Method of Babuška - Zlámal

Roman Heinzle

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Problem

We consider the following model problem:

Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f && \Omega \\ u &= 0 && \partial\Omega \end{aligned}$$

where Ω is a convex polygonal domain and $f \subset L^2(\Omega)$

Definitions

- Triangulation $\mathcal{T}_h = \{K\}$ of Ω
- Set of all edges $\varepsilon_h = \{e\}$
- $V_h := \{v_h \in L^2(\Omega) : v|_K \in P(K) \text{ for all } K \in \mathcal{T}_h\}$
- $\Sigma_h = \{\tau_h \in [L^2(\Omega)]^2 : \tau|_K \in \Sigma(K) \text{ for all } K \in \mathcal{T}_h\}$
- $T(\Gamma) := \prod_{K \in \mathcal{T}_h} L^2(\partial K)$, where Γ is the union of the boundaries of the elements K.

Definitions

For $q \in T(\Gamma)$ define the average $\{q\}$ and the jump $[[q]]$ as

$$\{q\} = \begin{cases} \frac{1}{2}(q_1 + q_2) & \text{on } e \in \varepsilon_h^\circ, \\ q & \text{on } e \in \varepsilon_h^\partial. \end{cases}$$

$$[\![q]\!] = \begin{cases} q_1 n_1 + q_2 n_2 & \text{on } e \in \varepsilon_h^\circ, \\ qn & \text{on } e \in \varepsilon_h^\partial. \end{cases}$$

where $q_i := q|_{\partial K_i}$. Analog for $\varphi \in [T(\Gamma)]^2$.

Flux Formulation

Rewriting the Problem in a first order system gives

$$\sigma = \nabla u \quad \Omega \qquad -\nabla \sigma = f \quad \Omega \qquad u = 0 \quad \partial\Omega.$$

Multiplying with test functions and integrating on K we have

$$\begin{aligned} \int_K \sigma \tau dx &= -\int_K u \nabla \cdot \tau dx + \int_{\partial K} u \, n_K \cdot \tau ds, \\ \int_K \sigma \cdot \nabla v \, dx &= \int_K f \, v \, dx + \int_{\partial K} \sigma \cdot n_K v \, ds, \end{aligned}$$

where n_K is the outward normal unit vector to ∂K.

Flux Formulation

Find $u_h \in V_h$ and $\sigma_h \in \Sigma_h$ such that

$$\int_K \sigma_h \tau dx = -\int_K u_h \nabla \cdot \tau dx + \int_{\partial K} \hat{u}_K n_K \cdot \tau ds \quad \forall \tau \in \Sigma(K), \quad (1)$$

$$\int_K \sigma_h \cdot \nabla v \, dx = \int_K f \, v \, dx + \int_{\partial K} \hat{\sigma}_K \cdot n_K v \, ds \quad \forall v \in P(K), \quad (2)$$

Babuška - Zlámal:

$$\begin{aligned} \hat{u}_K &= (u_h|_K)|_{\partial K}, \\ \hat{\sigma}_K &= -\mu [\![u_h]\!], \end{aligned}$$

where $\mu : \Gamma \to \mathbb{R}$ given by $\eta_e h_e^{-1}$ on each $e \in \varepsilon_h$ and η_e is a positive number.

Primal formulation

Adding over all elements in (1) and (2) gives

$$\int_\Omega \sigma_h \tau \, dx = -\int_\Omega u_h \nabla_h \cdot \tau \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{u}_K \, n_K \cdot \tau \, ds \quad \forall \tau \in \Sigma_h,$$

$$\int_\Omega \sigma_h \cdot \nabla_h v \, dx = \int_\Omega f \, v \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{\sigma}_K \cdot n_K v \, ds \quad \forall v \in V_h.$$

Using the average and the jump operators leads to

$$\int_\Omega \sigma_h \tau \, dx = -\int_\Omega u_h \nabla_h \cdot \tau \, dx + \int_\Gamma [\![\hat{u}]\!] \cdot \{\tau\} ds + \int_{\Gamma^\circ} \{\hat{u}\} [\![\tau]\!] ds, \quad (3)$$

$$\int_\Omega \sigma_h \cdot \nabla_h v \, dx - \int_\Gamma [\![\hat{\sigma}]\!] \cdot \{v\} ds - \int_{\Gamma^\circ} \{\hat{\sigma}\} [\![v]\!] ds = \int_\Omega f \, v \, dx. \quad (4)$$

Primal formulation

Expressing now σ_h in terms of u_h and inserting this in (3) gives the primal form

$$B_h(u_h, v) = \int_\Omega f\, v\, dx \quad \forall v \in V_h,$$

where

$$\begin{aligned} B_h(u_h, v) = & \int_\Omega \nabla_h u_h \cdot \nabla_h v\, dx \\ & + \int_\Gamma ([\![\hat{u} - u_h]\!] \cdot \{\nabla_h v\} - \{\hat{\sigma}\} \cdot [\![v]\!]) ds \qquad (5) \\ & + \int_{\Gamma^\circ} (\{\hat{u} - u_h\}[\![\nabla_h v]\!] - [\![\hat{\sigma}]\!]\{v\}) ds \end{aligned}$$

Primal formulation

Inserting the numerical fluxes for the method of Babuška - Zlámal leads to the primal formulation of Babuška - Zlámal

$$B_h(u_h, v) = \int_\Omega \nabla_h u_h \cdot \nabla_h v \, dx + \int_\Gamma (\mu [\![u_h]\!] \cdot [\![v]\!]) ds \qquad (6)$$

Stability

The method of Babuška - Zlámal is stable, this means

$$B_h(v, v) \geq C_s |||v|||^2 \quad \forall v \in V_h, \tag{7}$$

with C_s a positive constant and

$$|||v|||^2 = |v|_{1,h}^2 + \sum_K h_K^2 |v|_{2,K}^2 + |v|_*^2, \tag{8}$$

This norm is equivalent to the weaker norm

$$|v|_w^2 = |v|_{1,h}^2 + |v|_*^2, \tag{9}$$

where $|v|_*^2 = \sum_{e \in \varepsilon_h} \int_e \eta_e h_e^{-1} [\![v]\!]^2$

Proof : Stability

$$B_h(v, v) = \|\nabla_h v\|_{0,\Omega}^2 + \int_\Gamma \mu [\![v]\!]^2 \tag{10}$$

It can be shown that

$$\int_\Gamma \mu [\![v]\!]^2 \geq C_1 \eta_0 |v|_*^2 \quad \forall v \in V_h, \tag{11}$$

where $\eta_0 \equiv \inf_e \eta_e$.

Proof

This leads to

$$B_h(v,v) \geq |v|^2_{1,h} + C_1\eta_0|v|^2_* \geq C_s|v|^2_w \quad \forall v \in V_h \qquad (12)$$

and it can be shown for η_0 large enough that (7) holds, where we use the equivalence of the norms in (8) and (9). Further analysis shows that the stability of the method is guaranteed for any $\eta_0 > 0$.

Consistency

The method of Babuška - Zlámal is not consistent and not adjoint consistent. Instead of

$$B_h(u, v) = \int_\Omega f \, v \, dx \quad \forall v \in V_h, \tag{13}$$

the method satisfies

$$B_h(u, v) = \int_\Omega f \, v \, dx + \int_\Gamma \{\nabla u\} \cdot [\![v]\!] \, ds. \tag{14}$$

when u is the exact solution and $v \in H^2(\mathcal{T}_h)$.
Analogical with the adjoint problem.

Error estimates

Defining the norm in $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega) \subset H^2(\mathcal{T}_h)$ as

$$|||v|||_h^2 = |v|_{1,h}^2 + \sum_K h_K^2 |v|_{2,K}^2 + \alpha(v,v), \tag{15}$$

where

$$\alpha(v,v) = \sum_{e \in \varepsilon_h} \int_e \eta_e h_e^{-2p-1} [\![v]\!]^2 \, ds \tag{16}$$

$\Rightarrow$ superpenalty method

Error estimates

Because of stability and η_e sufficiently large we have

$$\begin{aligned} C_s \|\!| u_I - u_h \|\!|_h^2 &\leq B_h(u_I - u, u_I - u_h) + B_h(u - u_h, u_I - u_h) \\ &=: T_1 \qquad\qquad\qquad\qquad + T_2, \end{aligned} \tag{17}$$

where u_I is the continuous interpolant of u in V_h.

Error estimates

The approximation error is bounded by

$$|||u - u_I||| \leq C_1 h^p |u|_{p+1,\Omega} \tag{18}$$

For T_1 we get

$$T_1 \leq C |||u_I - u|||_h \|u_I - u_h\|_h \leq C h^p |||u_I - u_h|||_h |u|_{p+1,\Omega}. \tag{19}$$

Error estimates

The term T_2 comes from the inconsistency of the method. We have

$$\sum_{e\in\varepsilon_h}\int_e \{\nabla u\}[\![v]\!]\,ds = \sum_{e\in\varepsilon_h}\int_e (h_e^{2p+1})^{1/2}\{\nabla u\}[\![v]\!](h_e^{-2p-1})^{1/2}\,ds$$
$$\leq C|\!|\!|v|\!|\!|_h\Big(\sum_{e\in\varepsilon_h} h_e^{2p+1}\int_e |\{\nabla u\}\cdot n_e|^2\,ds\Big)^{1/2} \leq Ch^p|\!|\!|v|\!|\!|_h\|u\|_{2,h} \tag{20}$$

and so

$$T_2 = \int_\Gamma \{\nabla u\}[\![u_I - u_h]\!]\,ds \leq Ch^p|\!|\!|u_I - u_h|\!|\!|_h\|u\|_{2,\Omega}. \tag{21}$$

Error estimates

Now inserting (19) and (21) in (17), we get

$$|||u_I - u_h|||_h \leq Ch^p \|u\|_{p+1,\Omega} \tag{22}$$

and so the optimal order estimate is achieved via triangle inequality

$$|||u - u_h|||_h \leq Ch^p \|u\|_{p+1,\Omega}. \tag{23}$$

L^2-error estimates

If ψ is the solution of the adjoint problem $-\triangle\psi = u - u_h$ in Ω, $\psi = 0$ on $\partial\Omega$ then

$$\|u-u_h\|_{0,\Omega}^2 = B_h(u-u_h,\psi) - \int_\Gamma \{\nabla\psi\}\cdot[\![u-u_h]\!]\,ds =: T_1 + T_2. \quad (24)$$

The estimate for T_1 gives

$$\begin{aligned} T_1 &= B_h(u-u_h,\psi) = B_h(u-u_h,\psi-\psi_I) \\ &\leq C|\!|\!|u-u_h|\!|\!|_h |\!|\!|\psi-\psi_I|\!|\!|_h \leq Ch|\!|\!|u-u_h|\!|\!|_h \|u-u_h\|_{0,\Omega}. \end{aligned} \quad (25)$$

L^2-error estimates

The term T_2 arises from the adjoint inconsistency and can be estimated with (20) by

$$T_2 = -\int_\Gamma \{\nabla\psi\} \cdot [\![u - u_h]\!]\, ds \leq Ch^p |\!|\!| u - u_h |\!|\!|_h \|\psi\|_{2,\Omega} \\ \leq Ch^p |\!|\!| u - u_h |\!|\!|_h \|u - u_h\|_{0,\Omega}. \tag{26}$$

So the optimal estimate is achieved by inserting (25) and (26) in (24) and using (23)

$$|\!|\!| u - u_h |\!|\!|_{0,\Omega} \leq Ch^{p+1} \|u\|_{p+1,\Omega}. \tag{27}$$

Conclusion

The method of Babuška - Zlámal is

- inconsistent, but
- stable and
- there exist optimal error estimates for the L^2 norm and the $\|\cdot\|_h$ norm.

$\Rightarrow$ choosing the penalty term large enough reduces the consistency error (optimal order convergence with rate h^{p+1} and h^p)