are able to prove the existence of discrete solution and a priori estimates.

Proposition 6.3 Assume that the regular triangelation satisfy Hypotheses H1 and H2. Then, if H3 holds, for any $\mathbf{f}$ in $L^{4/3}(\Omega)^2$ and $\mu > 0$, the discrete Navier-Stokes problem (17),(18) has at least one solution $(\mathbf{U}, P)$ in $\mathbf{X}_h \times M_h$, and each solution satisfies the a priori estimates

$$
\begin{aligned}
[\![\mathbf{U}]\!] &\leq \frac{C(4)}{\mu K}\|\mathbf{f}\|_{L^{4/3}(\Omega)}, \\
\sum_{E\in\mathcal{E}_h}\int_{\partial E_-}|\{\mathbf{U}\}\cdot\mathbf{n}_E|\|\mathbf{U}^{int}-\mathbf{U}^{ext}\|^2 &+ \int_{\Gamma_+}|\mathbf{U}\cdot\mathbf{n}|\|\mathbf{U}\|^2 \leq \frac{2}{\mu K}C(4)^2\|\mathbf{f}\|^2_{L^{4/3}(\Omega)}, \\
\|P\|_0 &\leq C(\|\mathbf{f}\|_{L^{4/3}(\Omega)} + [\![\mathbf{U}]\!]^2).
\end{aligned}
$$

where $C(4)$ is the constant of Lemma 6.2 with exponent 4 and C is another constant that depends on β^* but is independent of h and μ.

6. Navier-Stokes problem

Now, we turn to error estimates. Let $(\mathbf{U}, P)$ be a solution of (17),(18), let $(\mathbf{u}, p)$ be a solution of (4),(2),(3), let $\mathbf{U}_I = \mathbf{P}_h(\mathbf{u})$ be the operator defined in Corollary 4.3 and let $P_I = r_h(p)$ be the operator defined in (9). Set $\chi = \mathbf{U} - \mathbf{U}_I$, $\xi = P - P_I$, we obtain the following equation:

$$\mu(a(\chi,\chi) + J_0(\chi,\chi)) + c(\mathbf{U}_I;\chi,\chi) + c(\chi;\mathbf{U},\chi) = \mu(a(\mathbf{u}-\mathbf{U}_I,\chi)$$
$$+J_0(\mathbf{u}-\mathbf{U}_I,\chi)) + b(\chi, p - P_I) + c(\mathbf{u}-\mathbf{U}_I;\mathbf{U}_I,\chi) + c(\mathbf{u};\mathbf{u}-\mathbf{U}_I,\chi).$$

The left-hand side is bounded below by

$$K\mu\,[\![\chi]\!]^2 + \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_{\partial E_-} |\{\mathbf{U}_I\cdot\mathbf{n}_e\}|\|\chi^{ext} - \chi^{int}\|^2 +$$
$$\frac{1}{2}\int_{\Gamma_+} |\mathbf{U}\cdot\mathbf{n}|\|\chi\|^2 + c(\chi;\mathbf{U},\chi).$$

Therefore, we must find an upper bound for $c(\chi;\mathbf{U},\chi)$.

Lemma 6.4 Assume that the regular mesh satisfies Hypothesis H2. There exists a constant C and for each $r > 2$, there exists a constant C_r, both independent of h, such that

$$\begin{aligned} \forall \mathbf{u}_h \in \mathbf{V}_h, \quad & \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{X}_h, \\ |c(\mathbf{u}_h; \mathbf{v}_h, \mathbf{w}_h)| \; \leq \; & C_r h^{2/r} [\![\mathbf{u}_h]\!] \, [\![\mathbf{v}_h]\!] \, [\![\mathbf{w}_h]\!] \\ + \; & C \|\mathbf{w}_h\|_{L^4(\Omega)} (\|\mathbf{u}_h\|_{L^4(\Omega)} [\![\mathbf{v}_h]\!] \\ + \; & \|\mathbf{v}_h\|_{L^4(\Omega)} J_0(\mathbf{u}_h, \mathbf{u}_h)^{1/2}). \end{aligned}$$

As is usual for the Navier-Stokes equations, we introduce the quantity

$$N(h) = \sup_{\mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h} \frac{c(\mathbf{w}_h; \mathbf{v}_h, \mathbf{w}_h)^2}{[\![\mathbf{v}_h]\!] \, [\![\mathbf{w}_h]\!]}.$$

According to Lemmas 6.1 and 6.2, $N(h)$ is bounded by a constant N independent of h. It is easy to check that the discrete Navier-Stokes problems (17),(18) has a unique solution if the data satisfy

$$\frac{N}{K^2 \mu^2} C(4) \|\mathbf{f}\|_{L^{4/3}(\Omega)} < 1.$$

6. Navier-Stokes problem

Theorem 6.5 Under the assumption of Theorem 5.1 and if the data $\mathbf{f}$ and μ satisfy:

$$\frac{N}{K^2\mu^2}C(4)\|\mathbf{f}\|_{L^{4/3}(\Omega)} < \frac{1}{2},$$

then, the solution $(\mathbf{U}, P)$ of (17),(18) satisfies the following a priori error estimates:

$$\begin{aligned} [\![\mathbf{u}-\mathbf{U}]\!] &\leq Ch^k((1+\frac{1}{\mu^2})|\mathbf{u}|_{k+1}+\frac{1}{\mu}|p|_k), \\ \|p-P\|_0 &\leq Ch^k(\mu|\mathbf{u}|_{k+1}+|p|_k)+\frac{C}{\mu}[\![\mathbf{u}-\mathbf{U}]\!]. \end{aligned}$$

where $(\mathbf{u}, p)$ is the solution of (4),(2),(3) and the constant C depends upon $\mathbf{f}$, but not on h or μ.

Theorem 6.6 Under the assumption of Theorem 6.5, and if Ω is convex, there is a constant C independent of h such that

$$\|\mathbf{u}-\mathbf{U}\|_{0,\Omega} \leq Ch^{k+1}.$$

Conclusion:

In this paper, the authors have established optimal a priori estimates for a totally discontinuous family of approximations of the steady incompressible Stokes and Navier-Stokes equations in two dimensions. Both symmetric and nonsymmetric cases are discussed. The scheme are locally conservative away from subdomain interfaces. This paper is the first analysis of discontinuous Galerkin methods with nonmatching domain decomposition for Stokes and Navier-Stokes equations in the primitive variables.