The Method of Baumann and Oden

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Contents

Problem

Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where

- $\Omega \subset \mathbb{R}^2$ convex, polygonal Lipschitz domain
- $f \in L^2(\Omega)$

Elliptic regularity gives

$$u \in H^2(\Omega)$$

Definitions

Triangulation $\mathcal{T}_h = \{K\}$ of Ω
Set of all edges $\mathcal{E}_h = \{E\}$
$\mathcal{T}_h$ shape-regular $\Rightarrow |E| \sim h$
Define the “broken Sobolev space”

$$V = V(\mathcal{T}_h) = \{v \in L^2(\Omega) : \ v|_K \in H^2(K) \text{ for all } K \in \mathcal{T}_h\}$$

and the finite element space

$$V_h = \{v_h \in V : v|_K \in P^p \text{ for all } K \in \mathcal{T}_h\}$$

for $p \geq 1$.

Definitions

For the edge $E = \bar{K}_1 \cap \bar{K}_2$, scalar function $v \in V$, vector function $\mu \in V^2$, define the

- jump

$$[\![v]\!] = \begin{cases} v_1 n_1 + v_2 n_2 & E \subset \text{int}(\Omega) \\ vn & E \subset \partial\Omega \end{cases}$$

$$[\![\mu]\!] = \begin{cases} \mu_1 \cdot n_1 + \mu_2 \cdot n_2 & E \subset \text{int}(\Omega) \\ \mu \cdot n & E \subset \partial\Omega \end{cases}$$

- average

$$\{v\} = \begin{cases} \frac{1}{2}(v_1 + v_1) & E \subset \text{int}(\Omega) \\ v & E \subset \partial\Omega \end{cases}$$

$$\{\mu\} = \begin{cases} \frac{1}{2}(\mu_1 + \mu_2) & E \subset \text{int}(\Omega) \\ \mu & E \subset \partial\Omega \end{cases}$$

Definitions

Norm on V (DG norm):

$$|||v|||^2 = \sum_K \int_K |\nabla v|^2 \, dx + h^{-1} \sum_E \int_E [\![v]\!]^2 \, ds$$

Seminorm on V:

$$|v|_{1,h}^2 = \sum_K \int_K |\nabla v|^2 \, dx$$

For $|.|_{1,h}$ there holds

$$|v|_{1,h} = 0 \qquad \text{for } v \text{ piecewise constant}$$

Variational Formulation

For $K \in \mathcal{T}_h$ the solution u satisfies

$$-\int_K \Delta u\, v\, dx = \int_K fv\, dx \qquad \forall v \in H^2(K).$$

Integration by parts:

$$\int_K \nabla u \cdot \nabla v\, dx - \int_{\partial K} v \nabla u \cdot n\, ds = \int_K fv\, dx \qquad \forall v \in H^2(K)$$

Summation over all elements $K \in \mathcal{T}_h$:

$$\sum_K \int_K \nabla u \cdot \nabla v\, dx - \sum_E \int_E [\![v \nabla u]\!] ds = \int_\Omega fv\, dx \qquad \forall v \in V$$

Variational Formulation

As $\nabla u \in H^1(\Omega)^2$, the trace of ∇u on edge E exists:

$$\int_E [\![v \nabla u]\!] ds = \int_E [\![v]\!] \cdot \nabla u \, ds$$
$$= \int_E [\![v]\!] \cdot \{\nabla u\} \, ds$$

The solution $u \in H^2(\Omega)$ is continuous, all jumps vanish

$$\int_E [\![u]\!] \cdot \mu \, ds = 0 \qquad \forall E \in \mathcal{E}_h, \mu \in L^2(E).$$

Set $\mu = \{\nabla v\}$, sum up equations

$$\sum_K \int_K \nabla u \cdot \nabla v \, dx - \sum_E \int_E \left([\![v]\!] \cdot \{\nabla u\} - [\![u]\!] \cdot \{\nabla v\} \right) ds$$
$$= \int_\Omega f v \, dx \qquad \forall v \in V$$

Method of Baumann and Oden

Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

where

$$a_h(u,v) = \sum_K \int_K \nabla u \cdot \nabla v \, dx - \sum_E \int_E \left([\![v]\!] \cdot \{\nabla u\} - [\![u]\!] \cdot \{\nabla v\} \right) ds$$

The method is

- consistent, for u there holds $a_h(u, v_h) = (f, v_h)$ for $v_h \in V_h$.
- not adjoint consistent, non-symmetric bilinear form
- not stable w. r. t. the DG norm $|\!|\!|.|\!|\!|$

Missing adjoint consistency

Let w satisfy adjoint equation,

$$\begin{aligned} -\Delta w &= g \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

For $g \in L^2(\Omega)$, we have $w \in H^2(\Omega)$.
The method is not adjoint consistent:
Solution w does not satisfy discrete adjoint equation

$$a_h(v_h, w) = (g, v_h) \qquad \forall v_h \in V_h.$$

Proof: Let $v_h \in V_h$:

$$
\begin{aligned}
a_h(v_h, w) &= \sum_K \int_K \nabla v_h \cdot \nabla w \, dx \\
&\quad - \sum_E \int_E \big(\underbrace{[\![w]\!]}_{=0} \cdot \{\nabla v_h\} - [\![v_h]\!] \cdot \{\nabla w\} \big) ds \\
&= - \sum_K \int_K v_h \cdot \Delta w \, dx + \sum_k \int_{\partial K} v_h \nabla w \cdot n \, ds \\
&\quad + \sum_E \int_E [\![v_h]\!] \cdot \{\nabla w\} \, ds \\
&= \int_\Omega g v_h \, dx + 2 \sum_E \int_E [\![v_h]\!] \cdot \{\nabla w\} \, ds \neq (g, v_h)
\end{aligned}
$$

□

Stability

a_h is bounded w.r.t. DG norm:

$$|a_h(u, v)| \leq \alpha_2 \|u\| \|v\| \qquad \forall u, v \in V$$

a_h is not coercive in V, for v piecewise constant we have

$$a_h(v, v) = \sum_K \int_K \nabla v \cdot \nabla v \, dx = 0.$$

Possible method of stabilization: Choose bilinear form

$$a_h(u, v) + \alpha \sum_E \int_E [\![u]\!] \cdot [\![v]\!] ds$$

for $\alpha \sim h^{-1}$ ($\rightarrow$ NIPG method).

H^1 estimate

For $v \in V$ there holds

$$a_h(v, v) = |v|_{1,h}^2$$

Therefore, we try to bound $|u - u_h|_{1,h}$.
Choose an interpolant $u_I \in V_h$ of u such that

$$\|u - u_I\| \leq ch^p |u|_{p+1}$$

and

$$\int_E \{\nabla_h(u - u_I)\}\, ds = 0 \quad \forall E \in \mathcal{E}_h.$$

This is possible for $p \geq 2$.
Triangle inequality:

$$|u - u_h|_{1,h} \leq |u - u_I|_{1,h} + |u_I - u_h|_{1,h}$$

H^1 estimate

Lemma

There holds

$$a_h(u - u_I, v_h) \leq \alpha \|u - u_I\| |v_h|_{1,h} \qquad \forall v_h \in V_h.$$

Proof. Let $v_0 = P_0 v_h$ be the L^2 projection of v_h onto the space of piecewise constant functions. The interpolate u_I satisfies

$$\sum_E \int_E \{\nabla(u - u_I)\} [\![v_0]\!] \, ds = 0.$$

and therefore

$$a_h(u - u_I, v_0) = \sum_K \int_K \nabla(u - u_I) \cdot \underbrace{\nabla v_0}_{=0} \, dx$$

$$- \sum_E \int_E \left([\![v_0]\!] \cdot \{\nabla(u - u_I)\} - [\![u - u_I]\!] \underbrace{\{\nabla v_0\}}_{=0} \right) ds = 0$$

H^1 estimate

The seminorm $|.|_{1,h}$ is a norm on

$$\{v_0 \in V_h : P_0 v_0 = 0\}.$$

Therefore, there exists a constant C such that

$$\|v_h - P_0 v_h\| \leq C|v_h|_{1,h}$$

and further

$$\begin{aligned} a_h(u - u_I, v) &= a_h(u - u_I, v_h - v_0) \\ &\leq \alpha_2 \|u - u_I\| \|v_h - v_0\| \\ &\leq \alpha_2 C \|u - u_I\| |v_h|_{1,h} \end{aligned}$$

$\square$

H^1 estimate

We can now prove an estimate for $|u_h - u_I|_{1,h}$.
The method is consistent: There holds

$$\begin{aligned} |u_h - u_I|_{1,h}^2 &= a_h(u_h - u_I, \underbrace{u_h - u_I}_{\in V_h}) \\ &= a_h(u - u_I, u_h - u_I). \end{aligned}$$

Lemma 1 gives

$$a_h(u - u_I, u_h - u_I) \leq C \| u - u_I \| |u_h - u_I|_{1,h}.$$

H^1 estimate

We have shown

$$|u_h - u_I|_{1,h}^2 \leq C\|u - u_I\|\,|u_h - u_I|_{1,h} \leq Ch^p|u|_{p+1}|u_h - u_I|_{1,h}.$$

Therefore we have an optimal order estimate

$$|u - u_h|_{1,h} \leq qCh^p|u|_{p+1}.$$

Note that the estimate is weak, it does not imply convergence in V. We need an additional L^2 estimate.

L^2 error

Method is not adjoint consistent $\Rightarrow$ no optimal order L^2 error.
We will prove suboptimal order estimate:

$$\|u - u_h\|_0 \leq ch^p |u|_{p+1}$$

Let w be the solution to the adjoint problem

$$\begin{aligned} -\Delta w &= u - u_h \quad \text{in } \Omega \\ w &= 0 \qquad\quad \text{on } \partial\Omega \end{aligned}$$

Elliptic regularity: $w \in H^2(\Omega)$, $\|w\|_2 \leq c\|u - u_h\|_0$.

L^2 error

There holds

$$
\begin{aligned}
\|u-u_h\|_0^2 &= (-\Delta w, u-u_h) = a_h(w, u-u_h) \\
&= a_h(w, u-u_h) + a_h(u-u_h, w) - a_h(u-u_h, w) \\
&= 2\sum_K \int_K \nabla w \cdot \nabla(u-u_h)dx - a_h(u-u_h, w) \\
&= 2\sum_K \int_K \nabla w \cdot \nabla(u-u_h)dx - a_h(u-u_h, w-w_I)
\end{aligned}
$$

1st term:

$$
\begin{aligned}
\sum_K \int_K \nabla w \cdot \nabla(u-u_h)dx &\leq |w|_1 |u-u_h|_{1,h} \\
&\leq C\|w\|_2 |u-u_h|_{1,h} \leq C\|u-u_h\|_0 h^p |u|_{p+1}
\end{aligned}
$$

L^2 error

2nd term

$$
\begin{aligned}
a_h(u-u_h, w-w_I) &= a_h(u-u_h-P_0(u-u_h), w-w_I) \\
&\leq \alpha_2 |||u-u_h-P_0(u-u_h)|||\,|||w-w_I||| \\
&\leq c|u-u_h|_{1,h} h \|w\|_2 \\
&\leq ch^{p+1}|u|_{p+1}\|u-u_h\|_0
\end{aligned}
$$

Collecting all results, we get

$$
\|u-u_h\|_0 \leq ch^p|u|_{p+1}.
$$

Numerical Results

Problem: Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= \sin(\pi x)\sin(\pi y) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with $\Omega = [0, 1]^2$.

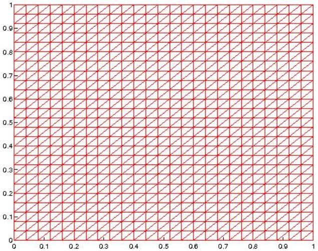


Figure: Triangulation

Solution

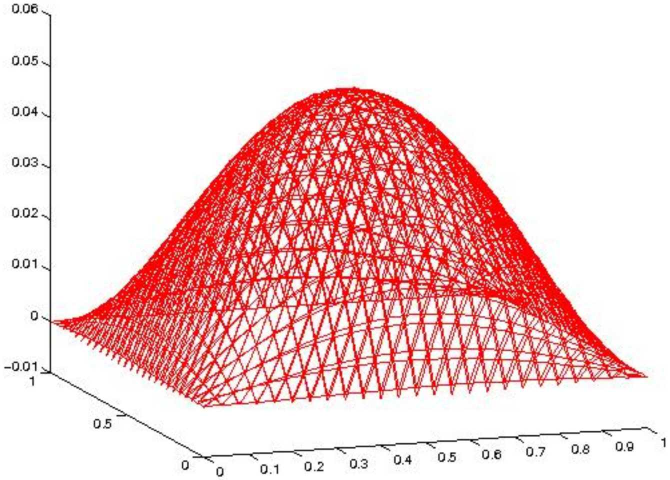


Figure: DG Solution, $p = 1$, h = 0.04

Solution

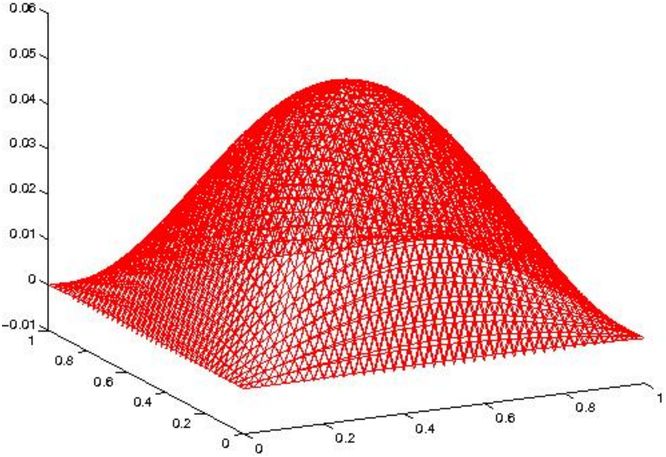
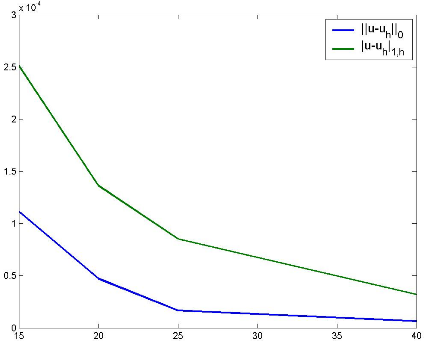


Figure: DG Solution, $p = 1$, h = 0.025

Error



	h	ndof	$\|u_h - u\|_0$	$\|u_h - u\|_{1,h}$
Mesh 1	0.07	1350	$1.1142 \cdot 10^{-4}$	$2.5120 \cdot 10^{-4}$
Mesh 2	0.05	2400	$4.7224 \cdot 10^{-5}$	$1.3616 \cdot 10^{-4}$
Mesh 3	0.04	3750	$1.6755 \cdot 10^{-5}$	$8.5133 \cdot 10^{-5}$
Mesh 4	0.025	9600	$6.5425 \cdot 10^{-6}$	$3.2029 \cdot 10^{-5}$

Conclusion

We have seen that the method of Baumann and Oden is

- consistent
- not adjoint consistent
- not stable, but can be stabilized by penalty terms ($\longrightarrow$ NIPG method)
- easy to implement

We have bounded the error $u - u_h$ in both L^2 and the broken H^1 space.