NIPG - The Nonsymmetric Interior Penalty Galerkin Method

Huidong Yang

Institute of Computational Mathematics
Johannes Kepler University Linz
huidong@numa.uni-linz.ac.at

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The Model Problem

Restrict ourselves to the model problem

$$
\begin{aligned}
-\Delta u &= f && \text{in } \Omega \\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Ω – a convex polygonal domain
f – a given function in $L^2(\Omega)$

Rewrite the problem as a first-order system

$$
\begin{aligned}
\sigma &= \nabla u && \text{in } \Omega && (1) \\
-\nabla \cdot \sigma &= f && \text{in } \Omega && (2)
\end{aligned}
$$

The Weak Formulation

Multiply (1)-(2) by test functions τ and v, and integrate on a subset K of Ω

$$\begin{aligned}
\int_K \sigma \cdot \tau dx &= -\int_K u \nabla \cdot \tau dx + \int_{\partial K} u n_K \cdot \tau ds \\
\int_K \sigma \cdot \nabla v dx &= \int_K f v dx + \int_{\partial K} \sigma \cdot n_K v ds
\end{aligned}$$

n_K – the outward normal unit vector to ∂K
This is the weak formulation we will use to define the DG methods.

The Flux Formulation

The finite element spaces associated with the triangulation $\Pi_h = \{K\}$ of Ω, $P(K) = P_p(K)$ $(p \geqslant 1)$, $\Sigma(K) = [P_p(K)]^2$

$$V_h := \{v \in L^2(\Omega) : v|_K \in P(K) \quad \forall K \in \Pi_h\}$$

$$\Sigma_h := \{\tau \in [L^2(\Omega)]^2 : \tau|_K \in \Sigma(K) \quad \forall K \in \Pi_h\}$$

Find $u_h \in V_h$ and $\sigma \in \Sigma_h$ such that for all $K \in \Pi_h$, we have

$$\int_K \sigma_h \cdot \tau dx = -\int_K u_h \nabla \cdot \tau dx + \int_{\partial K} \hat{u}_K n_K \cdot \tau ds \quad \forall \tau \in \Sigma(K) \quad (3)$$

$$\int_K \sigma_h \cdot \nabla v dx = \int_K f v dx + \int_{\partial K} \hat{\sigma}_K \cdot n_K v ds \quad \forall v \in P(K) \quad (4)$$

Numerical fluxes $\hat{\sigma}_K$ and $\hat{u}_K$ are approximations to $\sigma = \nabla u$ and to u on the boundary of K.

Traces and Numerical Fluxes

- The traces of functions in $H^1(\Pi_h)$ belong to $T(\Gamma) := \Pi_{K \in \Pi_h} L^2(\partial K)$, Γ, the union of the boundaries of the elements K of Π_h, functions in $T(\Gamma)$ are double-valued on $\Gamma^0 := \Gamma \backslash \partial\Omega$ and single-valued on $\partial\Omega$.

- Numerical fluxes are consistent if $\hat{u}(v) = v|_\Gamma$, $\hat{\sigma}(v, \nabla v) = \nabla v|_\Gamma$, conservative if $\hat{u}(\cdot)$ and $\hat{\sigma}(\cdot, \cdot)$ are single-valued on Γ.
- Trace operators
  - e, an interior edge shared by elements K_1 and K_2, n_1 and n_2, unit normal vector on e pointing exterior to K_1 and K_2, ε_h^o the set of interior edges e, ε_h^∂, the set of boundary edges.
  - For $q \in T(\Gamma)$, with $q_i := q|_{\partial K_i}$, $\{q\} = \frac{q_1+q_2}{2}$, $[\![q]\!] = q_1 n_1 + q_2 n_2$, on $e \in \varepsilon_h^o$
  - For $\varphi \in [T(\Gamma)]^2$, with $\varphi_i := \varphi|_{\partial K_i}$, $\{\varphi\} = \frac{\varphi_1+\varphi_2}{2}$, $[\![\varphi]\!] = \varphi_1 \cdot n_1 + \varphi_2 \cdot n_2$, on $e \in \varepsilon_h^o$
  - $[\![q]\!] = qn$, $\{\varphi\} = \varphi$, on $e \in \varepsilon_h^\partial$

The Primal Formulation

In (3)-(4), we add over all the element, use the average $\{\}$ and jump $[\![\,]\!]$ operators, express σ_h solely in terms of u_h, apply the integration by parts formula, we obtain

$$B_h(u_h, v) = \int_\Omega f v dx \quad \forall v \in V_h \tag{5}$$

$$B_h(u_h, v) := \int_\Omega \nabla_h u_h \cdot \nabla_h v dx + \int_\Gamma ([\![\hat{u} - u_h]\!] \cdot \{\nabla_h v\} - \{\hat{\sigma}\} \cdot [\![v]\!]) ds$$
$$+ \int_{\Gamma^0} (\{\hat{u} - u_h\} [\![\nabla_h v]\!] - [\![\hat{\sigma}]\!] \{v\}) ds \tag{6}$$

$\hat{u} = \hat{u}(u_h), \hat{\sigma} = \hat{\sigma}(u_h, \sigma_h(u_h))$
The primal form : $B_h(\cdot, \cdot) : H^2(\Pi_h) \times H^2(\Pi_h) \to R$
Equation 5 : The primal formulation of the method

The Primal Form of NIPG Method

Numerical flux of NIPG method

$$\hat{u}_K = \{u_h\} + n_K \cdot [\![u_h]\!]$$
$$\hat{\sigma}_K = \{\nabla_h u_h\} - \alpha_j([\![u_h]\!])$$

where $\alpha_j([\![u_h]\!]) = \eta_e h_e^{-1}[\![u_h]\!]$

$$B_h(u,v) = (\nabla_h u, \nabla_h v) - < \{\nabla_h u\}, [\![v]\!] > + < [\![u]\!], \{\nabla_h v\} > + \alpha^j(u,v) \tag{7}$$

where

$(a,b) = \int_\Omega abdx$
$< a, b >= \int_\Gamma abds$
$\alpha^j(u,v) = \sum_{e\in\varepsilon_h} \int_e \eta_e h_e^{-1}[\![u]\!] \cdot [\![v]\!]ds$
η_e – are bounded uniformly above and below by positive constant

Consistency of NIPG Method (1)

We have consistency of numerical fluxes of NIPG method

$$\hat{u}(u) = u|_\Gamma, \quad \hat{\sigma}(u, \nabla(u))|_\Gamma = \nabla u$$

Then, use numerical fluxes,

$$\hat{u} = \{u\} + n_K \cdot [\![u]\!], \quad \hat{\sigma} = \{\nabla_h u\} - \alpha_j([\![u]\!])$$

we have

$$[\![\hat{u}]\!] = 0, \quad \{\hat{u}\} = u$$
$$[\![\hat{\sigma}]\!] = 0, \quad \{\hat{\sigma}\} = \nabla u$$
$$\alpha^j(u, v) = 0$$

Consistency of NIPG Method (2)

Let u solve the model problem, by the integration by parts formula, we have for any $v \in H^2(\Pi_h)$ that

$$\int_\Omega \nabla_h u \cdot \nabla_h v dx = -\int_\Omega \Delta u v dx + \int_\Gamma \{\nabla_h u\} \cdot [\![v]\!] ds + \int_{\Gamma 0} [\![\nabla_h u]\!] \cdot \{v\} ds$$

With $\{u\} = u$, $[\![u]\!] = 0$, $\{\nabla_h u\} = \nabla_h u$, $[\![\nabla_h u]\!] = 0$, $-\Delta u = f$, we have

$$\begin{aligned} B_h(u,v) := & \int_\Omega f v dx + \int_\Gamma ([\![\hat{u}]\!] \cdot \{\nabla_h v\} + (\nabla u - \{\hat{\sigma}\}) \cdot [\![v]\!]) ds \\ & + \int_{\Gamma 0} (\{\hat{u} - u\} [\![\nabla_h v]\!] - [\![\hat{\sigma}]\!] \{v\}) ds + \alpha^j(u,v) \end{aligned} \tag{8}$$

If numerical fluxes are consistent, on Γ, $[\![\hat{u}]\!] = 0$, $\{\hat{u}\} = u$, $[\![\hat{\sigma}]\!] = 0$, $\{\hat{\sigma}\} = \nabla u$, we conclude that

$$B_h(u,v) = \int_\Omega f v dx \quad \forall v \in H^2(\Pi_h) \tag{9}$$

Adjoint Inconsistency of NIPG Methods (1)

Let ψ solve

$$-\Delta\psi = g \qquad \text{in } \Omega$$
$$\psi = 0 \qquad \text{on } \partial\Omega$$

If $B_h(v,\psi) = \int_\Omega vg\,dx$ for all $v \in H^2(\Pi_h)$, we say that the primal form is adjoint consistent.
Since $\psi \in H^2(\Omega), [\![\psi]\!] = 0, \{\psi\} = \psi, [\![\nabla\psi]\!] = 0, \{\nabla\psi\} = \nabla\psi$, by the integration by parts, we get

$$B_h(v,\psi) = \int_\Omega vg\,dx + \int_\Gamma [\![\hat{u}(v)]\!] \cdot \nabla\psi\,ds - \int_{\Gamma^0} [\![\hat{\sigma}(v,\sigma_h(v))]\!]\psi\,ds \quad \forall v \in H^2(\Pi_h) \tag{10}$$

Adjoint Inconsistency of NIPG Methods (2)

For the numerical fluxes of NIPG method, we calculate

$$
\begin{aligned}
[\![\hat{u}]\!] &= [\![\{v\} + n_K \cdot [\![v]\!]]\!] \\
&= [\![\{v\}]\!] + [\![n_K \cdot [\![v]\!]]\!] \\
&= n_1(n_1 \cdot (n_1 v_1 + n_2 v_2)) + n_2(n_2 \cdot (n_1 v_1 + n_2 v_2)) \\
&= n_1(v_1 - v_2) + n_2(v_2 - v_1) \\
&= 2(n_1 v_1 + n_2 v_2) = 2[\![v]\!] \\
[\![\hat{\sigma}]\!] &= [\![\{\nabla_h v\}]\!] - [\![\alpha_j([\![v]\!])]\!] \\
&= 0 - 0 = 0
\end{aligned}
$$

So, insert them into Equation (10)

$$B_h(v, \psi) = \int_\Omega v g dx + 2 \int_\Gamma [\![v]\!] \cdot \nabla \psi ds$$

Penalty Term and DG Norm

The penalty term of NIPG method

$$\alpha^j(u,v) = \sum_{e\in\varepsilon_h} \int_e \eta_e h_e^{-1} [\![u]\!] \cdot [\![v]\!] ds, \quad (C_{e,0} \le \eta_e \le C_{e,1})$$

Take the norm of NIPG method in $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega) \subset H^2(\Pi_h)$ as

$$|\!|\!| v |\!|\!|_h^2 = |v|_{1,h}^2 + \sum_{K\in\Pi_h} h_K^2 |v|_{2,K}^2 + |v|_*^2$$

where $|v|_{1,h}^2 = \sum_K |v|_{1,K}^2, \quad |v|_*^2 = \alpha^j(v,v).$

Boundedness of NIPG Method (1)

By noting if $u \in H^2(K)$ and e is an edge of K, we have trace inequality[1]

$\|\frac{\partial u}{\partial n}\|_{0,e}^2 \leq C(h_e^{-1}|u|_{1,K}^2 + h_e|u|_{2,K}^2)$, Then,

$\int_e |\frac{\partial u}{\partial n} q| ds \leq C(|u|_{1,K}^2 + h_e^2|u|_{2,K}^2)^{1/2} h_e^{-1/2} \|q\|_{0,e}$, for every $q \in L^2(e)$,

$$\int_\Gamma \{\nabla_h u\} \cdot [\![v]\!] ds = \sum_{e \in \varepsilon_h} \int_e \{\nabla_h u\} \cdot [\![v]\!] ds$$

$$\leq C[\sum_K (|u|_{1,K}^2 + h_K^2|u|_{2,K}^2)]^{1/2} [\text{sup}(1/\eta_e) \sum_{e \in \varepsilon_h} \eta_e h_e^{-1} \int_e |[\![v]\!]|^2 ds]^{1/2}$$

$$\leq \text{sup}(1/\eta_e) C \, |\!|\!| u |\!|\!|_h \, |v|_* \leq \text{sup}(1/\eta_e) C \, |\!|\!| u |\!|\!|_h \, |\!|\!| v |\!|\!|_h$$

[1] D.N.Arnold, An interior penalty finite element method with discontinuous elements, SIAM J. Numer. Anal., 19(1982), pp. 742-760.

Boundedness of NIPG Method (2)

Similarly, we have $< [\![u]\!], \{\nabla_h v\} >= \int_\Gamma [\![u]\!] \cdot \{\nabla_h v\} ds \leq C \, |\!|\!| u |\!|\!|_h \, |\!|\!| v |\!|\!|_h$
Obviously, we have $(\nabla_h u, \nabla_h v) \leq C |u|_{1,h} |v|_{1,h} \leq C \, |\!|\!| u |\!|\!|_h \, |\!|\!| v |\!|\!|_h$,

$$\alpha^j(u,v) = \sum_{e \in \varepsilon_h} \int_e \eta_e h_e^{-1} [\![u]\!] \cdot [\![v]\!] ds \leq \sum_{e \in \varepsilon_h} [\int_e \eta_e h_e^{-1} [\![u]\!]^2 ds]^{1/2} [\int_e \eta_e h_e^{-1} [\![v]\!]^2 ds]^{1/2}$$

$$\leq [\sum_{e \in \varepsilon_h} \int_e \eta_e h_e^{-1} [\![u]\!]^2]^{1/2} [\sum_{e \in \varepsilon_h} \int_e \eta_e h_e^{-1} [\![v]\!]^2]^{1/2}$$

$$= \alpha^j(u,u)^{1/2} \alpha^j(v,v)^{1/2} = |u|_* |v|_* \leq C \, |\!|\!| u |\!|\!|_h \, |\!|\!| v |\!|\!|_h$$

Collecting all the terms which are bounded by the DG norm of the NIPG method, we get

$$\begin{aligned} B_h(u,v) &= (\nabla_h u, \nabla_h v) - < \{\nabla_h u\}, [\![v]\!] > + < [\![u]\!], \{\nabla_h v\} > + \alpha^j(u,v) \\ &\leq C_1 |u|_{1,h} |v|_{1,h} + C_2 |u|_* \, |\!|\!| v |\!|\!|_h + C_3 \, |\!|\!| u |\!|\!|_h \, |v|_* + C_4 |u|_* |v|_* \\ &\leq C \, |\!|\!| u |\!|\!|_h \, |\!|\!| v |\!|\!|_h \end{aligned}$$

Stability Analysis of NIPG Method (1)

Equivalence of DG and weak norm

$$|v|_{\#} = (|v|_{1,h}^2 + |v|_*^2)^{1/2} = (|v|_{1,h}^2 + \alpha^j(v,v))^{1/2}$$

$$|||v|||_h^2 = |v|_{1,h}^2 + \sum_{K \in \Pi_h} h_K^2 |v|_{2,K}^2 + |v|_*^2$$

By inverse inequality

$$\|\nabla v\|_0 \leq h^{-1} \|v\|_0$$

So

$$C_1 |v|_{\#}^2 \leq |||v|||_h^2 \leq C_2 |v|_{\#}^2$$

Stability Analysis of NIPG Method (2)

We show that the NIPG method satisfies the stability condition

$$B_h(v,v) \geq C_s |v|_{\#}^2 \quad \forall v \in V_h$$

We define the week norm, the natural one for analyzing the stability of NIPG method

$$|v|_{\#} = (|v|_{1,h}^2 + |v|_*^2)^{1/2} = (|v|_{1,h}^2 + \alpha^j(v,v))^{1/2}$$

From the primal forms of the DG method, we have

$$\begin{aligned} B_h(v,v) &= (\nabla_h v, \nabla_h v) - < \{\nabla_h v\}, [\![v]\!] > + < [\![v]\!], \{\nabla_h v\} > + \alpha^j(v,v) \\ &= (\nabla_h v, \nabla_h v) + \alpha^j(v,v) \\ &= \|\nabla_h v\|_{0,\Omega}^2 + \alpha^j(v,v) \\ &= |v|_{\#}^2 \geq C_s |v|_{\#}^2 \end{aligned}$$

Error Estimates (1)

A bound on the approximation error $|||u - u_I|||$, where $u_I \in V_h$, the usual continuous interpolant, then

$$\alpha^j(u - u_I, u - u_I)) = \sum_{e \in \varepsilon_h} \int_e \eta_e h_e^{-2p-1} [\![u - u_I]\!] \cdot [\![u - u_I]\!] ds = 0$$

The norm of the NIPG method can be bounded by

$$|||u - u_I|||_h^2 = |u - u_I|_{1,h}^2 + \sum_{K \in \Pi_h} h_K^2 |u - u_I|_{2,K}^2 \leq C_a^2 h^{2p} |u|_{p+1,\Omega}^2$$

So

$$|||u - u_I|||_h \leq C_a h^p |u|_{p+1,\Omega}$$

By the stability of the NIPG method, we have

$$\begin{aligned} C_s\, |||u_I - u_h|||_h^2 &\leq B_h(u_I - u_h, u_I - u_h) \\ &= B_h(u_I - u, u_I - u_h) + B_h(u - u_h, u_I - u_h) \\ &= \{u_I - u_h \in V_h\} = B_h(u_I - u, u_I - u_h) \end{aligned}$$

Error Estimates (2)

Use the continuity of $u - u_I$, we have the estimate

$$\begin{aligned} C_s \, |||u_I - u_h|||_h^2 &= B_h(u_I - u, u_I - u_h) \\ &\le C_b \, |||u_I - u|||_h \, |||u_I - u_h|||_h \\ &\le C_b h^p \, |||u_I - u_h|||_h \, |u|_{p+1,\Omega} \end{aligned}$$

Thus by triangle inequality, we get

$$\begin{aligned} |||u - u_h|||_h &\le |||u - u_I|||_h + |||u_I - u_h|||_h \\ &\le C_a h^p |u|_{p+1,\Omega} + C_b h^p |u|_{p+1,\Omega} \\ &= C h^p |u|_{p+1,\Omega} \end{aligned}$$

Error Estimates (3)

For the L^2-error estimate of the NIPG method, let ψ is the solution of the adjoint problem

$$-\Delta\psi = u - u_h \quad \text{in } \Omega \qquad \psi = 0 \quad \text{on } \partial\Omega$$

We then have

$$\|u - u_h\|_{0,\Omega}^2 = B_h(u - u_h, \phi) - 2\int_\Gamma \{\nabla\phi\} \cdot [\![u - u_h]\!] ds =: T_1 + T_2$$

If ψ_I is the continuous interpolant of ψ in V_h, then $B_h(u, \psi_I) = (f, \psi_I)$, and

$$\begin{aligned} T_1 &= B_h(u - u_h, \psi) = B_h(u - u_h, \psi - \psi_I) \\ &\leq C \, |||u - u_h|||_h \, |||\psi - \psi_I|||_h \\ &\leq Ch \, |||u - u_h|||_h \, \|u - u_h\|_{0,\Omega} \end{aligned}$$

Error Estimates (4)

Use the definition of the penalty term and norm of the NIPG method, and apply the trace inequality, we get

$$\sum_{e\in\varepsilon_h}\int_e\{\nabla u\}\cdot[\![v]\!]ds = \sum_{e\in\varepsilon_h}\int_e(h_e^{2p+1})^{1/2}\{\nabla u\}\cdot[\![v]\!](h_e^{-2p-1})^{1/2}ds$$

$$\leq C\,|\!|\!|v|\!|\!|_h\,(\sum_{e\in\varepsilon_h}h_e^{2p+1}\int_e|\{\nabla u\}\cdot n_e|^2ds)^{1/2}\leq Ch^p\,|\!|\!|v|\!|\!|_h\,\|u\|_{2,h}$$

Again use the elliptic regularity, we have

$$T_2\leq Ch^p\,|\!|\!|u-u_h|\!|\!|_h\,\|\psi\|_{2,\Omega}\leq Ch^p\,|\!|\!|u-u_h|\!|\!|_h\,\|u-u_h\|_{0,\Omega}$$

Collect error estimates for T_1, T_2, we obtain the desired optimal estimate

$$\|u-u_h\|_{0,\Omega}\leq Ch^{p+1}\|u\|_{p+1,\Omega}$$