Domain Decomposition FEM-BEM Coupling and BEM-BEM Coupling Methods

Wael Elleithy

Higher Institute of Civil and Architectural Engineering, Cairo, Egypt

FEM and BEM

Powerful computational techniques
FEM: material is nonhomogeneous, behaves nonlinearly
BEM: infinite domains, predicting high gradients

FEM-BEM Coupling

in each sub-domain, according to its specific physics, the most appropriate solution technique is applied

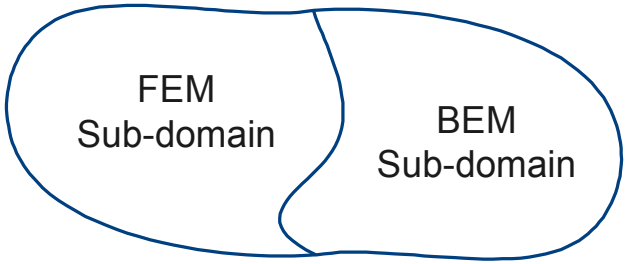
BEM-BEM Coupling

domains under consideration governed by individual differential equations
better computational efficiency and accuracy

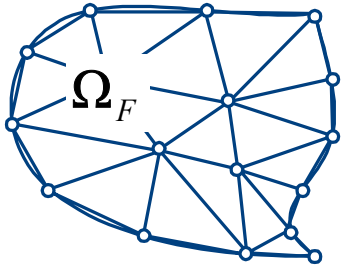
Outline

- FEM-BEM coupling
  - Conventional methods
  - Domain decomposition/interface relaxation
  - Overlapping DD coupling
  - Interface relaxation FEM-BEM coupling methods
  - Convergence/optimal convergence
  - IR FEM-BEM coupling in elasto-plasticity
- Interface relaxation BEM-BEM coupling

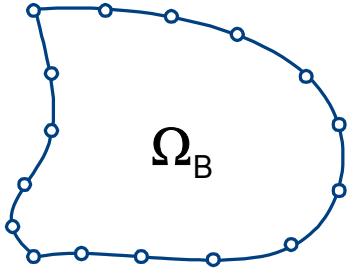
FEM-BEM coupling



Domain of the Original Problem



FEM Modeling



BEM Modeling

Proves advantageous in:

- Local nonlinearity
- Presence of infinite domains, singularities

Conventional methods

- equations for entire domain
  - FEM hosted
  - BEM hosted
- destroy desirable features existing in FEM matrices:
  - sparsity
  - symmetry
- implementation requires an integrated computational environment

Conventional methods

FEM hosted approach

BEM

$$[\mathbf{H}]\{\mathbf{u}_B\} = [\mathbf{G}]\{\mathbf{t}_B\}$$

$$[\mathbf{M}][\mathbf{G}]^{-1}[\mathbf{H}]\{\mathbf{u}_B\} = [\mathbf{M}]\{\mathbf{t}_B\}$$

$$[\mathbf{K}_B]\{\mathbf{u}_B\} = \{\mathbf{f}_B\}$$

FEM

$$[\mathbf{K}]\{\mathbf{u}_F\} = \{\mathbf{f}_F\}$$

coupled

BEM hosted approach

FEM

$$[\mathbf{K}]\{\mathbf{u}_F\} = \{\mathbf{f}_F\}$$

$$\begin{bmatrix}\mathbf{K}^F & \mathbf{K}_I^F\end{bmatrix}\begin{Bmatrix}\mathbf{u}^F \\ \mathbf{u}_I^F\end{Bmatrix} = \begin{bmatrix}\mathbf{M}^F & \mathbf{M}_I^F\end{bmatrix}\begin{Bmatrix}\mathbf{t}^F \\ \mathbf{t}_I^F\end{Bmatrix}$$

BEM

$$\begin{bmatrix}\mathbf{H}^B & \mathbf{H}_I^B\end{bmatrix}\begin{Bmatrix}\mathbf{u}^B \\ \mathbf{u}_I^B\end{Bmatrix} = \begin{bmatrix}\mathbf{G}^B & \mathbf{G}_I^B\end{bmatrix}\begin{Bmatrix}\mathbf{t}^B \\ \mathbf{t}_I^B\end{Bmatrix}$$

coupled

DD / interface relaxation

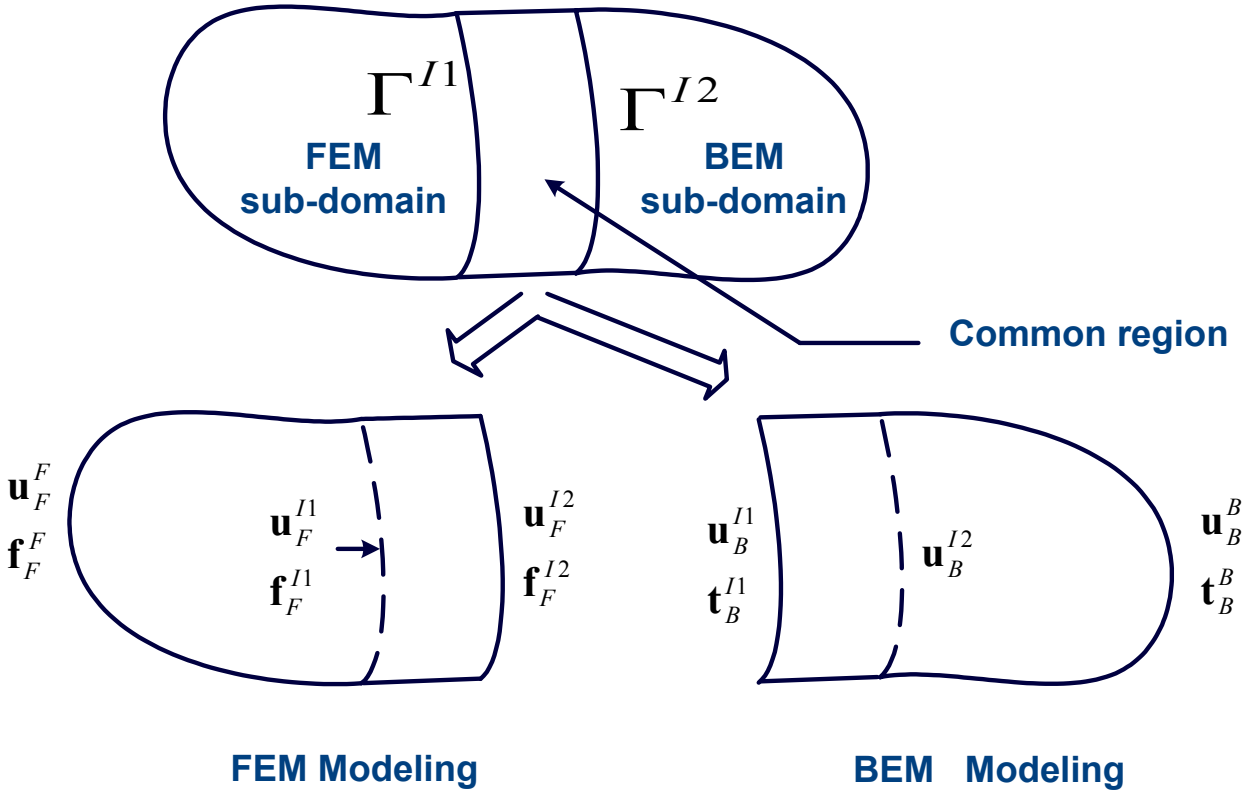
- **D**omain **D**ecomposition
  - three aspects are very often interconnected in practice
    - different physical models may be used in different regions
    - employ different approximation methods in different regions
    - solution of the algebraic systems arising from the approximation of the partial differential equation
  - overlapping or non-overlapping
  - non-overlapping: preconditioned and interface relaxation

* Langer, Parallel Iterative Solution of Symmetric Coupled FE/BE-Equations via Domain Decomposition, Contemporary Mathematics, 1994.

* Schnack and Türke, “Domain decomposition with BEM and FEM,” *Int. J. Num. Meth. Eng.*, 1997.

* Langer and Steinbach, “Coupled Boundary and Finite Element Tearing and Interconnecting Methods,” Proceedings of the Fifteenth International Conference on Domain Decomposition, 2003.

Overlapping DD coupling



* Stein and Kreienmeyer, “Coupling of BEM and FEM by a Multiplicative Schwarz Method and its Parallel Implementation,” *Eng. Comp.,* 1998.

* Elleithy and Al-Gahtani, "An Overlapping Domain Decomposition Approach for Coupling the Finite and Boundary Element Methods," *Eng. Anal. Bound. Elem.*, 2000.

Overlapping DD coupling (2)

Solution algorithm

BEM

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_B^B \\ \mathbf{u}_{B,n}^{I1} \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{t}_B^B \\ \mathbf{t}_{B,n}^{I1} \end{Bmatrix}$$

$$\mathbf{u}_{B,n}^{I2} = \begin{bmatrix} \mathbf{H}_{11}^* & \mathbf{H}_{12}^* \\ \mathbf{H}_{12}^* & \mathbf{H}_{22}^* \end{bmatrix} \begin{Bmatrix} \mathbf{u}_B^B \\ \mathbf{u}_{B,n}^{I1} \end{Bmatrix} + \begin{bmatrix} \mathbf{G}_{11}^* & \mathbf{G}_{12}^* \\ \mathbf{G}_{21}^* & \mathbf{G}_{22}^* \end{bmatrix} \begin{Bmatrix} \mathbf{t}_B^B \\ \mathbf{t}_{B,n}^{I1} \end{Bmatrix}$$

FEM

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{K}_{23} \\ \mathbf{K}_{31} & \mathbf{K}_{32} & \mathbf{K}_{33} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_F^F \\ \mathbf{u}_{F,n}^{I1} \\ \mathbf{u}_{B,n}^{I2} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_F^F \\ \mathbf{f}_F^{I1} \\ \mathbf{f}_F^{I2} \end{Bmatrix}$$

Convergence

NO

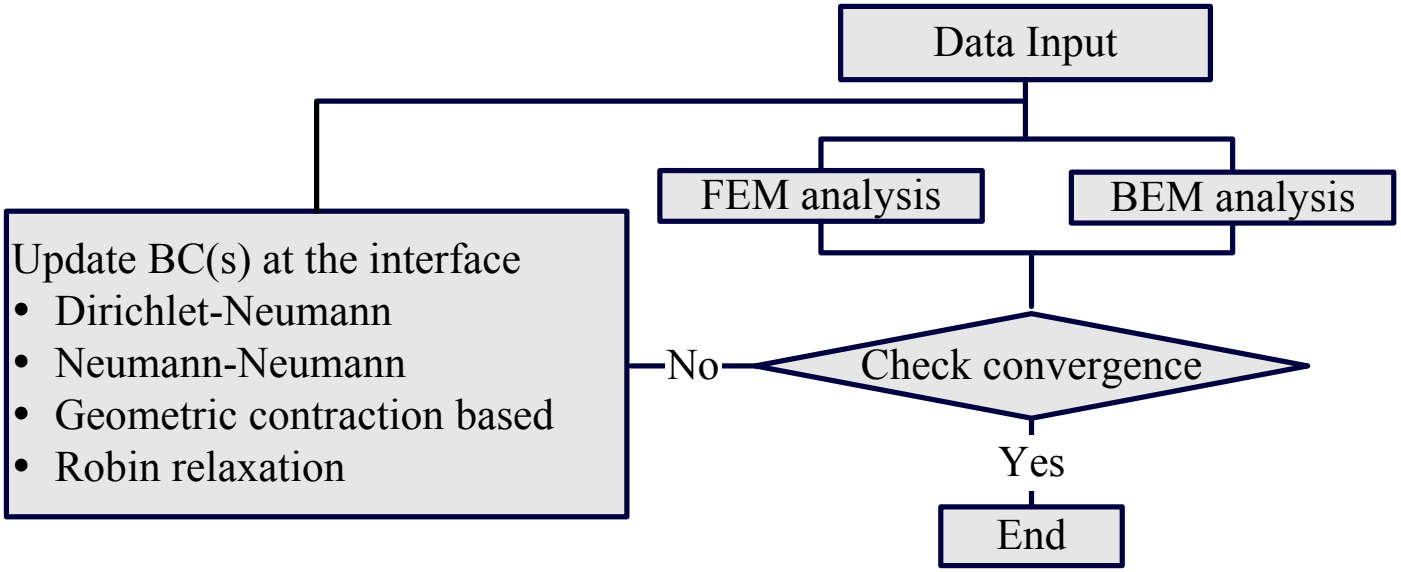
$$\mathbf{u}_{B,\,n+1}^{I1} = \mathbf{u}_{F,\,n}^{I1}$$
$$\mathbf{u}_{F,\,n+1}^{I2} = \mathbf{u}_{B,\,n+1}^{I2}$$

YES

Stop

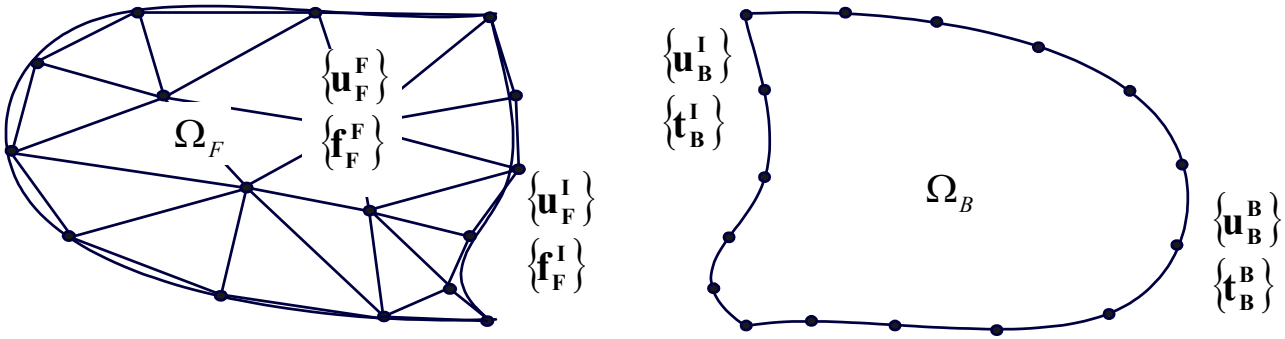
Interface relaxation FEM-BEM coupling

- separate computations in sub-domains
- successive update of degrees of freedom at the interface until convergence is reached
- offers many advantages over the conventional coupling methods
- several challenging questions concerning their application



IR FEM-BEM coupling methods

Dirichlet-Neumann FEM-BEM coupling *



Set initial guess $\mathbf{u}^I_{B,0}$ and $\mathbf{t}^I_{F,0}$

$$\{\mathbf{f}^I_{F,n}\} = [\mathbf{M}]\{\mathbf{t}^I_{F,n}\}$$

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^F_{F,n} \\ \mathbf{u}^I_{F,n} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^F_{F,n} \\ \mathbf{f}^I_{F,n} \end{Bmatrix}$$

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^B_{B,n} \\ \mathbf{u}^I_{B,n} \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{t}^B_{B,n} \\ \mathbf{t}^I_{B,n} \end{Bmatrix}$$

Convergence

NO

$$\{\mathbf{u}^I_{B,\,n+1}\} = (1-\gamma)\{\mathbf{u}^I_{B,\,n}\} + \gamma\ \{\mathbf{u}^I_{F,\,n}\}$$

$$\{\mathbf{t}^I_{F,\,n+1}\} = -\{\mathbf{t}^I_{B,\,n}\}$$

YES

Stop

IR FEM-BEM coupling methods (2)

Dirichlet-Neumann FEM-BEM coupling *

$$\left\{\mathbf{u}^{I}_{B,\,n+1}\right\} = (1-\gamma)\left\{\mathbf{u}^{I}_{B,\,n}\right\} + \gamma\left\{\mathbf{u}^{I}_{F,\,n}\right\}$$

$$\left\{\mathbf{t}^{I}_{F,\,n+1}\right\} = -\left\{\mathbf{t}^{I}_{B,\,n}\right\}$$

Neumann-Neumann FEM-BEM coupling *

$$\left\{\mathbf{t}^{I}_{B,\,n+1}\right\} = \left\{\mathbf{t}^{I}_{B,\,n}\right\} + \beta\left(\left\{\mathbf{u}^{I}_{F,\,n}\right\} - \left\{\mathbf{u}^{I}_{B,\,n}\right\}\right)$$

$$\left\{\mathbf{t}^{I}_{F,\,n+1}\right\} = -\left\{\mathbf{t}^{I}_{B,\,n+1}\right\}$$

* Kamiya, Iwase and Kita, "Parallel Computing for the Combination Method of BEM and FEM," *Eng. Anal. Bound. Elem.,* 1996.

* Lin, Lawton, Caliendo and Anderson, “An Iterative Finite Element-Boundary Element Algorithm,” *Comp. & Struc.*, 1996.

* Feng and Owen, "Iterative Solution of Coupled FE/BE Discretization for Plate-Foundation Interaction Problems," *Int. Jour. Num. Meth. Eng.,* 1996.

IR FEM-BEM coupling methods (3)

Geometric contraction based FEM-BEM coupling *

$$\left\{\mathbf{u}^{I}_{B,\,n+1}\right\} = \left\{\mathbf{u}^{I}_{B,\,n}\right\} - \alpha\left(\left\{\mathbf{t}^{I}_{B,n}\right\} + \left\{\mathbf{t}^{I}_{F,n}\right\}\right)$$

$$\left\{\mathbf{u}^{I}_{F,\,n+1}\right\} = \left\{\mathbf{u}^{I}_{B,\,n+1}\right\}$$

Robin relaxation FEM-BEM coupling *

$$\left\{\mathbf{g}^{I}_{B,\,n+1}\right\} = -\left\{\mathbf{t}^{I}_{F,\,n}\right\} + \rho\left\{\mathbf{u}^{I}_{F,\,n}\right\}$$

$$\left\{\mathbf{g}^{I}_{F,\,n+1}\right\} = -\left\{\mathbf{t}^{I}_{B,\,n}\right\} + \rho\left\{\mathbf{u}^{I}_{B,\,n}\right\}$$

* Elleithy and Tanaka, “Interface Relaxation Algorithms for BEM-BEM Coupling and FEM-BEM Coupling,” *Comp. Meth. Appl. Mech. & Eng.,* 2003.

Convergence / optimal convergence (1)

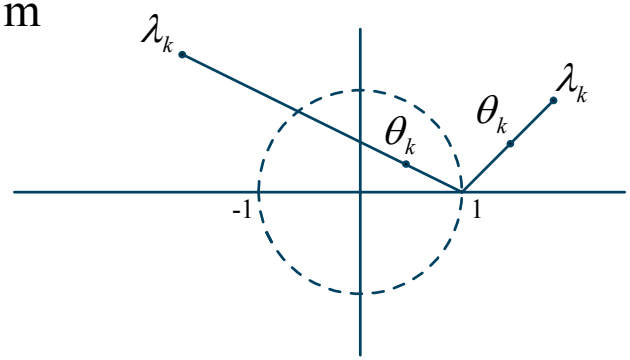
Sequential Dirichlet-Neumann FEM-BEM coupling *

$$u^{I}_{B,n+1} = \left[(1-\gamma)I + \gamma\, T\right] u^{I}_{B,n} + \gamma Q \qquad (1)$$

where $T = -F_{22} M A_{22}$ and $Q = F_{21} C_F - F_{22} M A_{21} C_B$

Eigenvalues of the system (1): $\theta_k = (1-\gamma) + \gamma\, \lambda_k$

where λ_k is the kth eigenvalue of T and $k = 1, 2, 3, \ldots, m$



$\exists\ \gamma_c$ s.t. $|\theta_k| < 1 \iff \mathrm{Re}(\lambda_k) < 1$

let $\lambda_k = x_k + i y_k$ then

$$\left[x_k - (1 - \frac{1}{\gamma})\right]^2 + y_k^{\,2} < \frac{1}{\gamma^2} \quad \Rightarrow \quad (1 - x_k)^2 + y_k^{\,2} < \frac{2(1 - x_k)}{\gamma}$$

* El-Gebeily, Elleithy and Al-Gahtani, "Convergence of the Domain Decomposition Finite Element-Boundary Element Coupling Methods," *Comp. Meth. App. Mech. Eng.*, 2002.

Convergence / optimal convergence (2)

Inequality implies: $\gamma < \min_{1 \le k \le m} \left\{ \frac{2(1 - x_k)}{(1 - x_k)^2 + y_k^{\,2}} \right\}$

Choice of the iteration parameter that minimizes the spectral radius of $[(1-\gamma)I + \gamma\, T]$

minimze $\max_{1 \le k \le m} \left| (1-\gamma) + \gamma\, \lambda_k \right|$

minimize $\| C(\gamma) \|_\infty$ where $C(\gamma) = \begin{bmatrix} (1-\gamma) + \gamma\, \lambda_1 \\ (1-\gamma) + \gamma\, \lambda_2 \\ \vdots \\ (1-\gamma) + \gamma\, \lambda_m \end{bmatrix}$

minimize $\| C(\gamma) \|_2$

$\| x \|_\infty$ is not a diffrentiable function and $\frac{1}{\sqrt{m}} \| x \|_2 \le \| x \|_\infty \le \| x \|_2$

Convergence / optimal convergence (3)

Let: $$F(\gamma) = \| C(\gamma) \|_2^{\,2}$$

$$F'(\gamma) = 2\sum_{k=1}^{m} \mathrm{Re}(\lambda_k - 1) + 2\gamma \sum_{k=1}^{m} |\lambda_k - 1|^2$$

$$F''(\gamma) = 2\sum_{k=1}^{m} |\lambda_k - 1|^2 > 0$$

$$\bar{\gamma} = -\frac{\sum_{k=1}^{m} \mathrm{Re}(\lambda_k - 1)}{\sum_{k=1}^{m} |\lambda_k - 1|}$$

$$F_{min} = F(\bar{\gamma}) = n - \frac{\left(\sum_{k=1}^{m} \mathrm{Re}(\lambda_k - 1)\right)^2}{\sum_{k=1}^{n} |\lambda_k - 1|^2}$$

If $F_{min} < 1 \Rightarrow \rho\left((1-\bar{\gamma})I + \bar{\gamma}\, T\right) < 1$ and convergence is achieved

Convergence / optimal convergence (4)

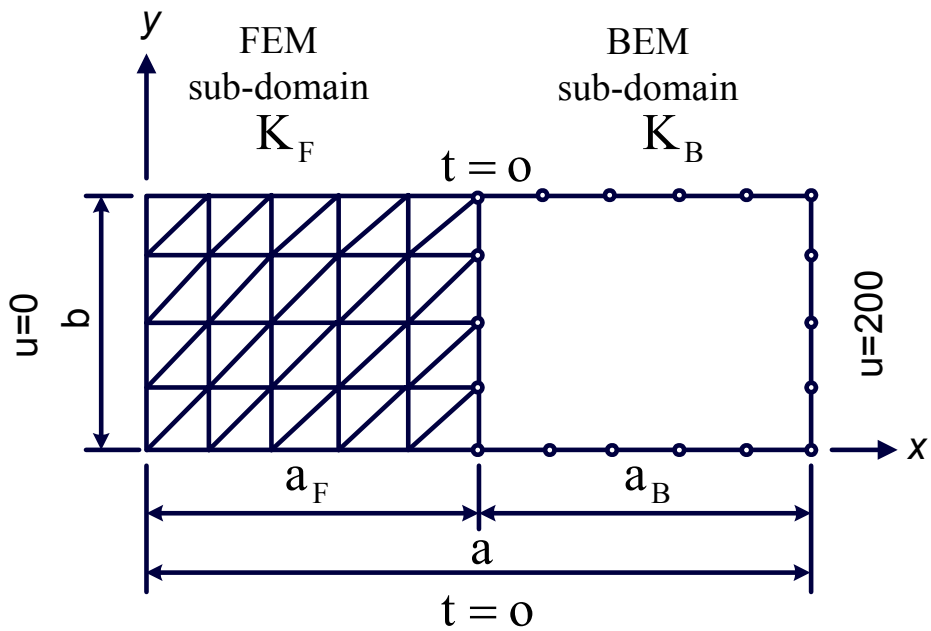
- (Optimal) **convergence** ensured by properly set relaxation parameters
  - **Static**
    - experimenting with different values
    - convergence/optimal convergence conditions
      - Sequential Dirichlet-Neumann, Parallel Dirichlet-Neumann, Neumann-Neumann FEM-BEM coupling *
      - Unified convergence analysis *
  - **Dynamic:** determined by minimizing square error functional of next and current iterations **

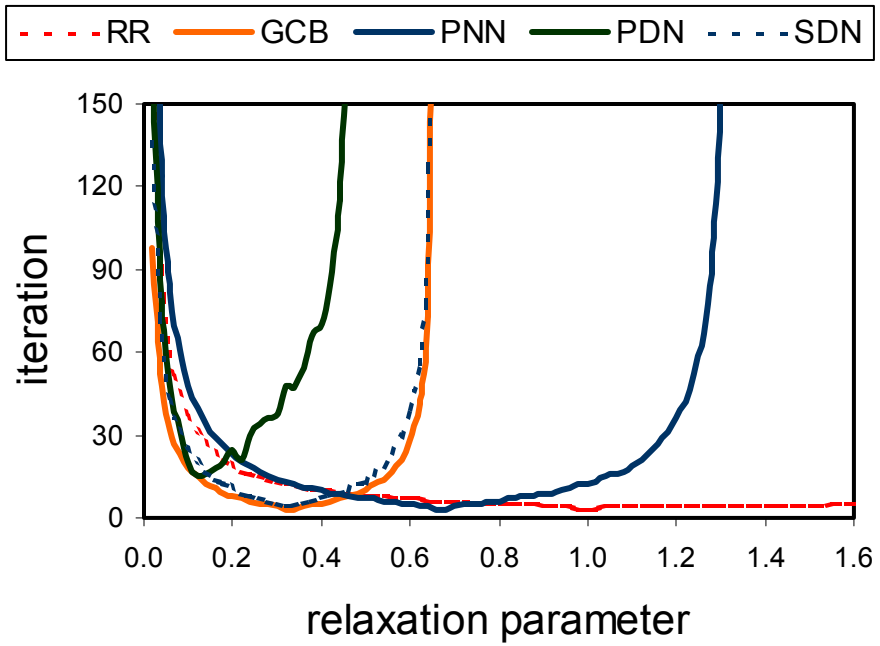
$$\mathbf{g}(\alpha) = \left\| \mathbf{u}^{\mathrm{I}}_{B,\,n+1}(\alpha) - \mathbf{u}^{\mathrm{I}}_{B,\,n}(\alpha) \right\|_2^2 \longrightarrow \alpha_{n+1} = \frac{(\mathbf{u}^{\mathrm{I}}_{\mathrm{B,\,n}} - \mathbf{u}^{\mathrm{I}}_{\mathrm{B,\,n-1}}) \cdot ((\mathbf{t}^{\mathrm{I}}_{\mathrm{B,\,n}} - \mathbf{t}^{\mathrm{I}}_{\mathrm{B,\,n-1}}) + (\mathbf{t}^{\mathrm{I}}_{\mathrm{F,\,n}} - \mathbf{t}^{\mathrm{I}}_{\mathrm{F,\,n-1}}))}{\left\| (\mathbf{t}^{\mathrm{I}}_{\mathrm{B,\,n}} - \mathbf{t}^{\mathrm{I}}_{\mathrm{B,\,n-1}}) + (\mathbf{t}^{\mathrm{I}}_{\mathrm{F,\,n}} - \mathbf{t}^{\mathrm{I}}_{\mathrm{F,\,n-1}}) \right\|_2^2}$$

* El-Gebeily, Elleithy and Al-Gahtani, "Convergence of the Domain Decomposition Finite Element-Boundary Element Coupling Methods," *Comp. Meth. App. Mech. Eng.,* 2002.

** Elleithy, Tanaka, & Guzik, “Interface Relaxation FEM-BEM Coupling Method for Elasto-Plastic Analysis,” *Eng. Anal. Bound. Elem.,* 2004.

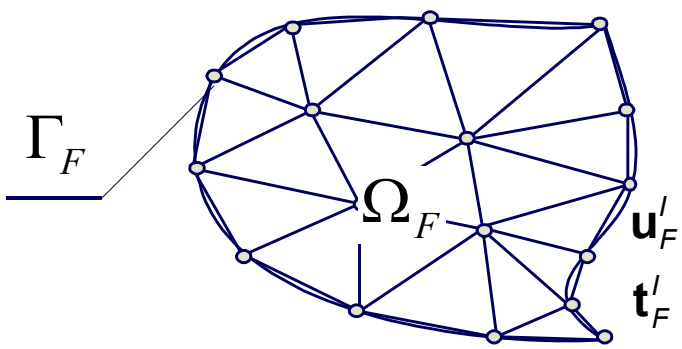
Convergence / optimal convergence (5)





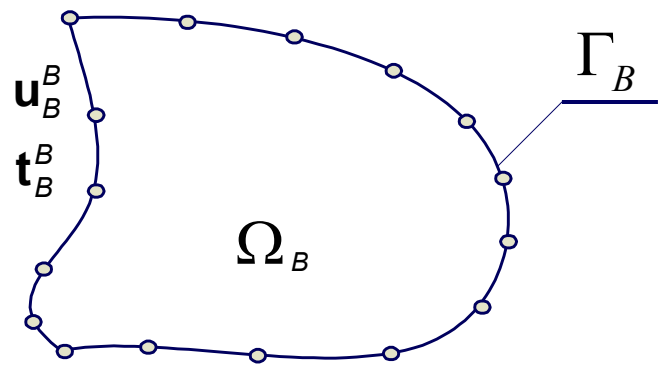
IR coupling in Elasto-Plasticity

FEM



$$\begin{Bmatrix} \Delta\boldsymbol{\psi}_F^F \\ \Delta\boldsymbol{\psi}_F^I \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{T11} & \mathbf{K}_{T12} \\ \mathbf{K}_{T21} & \mathbf{K}_{T22} \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{u}_F^F \\ \Delta\mathbf{u}_F^I \end{Bmatrix} - \begin{Bmatrix} \Delta\mathbf{f}_F^F \\ \Delta\mathbf{f}_F^I \end{Bmatrix}$$

BEM



$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_B^B \\ \mathbf{u}_B^I \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{t}_B^B \\ \mathbf{t}_B^I \end{Bmatrix}$$

at the interface

$$\{\mathbf{u}_B^I\} = \{\mathbf{u}_F^I\}$$ **equilibrium**

$$\{\mathbf{f}_F^I\} + [\mathbf{M}]\{\mathbf{t}_B^I\} = 0$$ **compatibility**

IR coupling in Elasto-Plasticity (2)

Solution algorithm

Data Input

Set Initial Guess $\mathbf{u}^I_{B,0}$ and $\mathbf{u}^I_{F,0}$

FEM

for $i = 1, 2, \ldots, m$

$$\begin{Bmatrix} \Delta\boldsymbol{\psi}^F_{F,n} \\ \Delta\boldsymbol{\psi}^I_{F,n} \end{Bmatrix}_i = \begin{bmatrix} \mathbf{K}_{T11} & \mathbf{K}_{T12} \\ \mathbf{K}_{T21} & \mathbf{K}_{T22} \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{u}^F_{F,n} \\ \Delta\mathbf{u}^I_{F,n} \end{Bmatrix}_i - \begin{Bmatrix} \Delta\mathbf{f}^F_{F,n} \\ \Delta\mathbf{f}^I_{F,n} \end{Bmatrix}_i$$

$$\left\{\mathbf{f}^I_{F,n}\right\}_{i+1} = \left\{\mathbf{f}^I_{F,n}\right\}_i + \left\{\Delta\mathbf{f}^I_{F,n}\right\}_i \Rightarrow \left\{\mathbf{f}^I_{F,n}\right\} \equiv \left\{\mathbf{f}^I_{F,n}\right\}_m$$

$$\left\{\mathbf{f}^I_{F,n}\right\} = [\mathbf{M}]\left\{\mathbf{t}^I_{F,n}\right\}$$

BEM

$$\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^B_{B,n} \\ \mathbf{u}^I_{B,n} \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{t}^B_{B,n} \\ \mathbf{t}^I_{B,n} \end{Bmatrix}$$

Convergence

NO

$$\left\{\mathbf{u}^I_{B,n+1}\right\} = \left\{\mathbf{u}^I_{B,n}\right\} - \alpha\left(\left\{\mathbf{t}^I_{B,n}\right\} + \left\{\mathbf{t}^I_{F,n}\right\}\right)$$

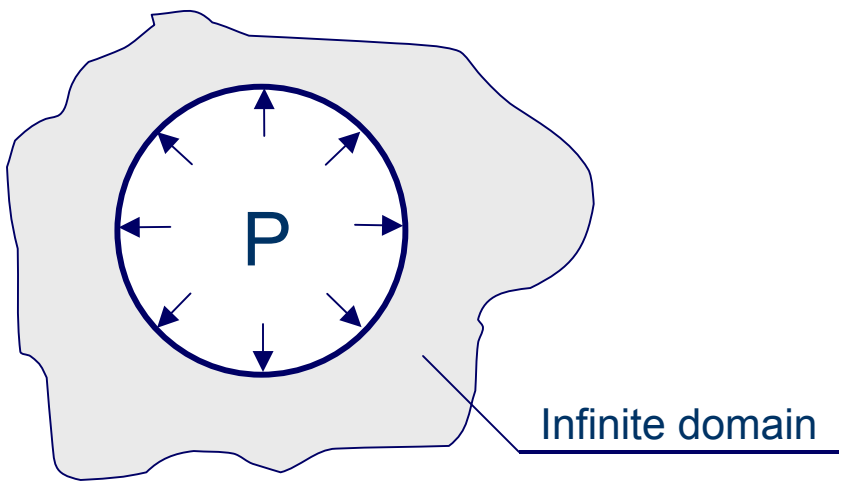
$$\left\{\mathbf{u}^I_{F,n+1}\right\} = \left\{\mathbf{u}^I_{B,n+1}\right\}$$

YES

Stop

IR coupling in Elasto-Plasticity (3)

Single tunnel test



tunnel radius = 1 m
P = 25 MPa

Domains

FEM – domain 1 – R= 4.30 m
FEM – domain 2 – R= 8.70 m
FEM – domain 3 – R= 15.0 m

FEM/BEM – R= 3.60 m

Mesh sizes

FEM – mesh 1 -- 288
FEM – mesh 2 -- 448
FEM – mesh 3 -- 608

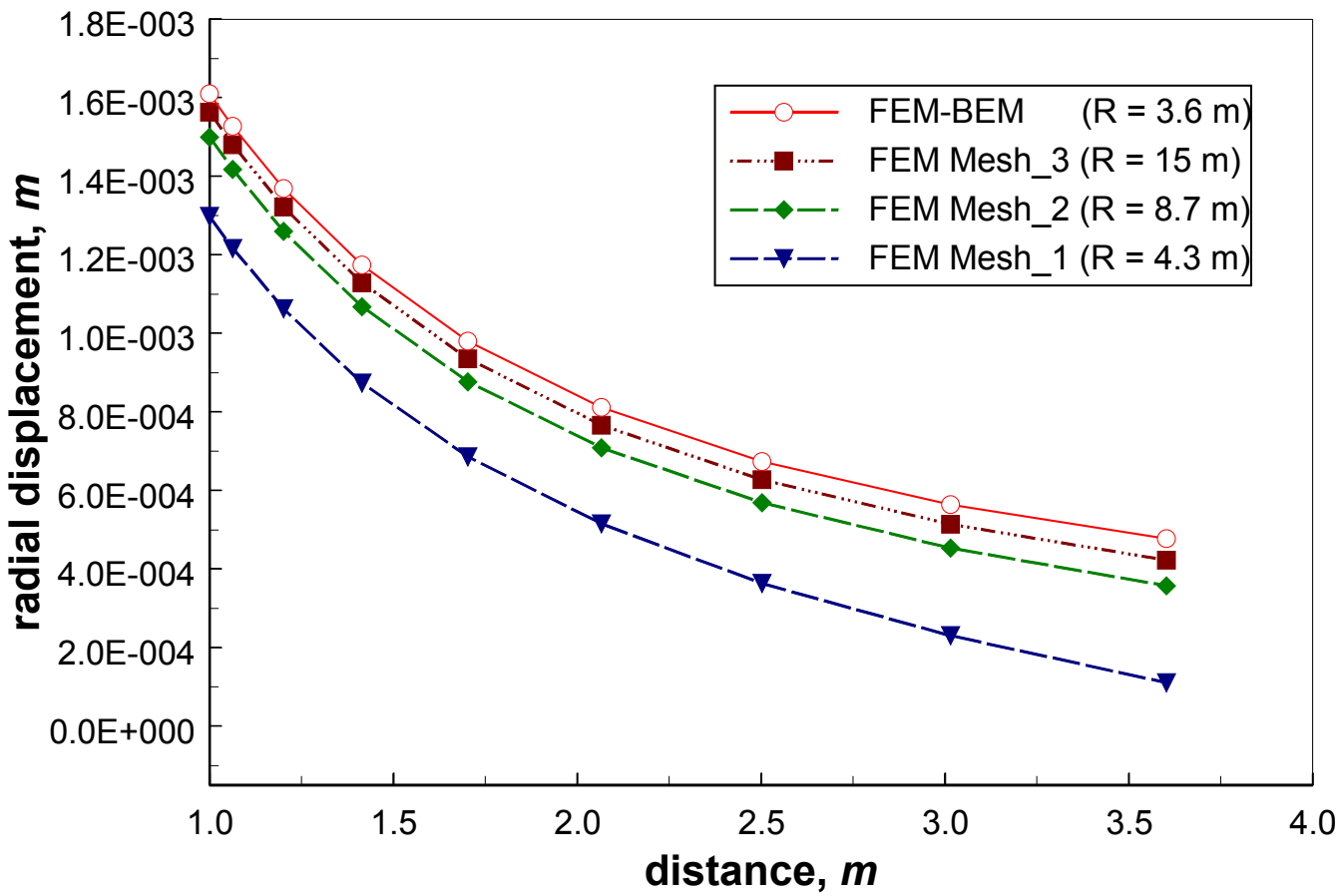
FEM/BEM -- 256/32

Plane strain loading conditions

- yield criterion - Drucker-Prager (c=10, ϕ=41)
- Young modulus E=21 GPa, Poisson's ratio ν=0.18

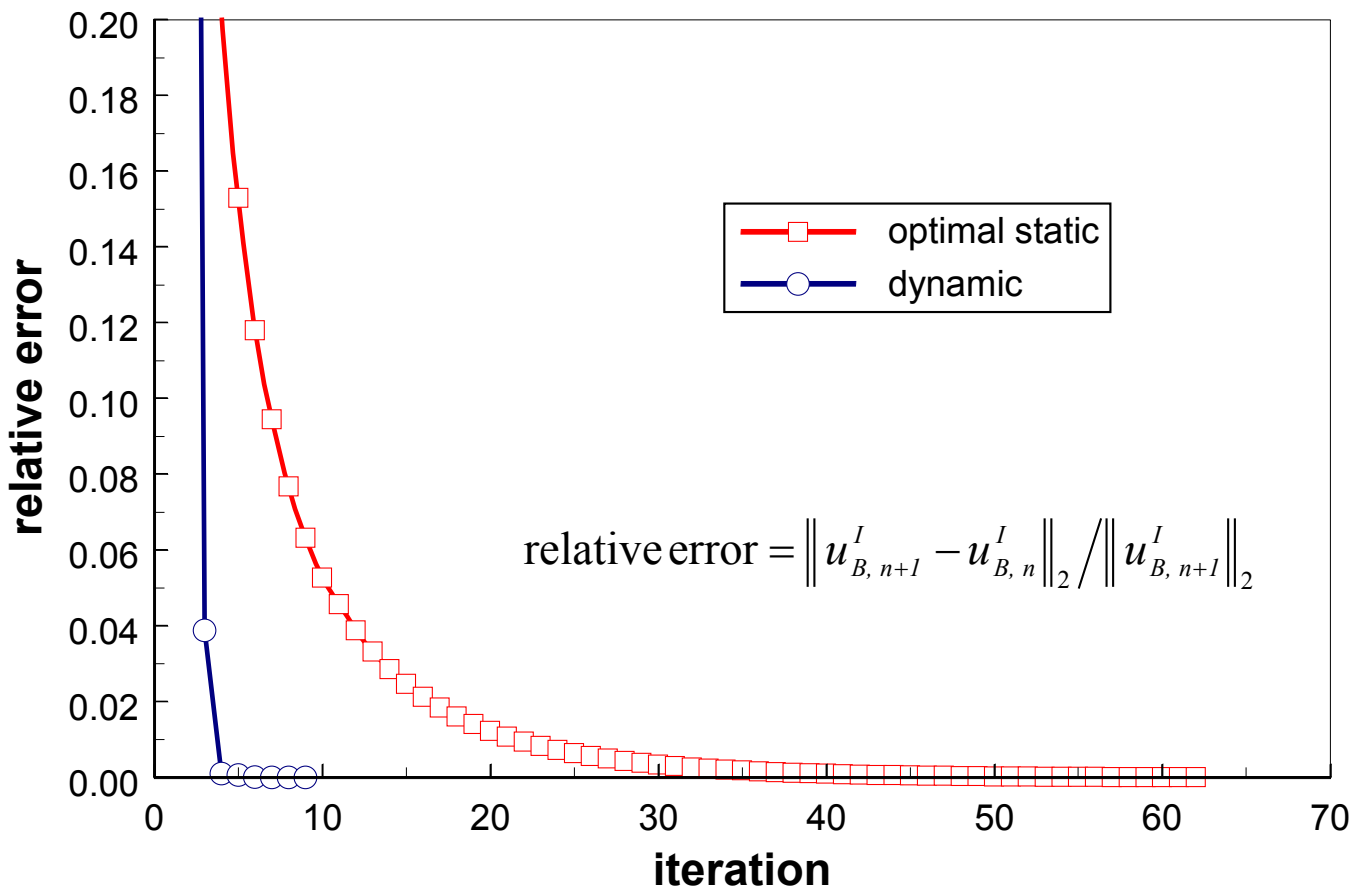
IR coupling in Elasto-Plasticity (4)

Displacements distribution



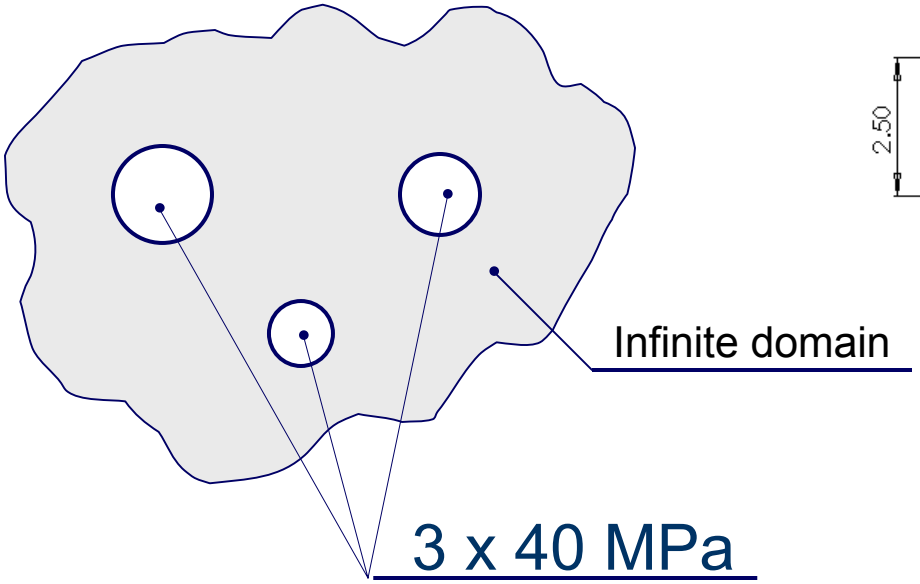
IR coupling in Elasto-Plasticity (5)

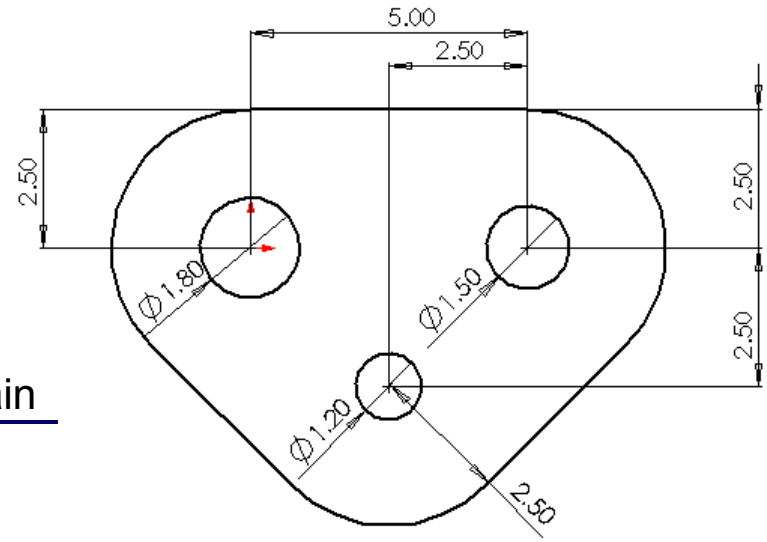
Comparison of relative errors



IR coupling in Elasto-Plasticity (6)

E-P deformations in tunnels system





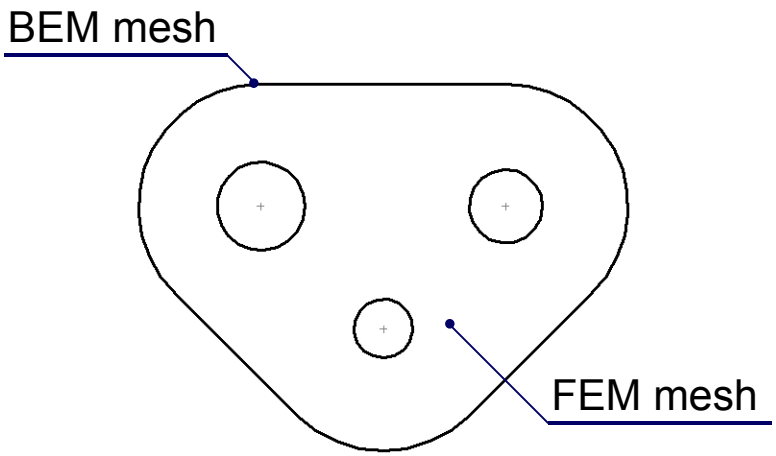
Plane strain loading conditions

- yield criterion - Drucker-Prager (c=41, ϕ=41)
- Young modulus E=21 GPa, Poisson's ratio ν=0.18

IR coupling in Elasto-Plasticity (7)

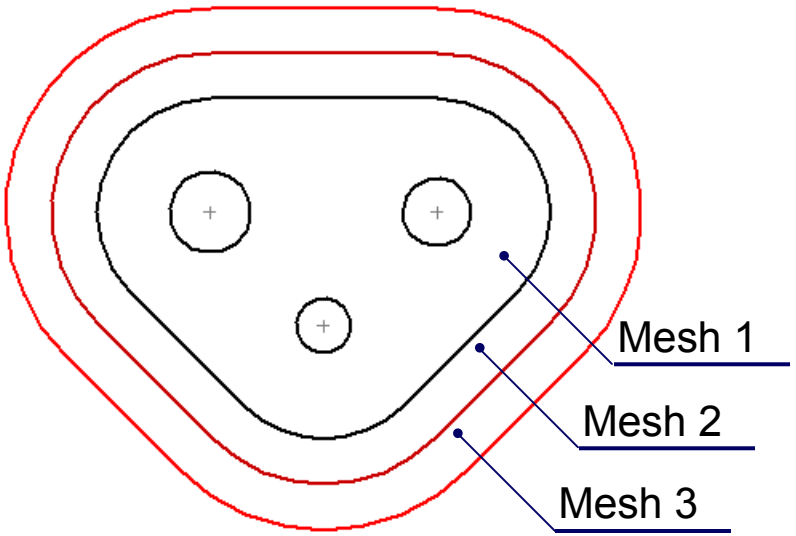
Tunnels system – domains

FEM/BEM model



138 boundary elements
1287 finite elements

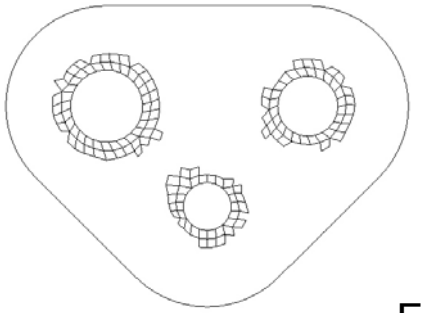
FEM (rigid truncated boundary) models



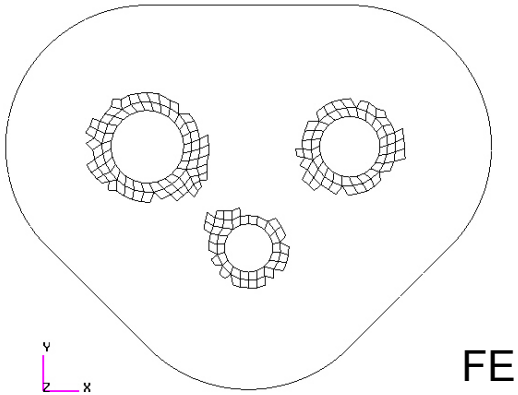
Mesh 1 -- 1287 elements
Mesh 2 -- 2123 elements
Mesh 3 -- 3170 elements

IR coupling in Elasto-Plasticity (8)

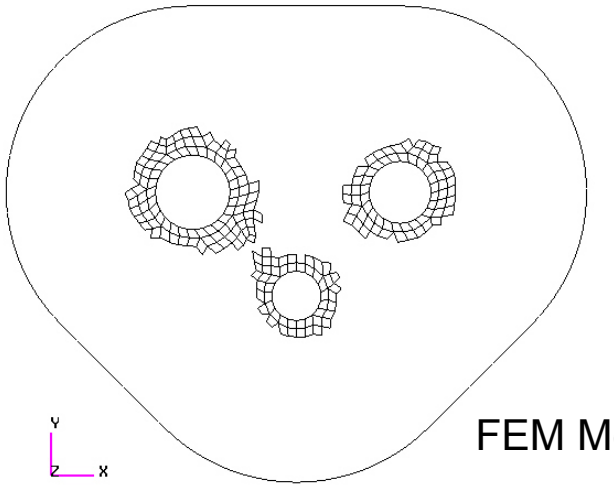
Yielding zones – increment 6



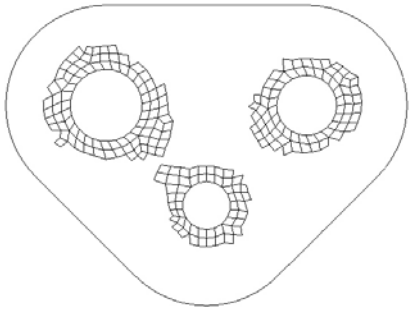
FEM Mesh 1



FEM Mesh 2



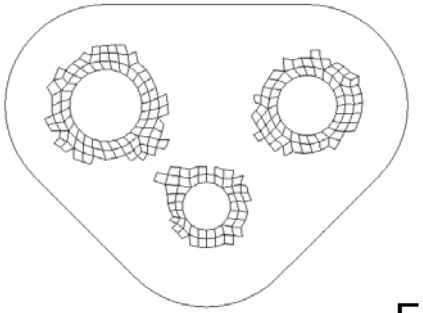
FEM Mesh 3



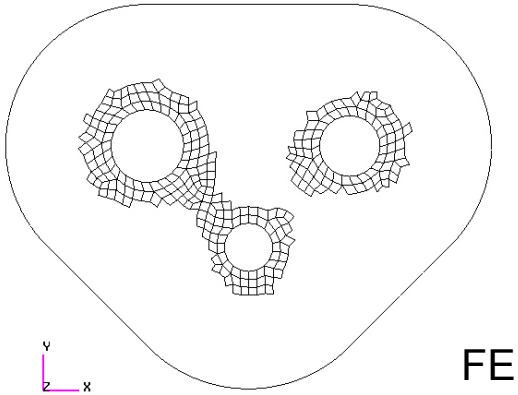
Coupled FEM/BEM

IR coupling in Elasto-Plasticity (9)

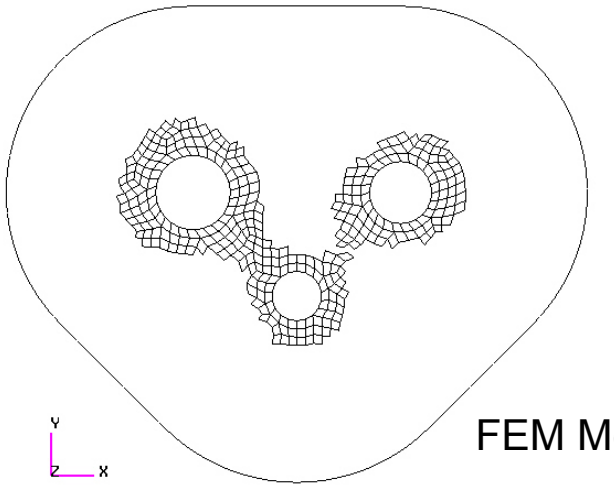
Yielding zones – increment 8 (total load)



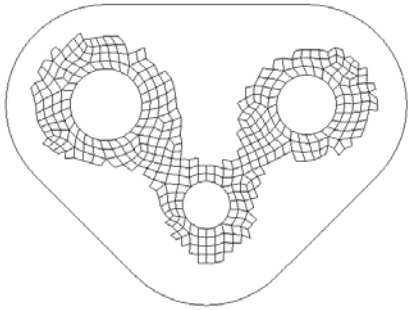
FEM Mesh 1



FEM Mesh 2



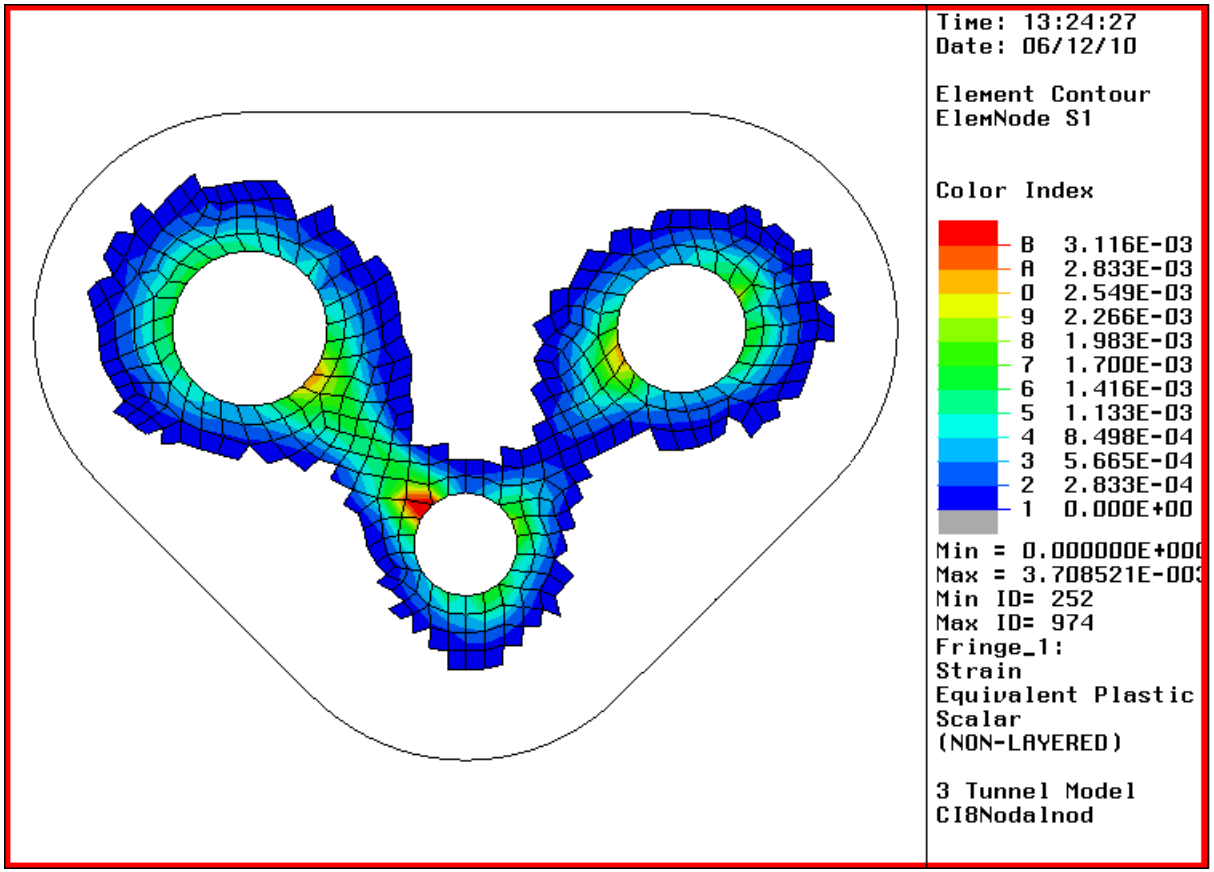
FEM Mesh 3



Coupled FEM/BEM

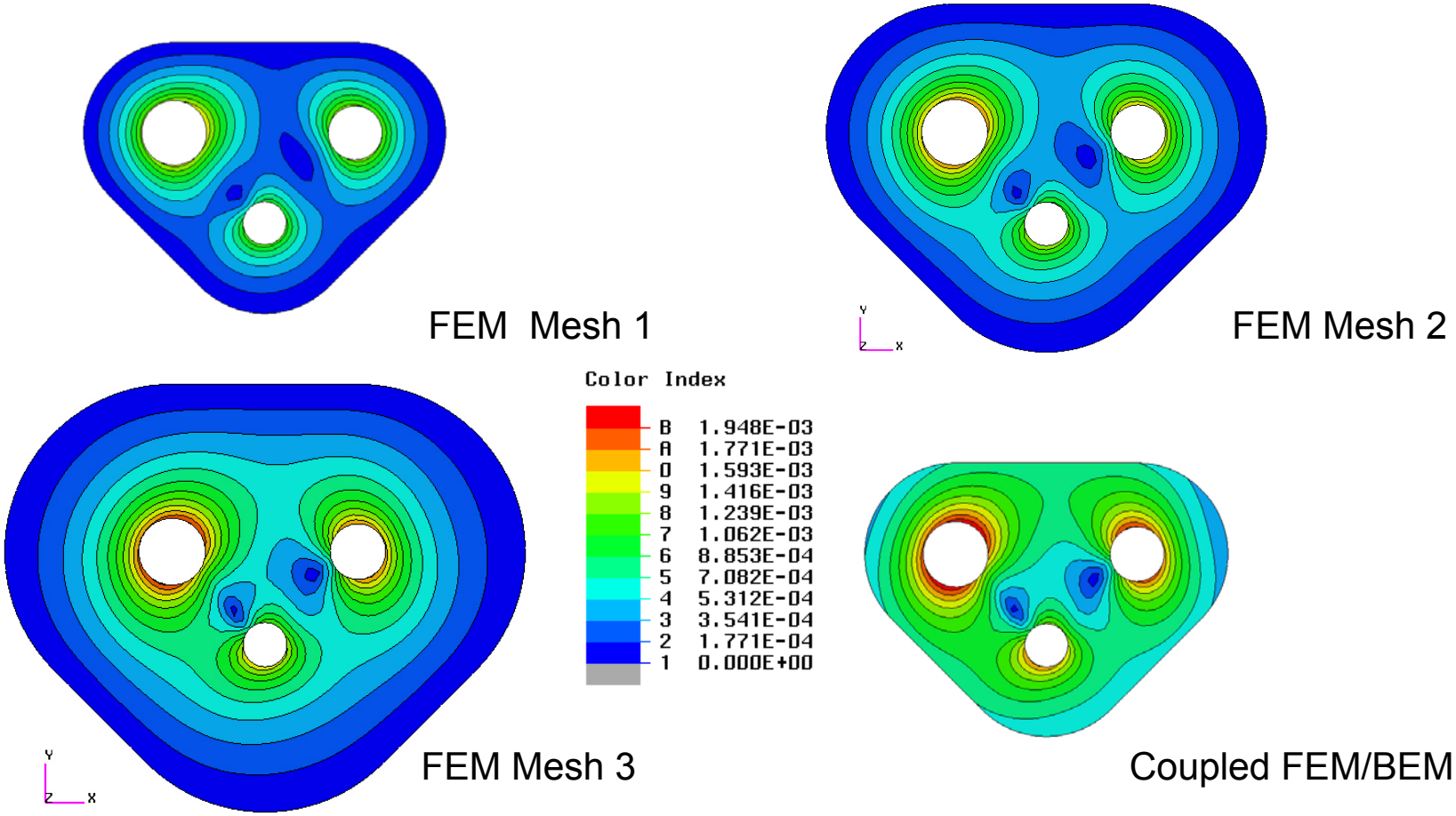
IR coupling in Elasto-Plasticity (10)

Yielding zone – load increment: 8 (total load)

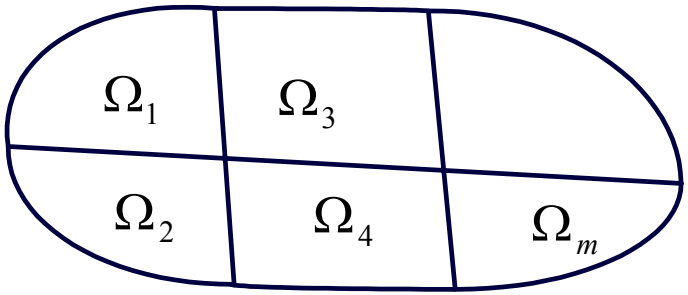


IR coupling in Elasto-Plasticity (11)

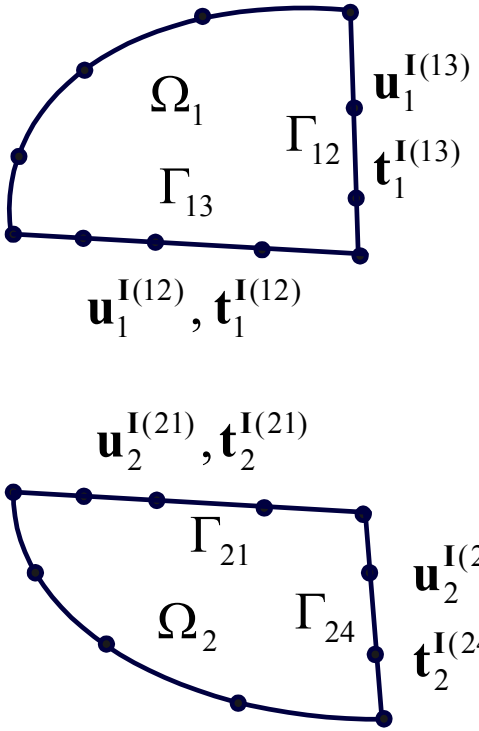
Displacements – increment 8 (total load)



Interface relaxation BEM-BEM coupling



Domain decomposed into m sub - domains



$$\left\{ \mathbf{u}_1^{\mathbf{I}} \right\} = \begin{Bmatrix} \mathbf{u}_1^{\mathbf{I}(12)} \\ \mathbf{u}_1^{\mathbf{I}(13)} \end{Bmatrix}$$

$$\mathbf{u}_i^{\mathrm{I}} = \bigcup \mathbf{u}_i^{\mathrm{I(ij)}}$$

$$\left\{ \mathbf{u}_2^{\mathbf{I}} \right\} = \begin{Bmatrix} \mathbf{u}_2^{\mathbf{I}(21)} \\ \mathbf{u}_2^{\mathbf{I}(24)} \end{Bmatrix}$$

BEM modeling for Ω_1 and Ω_2

IR BEM-BEM coupling (2)

Decompose the domain of the original problem into m sub-domains, $\Omega_1, \Omega_2, \ldots, \Omega_m$

Let $S(i)$ be the indices of sub-domains that are neighbors of sub-domain Ω_i

$$\mathbf{u}_i^I = \bigcup \mathbf{u}_i^{I(ij)} \text{ for all } j \in S(i)$$

discretize integral equations for the BEM sub-domains:

$$\begin{bmatrix} \mathbf{H}_{i,11} & \mathbf{H}_{i,12} \\ \mathbf{H}_{i,21} & \mathbf{H}_{i,22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_i^B \\ \mathbf{u}_i^I \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{i,11} & \mathbf{G}_{i,12} \\ \mathbf{G}_{i,21} & \mathbf{G}_{i,22} \end{bmatrix} \begin{Bmatrix} \mathbf{t}_i^B \\ \mathbf{t}_i^I \end{Bmatrix}$$

at the interface Γ_{kl}, compatability and equilibrium conditions:

$$\left\{ \mathbf{u}_k^{I(kl)} \right\} = \left\{ \mathbf{u}_l^{I(lk)} \right\}$$

$$\left\{ \mathbf{t}_k^{I(kl)} \right\} = -\left\{ \mathbf{t}_l^{I(lk)} \right\}$$

IR BEM-BEM coupling (3)

Neumann-Neumann BEM-BEM coupling *

Set initial guess $\left\{ \mathbf{t}_{i,0}^{I} \right\}$ for the BEM sub-domains $(i = 1, \ldots, m)$

Do for $n = 0, 1, 2, \ldots$ until convergence

for $i = 1, \ldots, m$

$$\text{solve} \begin{bmatrix} \mathbf{H}_{i,11} & \mathbf{H}_{i,12} \\ \mathbf{H}_{i,21} & \mathbf{H}_{i,22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{i,n}^{B} \\ \mathbf{u}_{i,n}^{I} \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_{i,11} & \mathbf{G}_{i,12} \\ \mathbf{G}_{i,21} & \mathbf{G}_{i,22} \end{bmatrix} \begin{Bmatrix} \mathbf{t}_{i,n}^{B} \\ \mathbf{t}_{i,n}^{I} \end{Bmatrix} \text{ for } \mathbf{u}_{i,n}^{I}$$

$$\text{apply } \left\{ \mathbf{t}_{i,\,n+1}^{I(ij)} \right\} = \left\{ \mathbf{t}_{i,\,n}^{I(ij)} \right\} + \beta \left(\left\{ \mathbf{u}_{j,n}^{I(ji)} \right\} - \left\{ \mathbf{u}_{i,n}^{I(ij)} \right\} \right) \text{ for all } j \in S(i)$$

where β, is a relaxation parameter to ensure and/or accelerate convergence

* Kamiya, Iwase and kita, “Parallel implementation of boundary element method with domain decomposition,” Eng. Anal. Bound. Elem., 1997.

IR BEM-BEM coupling (4)

Geometric contraction BEM-BEM based coupling *

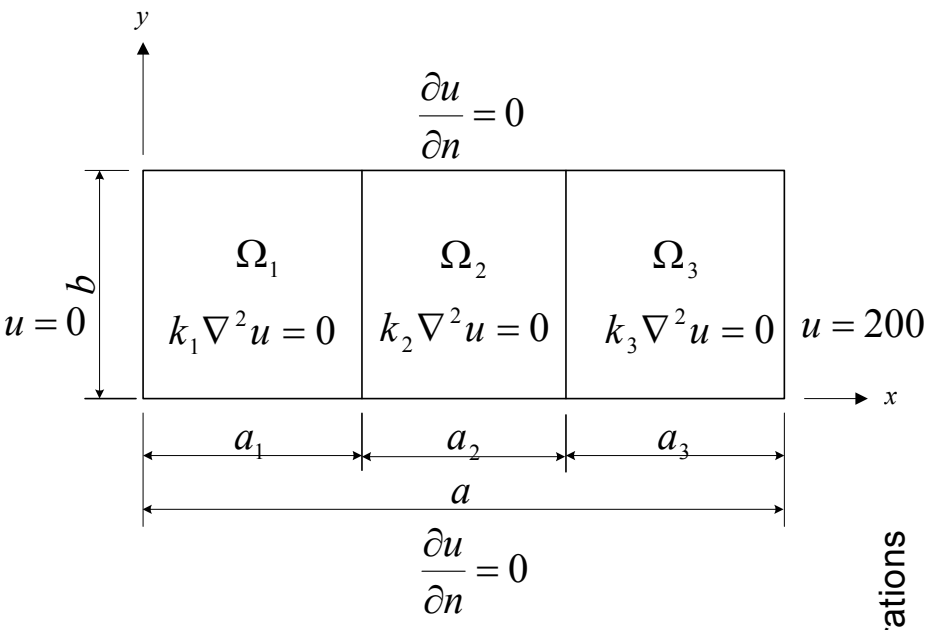
$$\left\{\mathbf{u}^{\mathbf{I}(ij)}_{i,\,\mathbf{n+1}}\right\} = \left\{\mathbf{u}^{\mathbf{I}(ij)}_{i,\,\mathbf{n}}\right\} - \boldsymbol{\alpha}\left(\left\{\mathbf{t}^{\mathbf{I}(ij)}_{i,\mathbf{n}}\right\} + \left\{\mathbf{t}^{\mathbf{I}(ji)}_{j,\mathbf{n}}\right\}\right) \text{ for all } \mathrm{j} \in \mathrm{S(i)}$$

Robin relaxation BEM-BEM coupling *

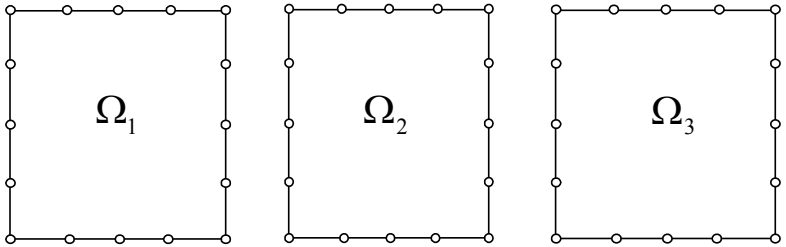
$$\left\{\mathbf{g}^{\mathbf{I}(ij)}_{i,\,n+1}\right\} = -\left\{\mathbf{q}^{\mathbf{I}(ji)}_{j,\,\mathbf{n}}\right\} + \rho\left\{\mathbf{u}^{\mathbf{I}(ji)}_{j,\,\mathbf{n}}\right\} \text{ for all } \mathrm{j} \in \mathrm{S(i)}$$

* Elleithy and Tanaka, “Interface Relaxation Algorithms for BEM-BEM Coupling and FEM-BEM Coupling,” *Comp. Meth. Appl. Mech. & Eng.,* 2003.

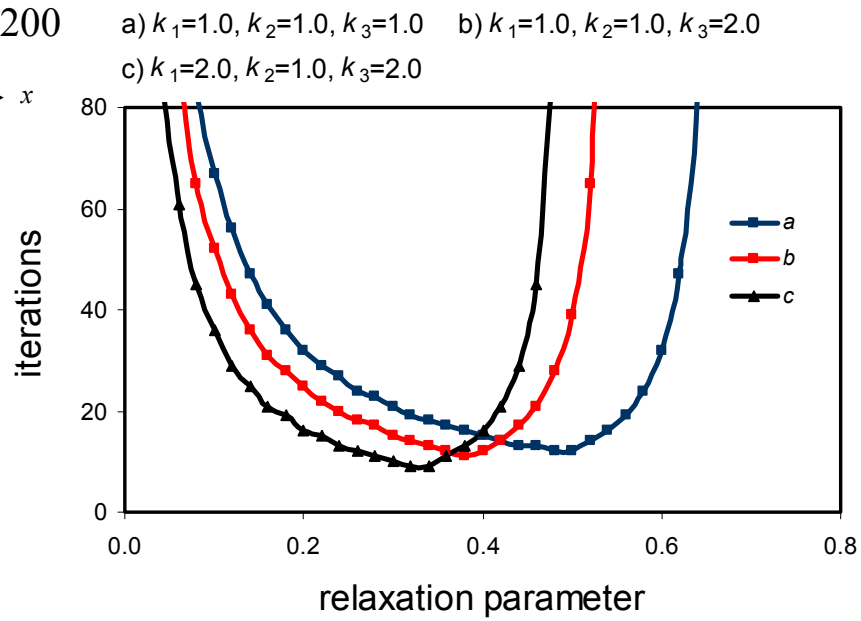
IR BEM-BEM coupling (5)



BEM-BEM coupling example



Domain decomposed into sub-domains



Conclusions

- **In certain cases it is beneficial to couple the FEM and BEM in analysis**
- **The IR coupling algorithms have the potential to work effectively**
  - No access to FEM/BEM matrices required
  - commercial packages may be used to solve sub-problems
  - smaller FEM mesh for plastic region
  - reduced number of FEM-BEM iterations with dynamic calculation of relaxation parameters
- **The IR algorithms exhibit different mathematical and computational properties**
- **The rate of convergence of IR algorithms is affected by certain problem parameters**
- **There is still much to learn about IR algorithms**