Piezoelectric stack actuators regarding temperature and exciting voltage frequencies

Miniworkshop: Direct and Inverse Problems in Piezoelectricity

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Winfried Geis

geis@ians.uni-stuttgart.de

Bosch, AE/EDP 5

Outline

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- Constitutive equations
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- Mathematical models
  - Full 2D-model
  - Simplified asymptotic 2D-model
  - Existence and uniqueness
- Numerical examples: electrical and mechanical fields dependent on
  - the angular frequency of the exciting voltage
  - the heating
- Conclusions and future prospects

Engineering problem

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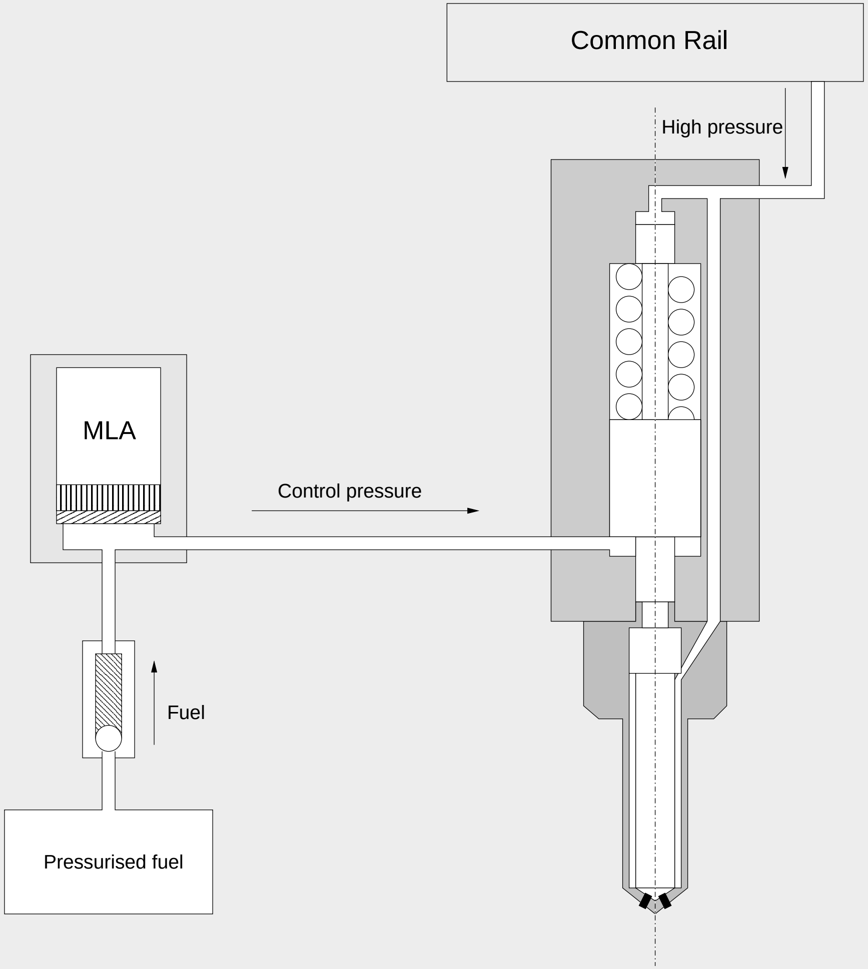
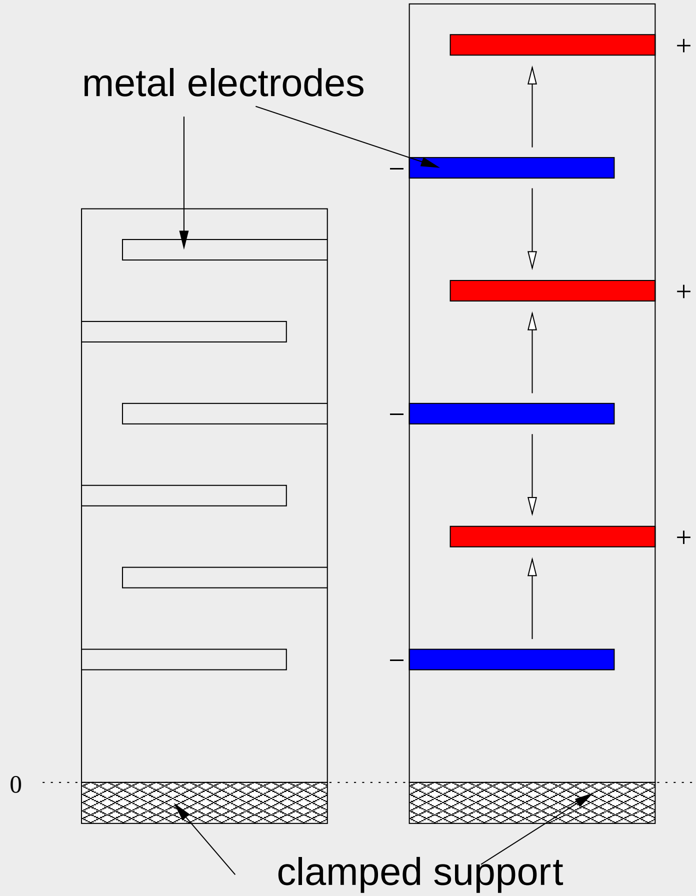


Piezoelectric injection nozzle of a common rail engine.



Growth of a piezoelectric multilayer-actuator (MLA). Common values: driving voltage: $U \approx \pm 200V$, number of layers $n > 80$.

Constitutive equations

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Thermopiezoelectricity in the ceramic

$$s = \frac{\rho c}{T_0} T + \lambda_{ij}\gamma_{ij} + \chi_m E_m$$

$$\sigma_{ij} = -\lambda_{ij} T + C_{ijkl}\gamma_{kl} - e_{mij} E_m$$

$$D_n = \chi_n T + e_{nij}\gamma_{ij} + \varepsilon_{mn} E_m$$

Constitutive equations

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$$D_n = \chi_n T + e_{nij}\gamma_{ij} + \varepsilon_{mn} E_m$$

T	difference of temperature: $T_a = T_0 + T$
$\underline{\underline{\boldsymbol{\gamma}}}$	linearised strain tensor: $\gamma_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$
$\underline{\boldsymbol{E}}$	electric vector field
s	entropy density
$\underline{\underline{\boldsymbol{\sigma}}}$	stress
$\underline{\boldsymbol{D}}$	dielectric displacement
ρ	mass density
c	specific heat per unit mass
$\underline{\underline{\boldsymbol{\lambda}}}$	thermal stress coefficient
$\underline{\boldsymbol{\chi}}$	pyroelectric coefficient
$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$	transversally isotropic (PZT-4) elasticity tensor
$\underline{\underline{\underline{\boldsymbol{e}}}}$	piezoelectric tensor (non-symmetric)
$\underline{\underline{\boldsymbol{\varepsilon}}}$	permittivity tensor (symmetric)

Constitutive equations

Thermopiezoelectricity in the ceramic

$$\sigma_{ij} = -\lambda_{ij} T + C_{ijkl}\gamma_{kl} + e_{mij}\partial_m \Phi$$

$$D_n = \chi_n T + e_{nij}\gamma_{ij} - \varepsilon_{mn}\partial_m \Phi$$

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$\underline{\underline{\underline{\boldsymbol{e}}}}$	piezoelectric tensor (non-symmetric)
$\underline{\underline{\boldsymbol{\varepsilon}}}$	permittivity tensor (symmetric)

Simplifications

- ➜ $\underline{\boldsymbol{E}}$ is curl free, $\underline{\boldsymbol{E}} = -\nabla\Phi$
- ➜ T is known

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Constitutive equations

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Thermoelasticity in the metal

$$s = \frac{\rho c}{T_0} T + \lambda_{ij} \gamma_{ij}$$

$$\sigma_{ij} = -\lambda_{ij} T + C_{ijkl} \gamma_{kl}$$

Constitutive equations

Thermoelasticity in the metal

$$s = \frac{\rho c}{T_0} T + \lambda_{ij}\gamma_{ij}$$

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$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$	isotropic (AgPd alloy) elasticity tensor

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Constitutive equations

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c	specific heat per unit mass
$\underline{\underline{\lambda}}$	thermal stress coefficient
$\underline{\underline{\underline{\underline{C}}}}$	isotropic (AgPd alloy) elasticity tensor

Simplification

➜ T is known

From now on, the two-index notation (Voigt mapping) is used.

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Equations of motion/Gauss' law

Notation

$\underline{\mathbf{u}}_C := r|_{\Omega_C} \underline{\mathbf{u}}$
$\underline{\mathbf{u}}_M := r|_{\Omega_M} \underline{\mathbf{u}}$
$\Phi_C := \Phi_C(\underline{\boldsymbol{x}}, t)$, electric potential

Force balance equations

$$\rho_C \partial_t^2 \underline{\mathbf{u}}_C - \operatorname{Div} \boldsymbol{\sigma}_C(\underline{\mathbf{u}}_C, \Phi, T) = \underline{\mathbf{0}}$$
$$\operatorname{div} \underline{\boldsymbol{D}}_C(\underline{\mathbf{u}}_C, \Phi, T) = 0$$
$$\rho_M \partial_t^2 \underline{\mathbf{u}}_M - \operatorname{Div} \boldsymbol{\sigma}_M(\underline{\mathbf{u}}_M, T) = \underline{\mathbf{0}}$$

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Equations of motion/Gauss' law

BOSCH

Notation

$\underline{\mathbf{u}}_C := r|_{\Omega_C} \underline{\mathbf{u}}$
$\underline{\mathbf{u}}_M := r|_{\Omega_M} \underline{\mathbf{u}}$

$\Phi_C := \Phi_C(\underline{\boldsymbol{x}}, t)$, electric potential, Φ_M is known in $Q_M^{(0,t^*)}$
$Q_C^{(0,t^*)} := \cup_{t \in (0,t^*)} \Omega_C^t$, time-space cylinder, $Q_M^{(0,t^*)}$ analogously defined

$$\mathcal{D}^\top := \mathrm{Div} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}$$

Force balance equations

$$\rho_C \underline{\ddot{\mathbf{u}}}_C - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\mathbf{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C = -\mathcal{D}^\top \underline{\boldsymbol{\lambda}}_C T \text{ in } Q_C^{(0,t^*)},$$

$$\mathrm{div}\left(\underline{\underline{\boldsymbol{e}}}\, \mathcal{D} \underline{\mathbf{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}}\, \nabla \Phi_C\right) = -\mathrm{div}\, \chi t \quad \text{in } Q_C^{(0,t^*)},$$

$$\rho_M \underline{\ddot{\mathbf{u}}}_M - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\mathbf{u}}_M = -\mathcal{D}^\top \underline{\boldsymbol{\lambda}}_M T \text{ in } Q_M^{(0,t^*)},$$

Simplifications

- ➜ T is known
- ➜ Two-index notation

Equations of motion/Gauss' law

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Notation

$\underline{\mathbf{u}}_C(\underline{\boldsymbol{x}}, t) := r_{|\Omega_C} \underline{\mathbf{u}}(\underline{\boldsymbol{x}}, t) = e^{\tau t} \underline{\boldsymbol{u}}_C(\underline{\boldsymbol{x}})$

$\underline{\mathbf{u}}_M := r_{|\Omega_M} \underline{\mathbf{u}} = e^{\tau t} \underline{\boldsymbol{u}}_M(\underline{\boldsymbol{x}})$

$\Phi_C := \Phi_C(\underline{\boldsymbol{x}}, t) = e^{\tau t} \Phi_C(\underline{\boldsymbol{x}})$, electric potential, $\Phi_M = e^{\tau t} \Phi_M(\underline{\boldsymbol{x}})$ is known in $Q_M^{(0,t^*)}$

$Q_C^{(0,t^*)} := \cup_{t \in (0,t^*)} \Omega_C^t$, time-space cylinder, $Q_M^{(0,t^*)}$ analogously defined

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Force balance equations

Pseudo oscillation equations: $\tau = s + i\omega$

$$\rho_C \underline{\ddot{\mathbf{u}}}_C - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\mathbf{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C = -\mathcal{D}^\top \underline{\boldsymbol{\lambda}}_C T \text{ in } Q_C^{(0,t^*)},$$

$$\mathrm{div}\,(\underline{\underline{\boldsymbol{e}}}\, \mathcal{D} \underline{\mathbf{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}} \nabla \Phi_C) = -\mathrm{div}\, \chi T \quad \text{ in } Q_C^{(0,t^*)},$$

$$\rho_M \underline{\ddot{\mathbf{u}}}_M - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\mathbf{u}}_M = -\mathcal{D}^\top \underline{\boldsymbol{\lambda}}_M T \text{ in } Q_M^{(0,t^*)},$$

Simplifications

- T is known
- Two-index notation
- **Ansatz:** All functions are harmonic time dependent, pseudo oscillation equations

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Equations of motion/Gauss' law

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Notation

$\underline{\mathbf{u}}_C(\underline{\boldsymbol{x}}, t) := r_{|\Omega_C}\underline{\mathbf{u}}(\underline{\boldsymbol{x}}, t) = e^{\tau t}\underline{\boldsymbol{u}}_C(\underline{\boldsymbol{x}})$

$\underline{\mathbf{u}}_M := r_{|\Omega_M}\underline{\mathbf{u}} = e^{\tau t}\underline{\boldsymbol{u}}_M(\underline{\boldsymbol{x}})$

$\Phi_C := \Phi_C(\underline{\boldsymbol{x}}, t) = e^{\tau t}\Phi_C(\underline{\boldsymbol{x}})$, electric potential, $\Phi_M = e^{\tau t}\Phi_M(\underline{\boldsymbol{x}})$ is known in Ω_M

$$\mathcal{D}^\top := \mathrm{Div} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}$$

Force balance equations

Steady oscillation equations: $\tau = i\omega$

$$-\rho_C\omega^2\underline{\boldsymbol{u}}_C - \mathcal{D}^T\underline{\underline{\boldsymbol{C}}}_C\mathcal{D}\underline{\boldsymbol{u}}_C - \mathcal{D}^\top\underline{\underline{\boldsymbol{e}}}^\top\nabla\Phi_C = -\mathcal{D}^\top\underline{\boldsymbol{\lambda}}_C T \text{ in } \Omega_C,$$

$$\mathrm{div}\,(\underline{\underline{\boldsymbol{e}}}\,\mathcal{D}\underline{\boldsymbol{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}}\,\nabla\Phi_C) = -\mathrm{div}\,\chi T \quad \text{in } \Omega_C,$$

$$-\rho_M\omega^2\underline{\boldsymbol{u}}_M - \mathcal{D}^T\underline{\underline{\boldsymbol{C}}}_M\mathcal{D}\underline{\boldsymbol{u}}_M = -\mathcal{D}^\top\underline{\boldsymbol{\lambda}}_M T \text{ in } \Omega_M$$

Simplifications

- ➜ T is known
- ➜ Two-index notation
- ➜ **Ansatz:** All functions are harmonic time dependent, steady oscillation equations (Helmholtz type)

Equations of motion/Gauss' law

BOSCH

Notation

$\underline{\mathbf{u}}_C(\underline{\boldsymbol{x}}, t) := r_{|\Omega_C} \underline{\mathbf{u}}(\underline{\boldsymbol{x}}, t) = \underline{\boldsymbol{u}}_C(\underline{\boldsymbol{x}})$

$\underline{\mathbf{u}}_M := r_{|\Omega_M} \underline{\mathbf{u}} = \underline{\boldsymbol{u}}_M(\underline{\boldsymbol{x}})$

$\Phi_C := \Phi_C(\underline{\boldsymbol{x}}, t) = \Phi_C(\underline{\boldsymbol{x}})$, electric potential, $\Phi_M = \Phi_M(\underline{\boldsymbol{x}})$ is known in Ω_M

$$\mathcal{D}^\top := \mathrm{Div} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}$$

Force balance equations

Static equations: $\tau = 0$

$$-\mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\boldsymbol{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C = -\mathcal{D}^\top \underline{\boldsymbol{\lambda}}_C T \text{ in } \Omega_C,$$

$$\mathrm{div}\,(\underline{\underline{\boldsymbol{e}}} \mathcal{D} \underline{\boldsymbol{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}} \nabla \Phi_C) = -\mathrm{div}\, \chi T \quad \text{ in } \Omega_C,$$

$$-\mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\boldsymbol{u}}_M = -\mathcal{D}^\top \underline{\boldsymbol{\lambda}}_M T \text{ in } \Omega_M$$

Simplifications

- ➜ T is known
- ➜ Two-index notation
- ➜ **Ansatz:** All functions are time independent, static equations

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Equations of motion/Gauss' law (2D)

BOSCH

Expanded force balance equation system (Thermopiezoelasticity)

$$-\rho\omega^2 u_1 - C_{11}\partial_1^2 u_1 - C_{12}\partial_1\partial_2 u_2 - C_{13}\partial_1\partial_3 u_3 - C_{44}(\partial_3^2 u_1 + \partial_3\partial_1 u_3)$$
$$-\frac{C_{11}-C_{12}}{2}(\partial_2^2 u_1 + \partial_2\partial_1 u_2) - e_{31}\partial_1\partial_3\Phi - e_{15}\partial_3\partial_1\Phi = -\partial_1\lambda_1 T$$

$$-\rho\omega^2 u_2 - C_{12}\partial_2\partial_1 - C_{11}\partial_2^2 u_2 - C_{13}\partial_2\partial_3 u_3 - C_{44}(\partial_3^2 u_2 + \partial_3\partial_2 u_3)$$
$$-\frac{C_{11}-C_{12}}{2}(\partial_1\partial_2 u_1\partial_1^2 u_2) - e_{31}\partial_2\partial_3\Phi - e_{15}\partial_3\partial_2\Phi = -\partial_2\lambda_1 T$$

$$-\rho\omega^2 u_3 - C_{13}\partial_3\partial_1 u_1 - C_{13}\partial_3\partial_2 u_2 - C_{33}\partial_3^2 u_3 - C_{44}(\partial_2\partial_3 u_2 + \partial_2^2 u_3)$$
$$-C_{44}(\partial_1\partial_3 u_1 + \partial_1^2 u_3) - e_{33}\partial_3^2\Phi - e_{15}\partial_2^2\Phi - e_{15}\partial_1^2\Phi = -\partial_3\lambda_3 T$$

$$e_{15}\partial_1\partial_3 u_1 + e_{15}\partial_1^2 u_3 + e_{15}\partial_2\partial_3 u_2 + e_{15}\partial_2^2 u_3 + e_{31}\partial_3\partial_1 u_1$$
$$+e_{31}\partial_3\partial_2 u_2 + e_{33}\partial_3^2 u_3 - \varepsilon_{11}\partial_1^2\Phi - \varepsilon_{11}\partial_2^2\Phi - \varepsilon_{33}\partial_3^2\Phi = -(\partial_1 p + \partial_2 p$$
$$+ \partial_3 p)T$$

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Equations of motion/Gauss' law (2D)

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Expanded force balance equation system (Thermopiezoelasticity) with plane strain assumption

$\underline{\boldsymbol{u}} = \underline{\boldsymbol{u}}(x_1, x_3),\ \Phi = \Phi(x_1, x_3)$

$$-\rho\omega^2 u_1 - C_{11}\partial_1^2 u_1 - C_{13}\partial_1\partial_3 u_3 - C_{44}(\partial_3^2 u_1 + \partial_3\partial_1 u_3) - e_{31}\partial_1\partial_3\Phi - e_{15}\partial_3\partial_1\Phi = -\partial_1\lambda_1 T \qquad (1)$$

$$-\rho\omega^2 u_2 - C_{44}\partial_3^2 u_2 - \frac{C_{11} - C_{12}}{2}\partial_1^2 u_2 = -\partial_2\lambda_1 T \qquad (2)$$

$$-\rho\omega^2 u_3 - C_{13}\partial_3\partial_1 u_1 - C_{33}\partial_3^2 u_3 - C_{44}(\partial_1\partial_3 u_1 + \partial_1^2 u_3) - e_{33}\partial_3^2\Phi - e_{15}\partial_1^2\Phi = -\partial_3\lambda_3 T \qquad (3)$$

$$e_{15}\partial_1\partial_3 u_1 + e_{15}\partial_1^2 u_3 + e_{31}\partial_3\partial_1 u_1 + e_{33}\partial_3^2 u_3 - \varepsilon_{11}\partial_1^2\Phi - \varepsilon_{33}\partial_3^2\Phi = -(\partial_1 p + \partial_2 p + \partial_3 p)T \qquad (4)$$

Equation system (1),(3),(4) and equation (2) decouple.

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Equations of motion/Gauss' law (2D)

BOSCH

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$$-\rho\omega^2 u_2 - C_{44}\partial_3^2 u_2 - \frac{C_{11} - C_{12}}{2}\partial_1^2 u_2 = -\partial_2\lambda_1 T \qquad (2)$$

$$-\rho\omega^2 u_3 - C_{13}\partial_3\partial_1 u_1 - C_{33}\partial_3^2 u_3 - C_{44}(\partial_1\partial_3 u_1 + \partial_1^2 u_3) - e_{33}\partial_3^2\Phi - e_{15}\partial_1^2\Phi = -\partial_3\lambda_3 T \qquad (3)$$

$$e_{15}\partial_1\partial_3 u_1 + e_{15}\partial_1^2 u_3 + e_{31}\partial_3\partial_1 u_1 + e_{33}\partial_3^2 u_3 - \varepsilon_{11}\partial_1^2\Phi - \varepsilon_{33}\partial_3^2\Phi = -(\partial_1 p + \partial_2 p + \partial_3 p)T \qquad (4)$$

Equation system (1),(3),(4) and equation (2) decouple.

From now on, the 2D-system (1),(3),(4) will be considered.

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Notation

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System (1,3,4) can be written shortly as:

$$-\rho_C \omega^2 \underline{\boldsymbol{u}}_C - \underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}_C \underline{\underline{\boldsymbol{B}}} \underline{\boldsymbol{U}}_C = \underline{\boldsymbol{F}}_C$$

The corresponding elastic system reads:

$$-\rho_M \omega^2 \underline{\boldsymbol{u}}_M - \underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}_M \underline{\underline{\boldsymbol{B}}} \underline{\boldsymbol{U}}_M = \underline{\boldsymbol{F}}_M$$

Notation

BOSCH

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The corresponding elastic system reads:

$$-\rho_M \omega^2 \underline{\boldsymbol{u}}_M - \underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}_M \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{U}}_M = \underline{\boldsymbol{F}}_M$$

Generalised material matrix $\underline{\underline{\boldsymbol{A}}}_C, \underline{\underline{\boldsymbol{A}}}_M$:

$$\underline{\underline{\boldsymbol{A}}}_C = \begin{pmatrix} c_{11} & c_{13} & 0 & 0 & -e_{31} \\ c_{13} & c_{33} & 0 & 0 & -e_{33} \\ 0 & 0 & c_{44} & -e_{15} & 0 \\ 0 & 0 & e_{15} & \varepsilon_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & \varepsilon_{33} \end{pmatrix}, \quad \underline{\underline{\boldsymbol{A}}}_M = \begin{pmatrix} \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Differential operator $\underline{\underline{\boldsymbol{B}}}$ and generalised displacement vectors $\underline{\boldsymbol{U}}_i$:

$$\underline{\underline{\boldsymbol{B}}} = \begin{pmatrix} \mathcal{D} & \mathbf{0} \\ \underline{\mathbf{0}} & -\nabla_{13} \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} \partial_1 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_1 \end{pmatrix}^\top, \quad \underline{\boldsymbol{U}}_C = \begin{pmatrix} u_{C,1} \\ u_{C,3} \\ \Phi_C \end{pmatrix}, \quad \underline{\boldsymbol{U}}_M = \begin{pmatrix} u_{M,1} \\ u_{M,3} \\ \mp\Phi_a \end{pmatrix}$$

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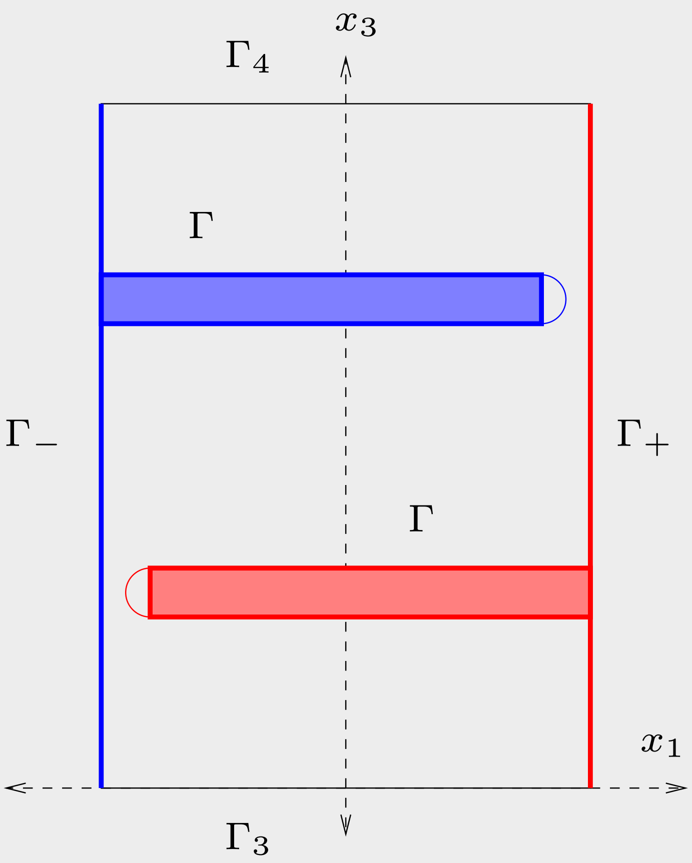
Mathematical models



Model of a simple stack actuator,

$\bar{\Omega} = \bar{\Omega}_M \cup \bar{\Omega}_C$
$\Gamma = \partial\Omega_C \cap \partial\Omega_M$

Linear Voigt Model (ceramic) & Hooke's law (metal-electrode) for the composite (2D,plane strain)

$$-\rho_C \omega^2 \underline{\boldsymbol{u}}_C - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\boldsymbol{u}}_C - \mathcal{D}^T \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C = \underline{\boldsymbol{F}}_C^u \quad \text{in } \Omega_C,$$

$$\operatorname{div}\left(\underline{\underline{\boldsymbol{e}}}\, \mathcal{D} \underline{\boldsymbol{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}}\, \nabla \Phi_C\right) = F_C^\Phi \quad \text{in } \Omega_C,$$

$$-\rho_M \omega^2 \underline{\boldsymbol{u}}_M - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\boldsymbol{u}}_M = \underline{\boldsymbol{F}}_M^u \quad \text{in } \Omega_M$$

$$\Phi_M = \pm \Phi_a \text{ known in } \Omega_M .$$

Boundary conditions

$$\boldsymbol{\sigma}_{C_n}(\underline{\mathbf{u}}_C, \Phi_C) = \underline{\mathbf{0}} \qquad \text{on } \partial\Omega \setminus \Gamma_3$$

$$\underline{\mathbf{u}}_C = \underline{\mathbf{0}} \qquad \text{on } \Gamma_3$$

$$D_{C_n}(\underline{\mathbf{u}}_C, \Phi_C) = 0 \qquad \text{on } \partial\Omega \cap \partial\Omega_C \setminus \Gamma_\pm$$

$$\Phi_C = \pm \Phi_a \qquad \text{on } \Gamma_\pm \cup \Gamma$$

Transmission conditions on Γ:

$$\underline{\mathbf{u}}_C = \underline{\mathbf{u}}_M, \qquad \boldsymbol{\sigma}_{C_n}(\underline{\mathbf{u}}_C, \Phi_C) = \boldsymbol{\sigma}_{M_n}(\underline{\mathbf{u}}_M)$$

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Mathematical model (full problem)

Mathematical models

For real-life actuator geometries, the electrode height is small in comparison with the layer height $\Rightarrow$ large number of nodes in the FEM-simulation.

Linear Voigt Model (ceramic) & Hooke's law (metal-electrode) for the composite (2D,plane strain)

$$
\begin{aligned}
-\rho_C \omega^2 \underline{\boldsymbol{u}}_C - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\boldsymbol{u}}_C - \mathcal{D}^T \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C &= \underline{\boldsymbol{F}}_C^u \quad \text{in } \Omega_C, \\
\operatorname{div}\left(\underline{\underline{\boldsymbol{e}}}\, \mathcal{D} \underline{\boldsymbol{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}} \nabla \Phi_C\right) &= F_C^\Phi \quad \text{in } \Omega_C, \\
-\rho_M \omega^2 \underline{\boldsymbol{u}}_M - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\boldsymbol{u}}_M &= \underline{\boldsymbol{F}}_M^u \quad \text{in } \Omega_M \\
\Phi_M &= \pm \Phi_a \quad \text{known in } \Omega_M .
\end{aligned}
$$

Boundary conditions

$$
\begin{aligned}
\boldsymbol{\sigma}_{C_n}(\underline{\mathbf{u}}_C, \Phi_C) &= \underline{\mathbf{0}} && \text{on } \partial\Omega \setminus \Gamma_3 \\
\underline{\mathbf{u}}_C &= \underline{\mathbf{0}} && \text{on } \Gamma_3 \\
D_{C_n}(\underline{\mathbf{u}}_C, \Phi_C) &= 0 && \text{on } \partial\Omega \cap \partial\Omega_C \setminus \Gamma_\pm \\
\Phi_C &= \pm \Phi_a && \text{on } \Gamma_\pm \cup \Gamma
\end{aligned}
$$

Transmission conditions on Γ:

$$
\underline{\mathbf{u}}_C = \underline{\mathbf{u}}_M, \qquad \boldsymbol{\sigma}_{C_n}(\underline{\mathbf{u}}_C, \Phi_C) = \boldsymbol{\sigma}_{M_n}(\underline{\mathbf{u}}_M)
$$

Asymptotic procedure

Idea: Exploitation of the small geometrical quantity (electrode height h) in the original problem: reduction to a multifield problem **only** in the ceramic domain by replacing the metallic electrodes by non-standard interface conditions on the middle lines Γ_M of the electrodes.

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Asymptotic procedure

BOSCH

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Proceeding (perturbed problem)

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Asymptotic procedure

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Asymptotic procedure

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4. Assumption: $\underline{\boldsymbol{u}}_C, \underline{\boldsymbol{u}}_M$ and Φ_C can be written as power series with respect to ϵ
5. Inserting the splitted operator and the power series into the PDE system and the transmission conditions and comparing coefficients.
   $\Rightarrow$ Taylor series of new transmission conditions around electrodes of thickness zero

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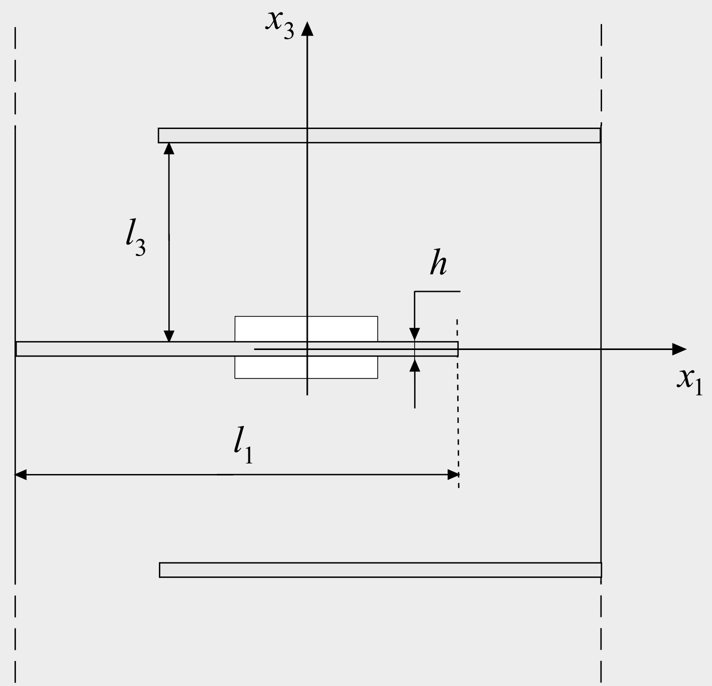
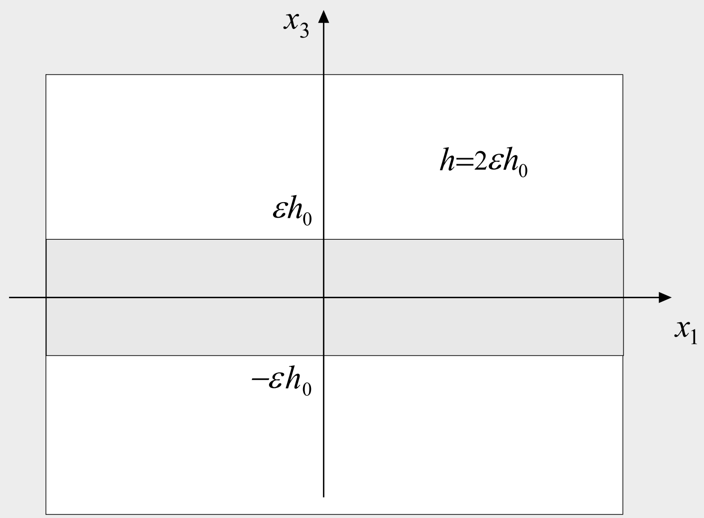
Asymptotic procedure

1. We select one electrode $\eta = \eta_j$, $\Omega M = \cup_{j=1}^{n} \eta_j$ with a local coordinate system in a neighbourhood $U(\eta)$:

$$x_3 = \epsilon\xi, \quad \xi \in [-h_0, h_0], \quad h_0 \sim l_3, \quad 0 \leq \epsilon \leq 1,$$

$$\underline{\boldsymbol{u}}_{\epsilon}(x_1, \xi) := \underline{\boldsymbol{u}}_M(x_1, x_3).$$





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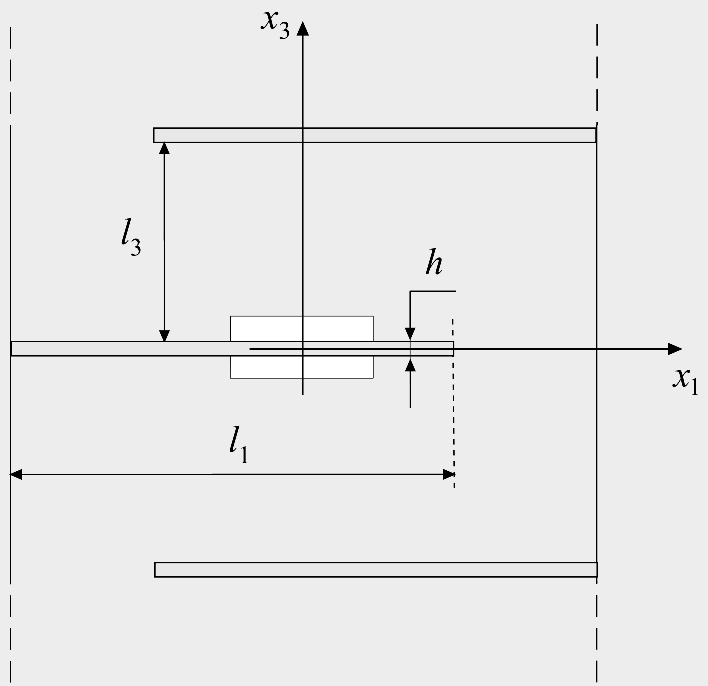

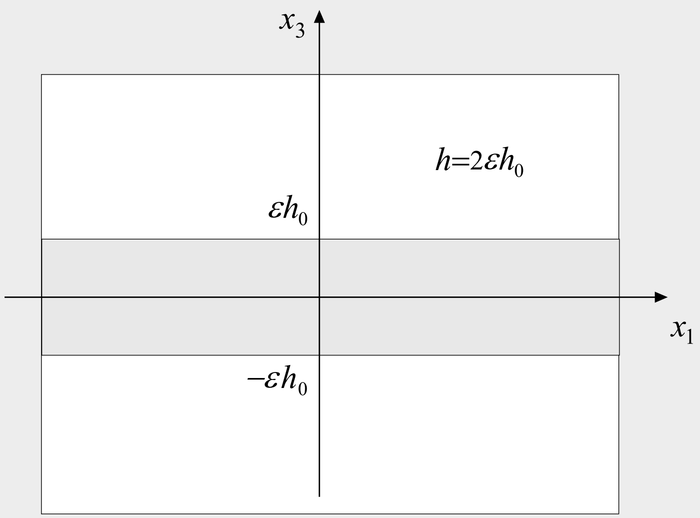

2. Assumption: $\underline{\boldsymbol{U}}_{C}$ is known in $U(\eta)$.

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Asymptotic procedure

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3. Splitting of the differential operator

$$\mathcal{D} = \begin{pmatrix} \partial_1 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_1 \end{pmatrix}^\top$$

into

$$\mathcal{A}_1^\top = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & 0 & \partial_1 \end{pmatrix}, \quad \underline{\underline{\boldsymbol{A}}}_0^\top = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{A}_0^\top = \underline{\underline{\boldsymbol{A}}}_0^\top \partial_\xi = \begin{pmatrix} 0 & 0 & \partial_\xi \\ 0 & \partial_\xi & 0 \end{pmatrix}.$$

Asymptotic procedure

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The differential operator $\mathcal{D}$ locally (in η) reads:

$$\mathcal{D} = \epsilon^{-1}\mathcal{A}_0 + \mathcal{A}_1 \,:\, \mathsf{H}^1(\eta) \to L_2(\eta)\,.$$

Asymptotic procedure

BOSCH

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The differential operator $\mathcal{D}$ locally (in η) reads:

$$\mathcal{D} = \epsilon^{-1} \mathcal{A}_0 + \mathcal{A}_1 : \mathrm{H}^1(\eta) \to L_2(\eta)\,.$$

4. Assumption: the solutions $\underline{\boldsymbol{u}}_C$, $\underline{\boldsymbol{u}}_M$, Φ_C of the PDE system, given in the ceramic and the metal domain can be written as asymptotic series within the neighbourhood of the electrode η:

$$\underline{\boldsymbol{u}}_C(x_1, x_3) = \sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{w}}_j(x_1, x_3), \quad \Phi_C(x_1, x_3) = \sum_{j=0}^{\infty} \epsilon^j \Phi_j(x_1, x_3),$$

$$\underline{\boldsymbol{u}}_\epsilon(x_1, x_2, \xi) = \sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{u}}_j(x_1, \xi),$$

Asymptotic procedure

5. Partial differential equation system (elasticity)

$$\left\{\mathcal{A}_0^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{A}_0 + \epsilon \left(\mathcal{A}_0^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{A}_1 + \mathcal{A}_1^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{A}_0\right) + \epsilon^2 \mathcal{A}_1^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{A}_1\right\} \sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{u}}_j = \underline{\boldsymbol{F}}_M \quad \text{in } \eta.$$

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Transmission conditions

$$\underline{\boldsymbol{u}}_\epsilon|_{\xi=\pm h_0} = \underline{\boldsymbol{u}}_C|_{x_3=\pm\epsilon h_0},$$

$$\sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{u}}_j|_{\xi=\pm h_0} = \sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{w}}_j|_{x_3=\pm\epsilon h_0},$$

$$\sigma_{M_n}(\underline{\boldsymbol{u}}_\epsilon)|_{\xi=\pm h_0} = \sigma_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C)|_{x_3=\pm\epsilon h_0}$$

$$\epsilon^{-1} \underline{\underline{\boldsymbol{A}}}_0^\top \underline{\underline{\boldsymbol{C}}}_M \{\mathcal{A}_0 + \epsilon\mathcal{A}_1\} \underline{\boldsymbol{u}}_\epsilon|_{\xi=\pm h_0} = \sigma_{C_n}\left(\sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{w}}_j, \sum_{j=0}^{\infty} \epsilon^j \Phi_j\right)|_{x_3=\pm\epsilon h_0}.$$

$\Rightarrow$ Series of limit problems ($\epsilon = 0$) with non-standard interface conditions.

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Example: Interface conditions in the first limit problem

$$[\underline{\boldsymbol{\sigma}}_n(\underline{\boldsymbol{u}}_C, \Phi_C)] = \underline{\boldsymbol{0}} \quad \text{on } \Gamma_M$$

$$[\underline{\boldsymbol{u}}_C] = \underline{\boldsymbol{0}} \quad \text{on } \Gamma_M$$

Asymptotic procedure

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Transmission conditions

$$\underline{\boldsymbol{u}}_\epsilon|_{\xi=\pm h_0} = \underline{\boldsymbol{u}}_C|_{x_3=\pm\epsilon h_0},$$

$$\sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{u}}_j|_{\xi=\pm h_0} = \sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{w}}_j|_{x_3=\pm\epsilon h_0},$$

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$$\epsilon^{-1} \underline{\underline{\boldsymbol{A}}}_0^\top \underline{\underline{\boldsymbol{C}}}_M \{\mathcal{A}_0 + \epsilon\mathcal{A}_1\} \underline{\boldsymbol{u}}_\epsilon|_{\xi=\pm h_0} = \sigma_{C_n}\left(\sum_{j=0}^{\infty} \epsilon^j \underline{\boldsymbol{w}}_j, \sum_{j=0}^{\infty} \epsilon^j \Phi_j\right)|_{x_3=\pm\epsilon h_0}.$$

$\Rightarrow$ Series of limit problems ($\epsilon = 0$) with non-standard interface conditions.

Example: Interface conditions in the second limit problem (here: $\underline{\underline{\boldsymbol{T}}} = \underline{\underline{\boldsymbol{T}}}\left(\underline{\underline{\boldsymbol{C}}}_M\right)$)

$$[\underline{\boldsymbol{w}}_1] = 2h_0 \underline{\underline{\boldsymbol{T}}}^{-1}\left(\underline{\boldsymbol{\sigma}}_{C_n}(\underline{\boldsymbol{w}}_0, \Phi_0) - \underline{\underline{\boldsymbol{A}}}_0^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{A}_1 \underline{\boldsymbol{w}}_0(x_1, x_2, 0)\right) - 2h_0 \langle \partial_3 \underline{\boldsymbol{w}}_0 \rangle,$$

$$[\underline{\boldsymbol{\sigma}}_{C_n}(\underline{\boldsymbol{w}}_1, \Phi_1)] = -2h_0 \langle \partial_3 \underline{\boldsymbol{\sigma}}_{C_n}(\underline{\boldsymbol{w}}_0, \Phi_0)\rangle - 2h_0 \mathcal{A}_1^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{A}_1 \underline{\boldsymbol{w}}_0(x_1, x_2, 0)$$
$$- 2h_0 \mathcal{A}_1^\top \underline{\underline{\boldsymbol{C}}}_M \underline{\underline{\boldsymbol{A}}}_0 \underline{\underline{\boldsymbol{T}}}_M^{-1}\left(\underline{\boldsymbol{\sigma}}_{C_n}(\underline{\boldsymbol{w}}_0, \Phi_0) - \underline{\underline{\boldsymbol{A}}}_0^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{A}_1 \underline{\boldsymbol{w}}_0(x_1, x_2, 0)\right)$$

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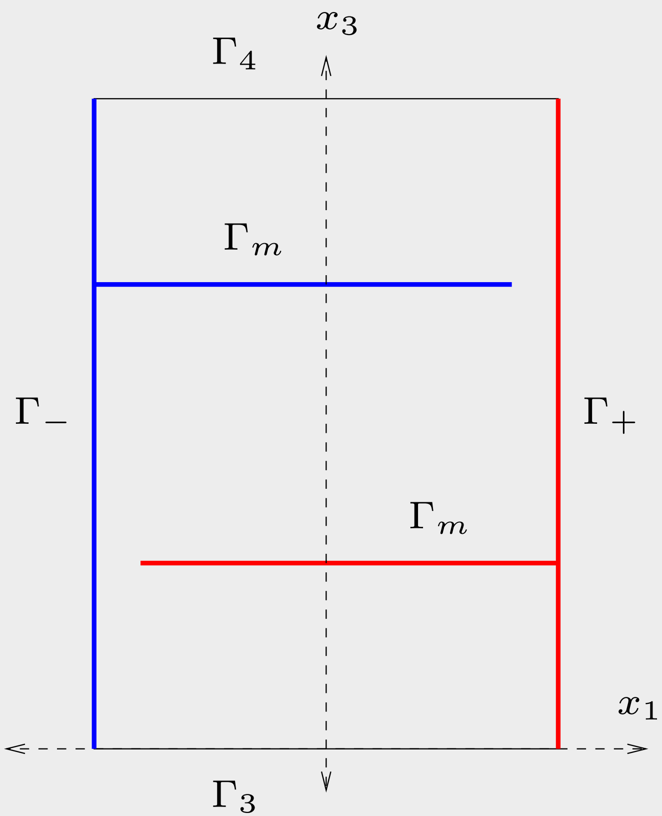

Model of a simple stack actuator with electrodes of thickness 0,

$\bar{\Omega} = \bar{\Omega}_C$

Linear Voigt Model (ceramic) for the simplified 2D model (plane strain)

$$\rho_C \omega^2 \underline{\boldsymbol{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\boldsymbol{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi = \underline{\boldsymbol{F}}_C^u \text{ in } \Omega_C$$

$$\operatorname{div}(\underline{\underline{\boldsymbol{e}}} \mathcal{D} \underline{\boldsymbol{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}} \nabla \Phi) = 0 \quad \text{in } \Omega_C$$

Boundary conditions

$$\boldsymbol{\sigma}_n(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\mathbf{0}} \quad \text{on } \partial\Omega \setminus \Gamma_3$$

$$\underline{\boldsymbol{u}}_C = \underline{\mathbf{0}} \quad \text{on } \Gamma_3$$

$$D_n(\underline{\boldsymbol{u}}_C, \Phi_C) = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_C \setminus \Gamma_\pm$$

$$\Phi = \pm \Phi_a \quad \text{on } \Gamma_\pm \cup \Gamma_m$$

Transmission conditions on Γ_m:

$$[\underline{\boldsymbol{u}}_C] = \underline{\mathbf{0}}, \qquad [\boldsymbol{\sigma}_n(\underline{\boldsymbol{u}}_C, \Phi_C)] = \underline{\mathbf{0}}$$

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Weak formulation of the boundary-transmission problem in the composite and the simplified model

Appropriate Sobolev spaces:

$$\mathcal{V} := \left\{ \underline{\boldsymbol{V}} = \left(\begin{smallmatrix} \underline{\boldsymbol{v}} \\ \Psi \end{smallmatrix}\right) \in \left[\mathrm{H}^1(\Omega)\right]^3, r|_{\Gamma_3}\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \text{ and } r|_{\Gamma\cup\Gamma^\pm}\Phi = 0 \right\}$$

$$\tilde{\mathcal{V}} := \left\{ \underline{\boldsymbol{V}} = \left(\begin{smallmatrix} \underline{\boldsymbol{v}} \\ \Psi \end{smallmatrix}\right) \in \left[\mathrm{H}^1(\Omega\backslash\Gamma_m)\right]^3, r|_{\Gamma_3}\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \text{ and } r|_{\Gamma_m\cup\Gamma^\pm}\Phi = 0 \right\}$$

Bilinear form:

$$a\left(\underline{\boldsymbol{U}}_0, \underline{\boldsymbol{V}}\right) := -\rho\omega^2 \int_\Omega \underline{\boldsymbol{u}}_0 \cdot \underline{\boldsymbol{v}}\, \mathrm{d}\underline{\boldsymbol{x}} + \int_\Omega \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{U}}_0 \cdot \underline{\underline{\boldsymbol{B}}}\,\underline{\boldsymbol{V}}\, \mathrm{d}\underline{\boldsymbol{x}}$$

$$= -\rho\omega^2 \int_\Omega \underline{\boldsymbol{u}}_0 \cdot \underline{\boldsymbol{v}}\, \mathrm{d}\underline{\boldsymbol{x}} + \int_\Omega \begin{pmatrix} \underline{\underline{\boldsymbol{C}}} & -\underline{\underline{\boldsymbol{e}}}^\top \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix} \begin{pmatrix} \gamma\left(\underline{\boldsymbol{u}}_0\right) \\ -\nabla\Phi \end{pmatrix} : \begin{pmatrix} \gamma\left(\underline{\boldsymbol{v}}\right) \\ -\nabla\Psi \end{pmatrix}$$

Linear form:

$$f\left(\underline{\boldsymbol{V}}\right) := \int_\Omega \underline{\boldsymbol{F}}\,(T) \cdot \underline{\boldsymbol{V}}\, \mathrm{d}\underline{\boldsymbol{x}}$$

Existence and uniqueness

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Transformation to homogeneous Dirichlet data: $\underline{\boldsymbol{U}} = \left(\begin{smallmatrix} \underline{\boldsymbol{u}} \\ \Phi \end{smallmatrix}\right) = \underline{\boldsymbol{U}}_0 - \underline{\boldsymbol{W}}$, such that $\underline{\boldsymbol{U}}_0 \in \mathcal{V}$.

Resulting weak formulation:

$$a(\underline{\boldsymbol{U}}_0, \underline{\boldsymbol{V}}) = a(\underline{\boldsymbol{W}}, \underline{\boldsymbol{V}}) + f(\underline{\boldsymbol{V}}) \tag{1}$$

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Existence and uniqueness

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Theorem.

The weak formulated multifield problem (1) in the composite and the simplified problem have unique solutions.

Existence and uniqueness

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Two different approaches, to show existence and uniqueness of weak solutions:

1. Use the Fredholm alternative (e.g. Mercier/Nicaise, 2005): *"There exists a discrete set S_0 (spectrum) such that for $\omega^2 \notin S_0$, the problem (1) has a unique solution for any right hand side F."*

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   - \+ General result
   - \- S_0 is not known explicitly
2. Choose appropriate Sobolev spaces $\mathcal{V}$ and $\tilde{\mathcal{V}}$ and prove the conditions of the Lax-Milgram lemma (ellipticity and continuity).
   - \- No general result (only valid for ω below the first eigenfrequency)
   - \+ The proof makes use of Korn's constant (dependent on the geometry of Ω), which gives a hint to the location of the first eigenfrequency.

Sketch of proof.

Ellipticity

$$a\left(\underline{\boldsymbol{U}}, \underline{\boldsymbol{U}}\right) = -\int_\Omega \rho\omega^2 \underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{u}} \, \mathrm{d}\underline{\boldsymbol{x}} + \int_\Omega \begin{pmatrix} \underline{\boldsymbol{\gamma}} \\ \underline{\boldsymbol{E}} \end{pmatrix}^\top \begin{pmatrix} \underline{\underline{\boldsymbol{C}}} & -\underline{\underline{\boldsymbol{e}}}^\top \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{\gamma}} \\ \underline{\boldsymbol{E}} \end{pmatrix} \mathrm{d}\underline{\boldsymbol{x}}$$

$$= -\rho\omega^2 \left\|\underline{\boldsymbol{u}}\right\|_{[L_2(\Omega)]^2} \int_\Omega \underline{\boldsymbol{\gamma}}\, \underline{\underline{\boldsymbol{C}}}\, \underline{\boldsymbol{\gamma}} + \underline{\boldsymbol{E}}^\top \underline{\underline{\boldsymbol{\epsilon}}}\, \underline{\boldsymbol{E}} \, \mathrm{d}\underline{\boldsymbol{x}}$$

Existence and uniqueness

Mechanical part

$$-\rho\omega^2 \left\|\underline{\boldsymbol{u}}\right\|^2_{[L_2(\Omega)]^2} + \int_\Omega \gamma^\top \underline{\underline{\boldsymbol{C}}}\,\gamma \,\mathrm{d}\underline{\boldsymbol{x}} \geq C_0 \left\|\gamma\right\|^2_{[L_2(\Omega)]^3} - \rho\omega^2 \left\|\underline{\boldsymbol{u}}\right\|^2_{[L_2(\Omega)]^2}$$

$$\overset{\text{Korn}}{\geq} C_{0,\text{Korn}}(C_0, \Omega, \Gamma_M^D) \left\|\underline{\boldsymbol{u}}\right\|^2_{[L_2(\Omega)]^2} - \rho\omega^2 \left\|\underline{\boldsymbol{u}}\right\|^2_{[L_2(\Omega)]^2}$$

$$\geq \tilde{C}_0 \left\|\underline{\boldsymbol{u}}\right\|^2_{\tilde{\mathcal{V}}},$$

with $\tilde{C}_0 > 0$ for $C_{0,\text{Korn}} > \rho\omega^2$ and ω small.

Existence and uniqueness

Mechanical part

$$-\rho\omega^2 \|\underline{\boldsymbol{u}}\|^2_{[L_2(\Omega)]^2} + \int_\Omega \gamma^\top \underline{\underline{\boldsymbol{C}}}\, \gamma \,\mathrm{d}\underline{\boldsymbol{x}} \geq C_0 \|\gamma\|^2_{[L_2(\Omega)]^3} - \rho\omega^2 \|\underline{\boldsymbol{u}}\|^2_{[L_2(\Omega)]^2}$$

$$\overset{\text{Korn}}{\geq} C_{0,\text{Korn}}(C_0, \Omega, \Gamma_M^D) \|\underline{\boldsymbol{u}}\|^2_{[L_2(\Omega)]^2} - \rho\omega^2 \|\underline{\boldsymbol{u}}\|^2_{[L_2(\Omega)]^2}$$

$$\geq \tilde{C}_0 \|\underline{\boldsymbol{u}}\|^2_{\tilde{\mathcal{V}}} ,$$

with $\tilde{C}_0 > 0$ for $C_{0,\text{Korn}} > \rho\omega^2$ and ω small.

Electrical part

$$\int_\Omega \underline{\boldsymbol{E}}^\top \underline{\underline{\boldsymbol{\varepsilon}}}\, \underline{\boldsymbol{E}} \,\mathrm{d}\underline{\boldsymbol{x}} \geq \varepsilon_0 \int_\Omega \nabla\Phi\nabla\Phi \,\mathrm{d}\underline{\boldsymbol{x}}$$

$$\overset{\text{Friedrichs}}{\geq} \varepsilon_{0,\text{Friedrichs}}(\varepsilon_0, \Omega, \Gamma_e^D) \|\Phi\|^2_{\mathcal{V}} .$$

BOSCH

Resulting block-LES:

$$\begin{pmatrix} \underline{\underline{\boldsymbol{C}}} & -\underline{\underline{\boldsymbol{E}}}^\top \\ \underline{\underline{\boldsymbol{E}}} & \underline{\underline{\boldsymbol{EPS}}} \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{U}} \\ \underline{\boldsymbol{\Phi}} \end{pmatrix} = \begin{pmatrix} \underline{\boldsymbol{F}}_1 \\ \underline{\boldsymbol{F}}_2 \end{pmatrix} \tag{2}$$

The skew-symmetric block-system (2) is solved with the Bramble Pasciak CG (BPCG) (see e.g. O. Steinbach: Numerische Näherungsverfahren für elliptische Randwertprobleme)

Steady oscillation case

BOSCH

- ➜ For small frequencies, we can neglect the term $\rho\omega^2 \underline{\boldsymbol{u}}$:
- $\Rightarrow$ Stationary boundary-transmission-problem
- ➜ For large frequencies, the term $\rho\omega^2 \underline{\boldsymbol{u}}$ should be taken into account.

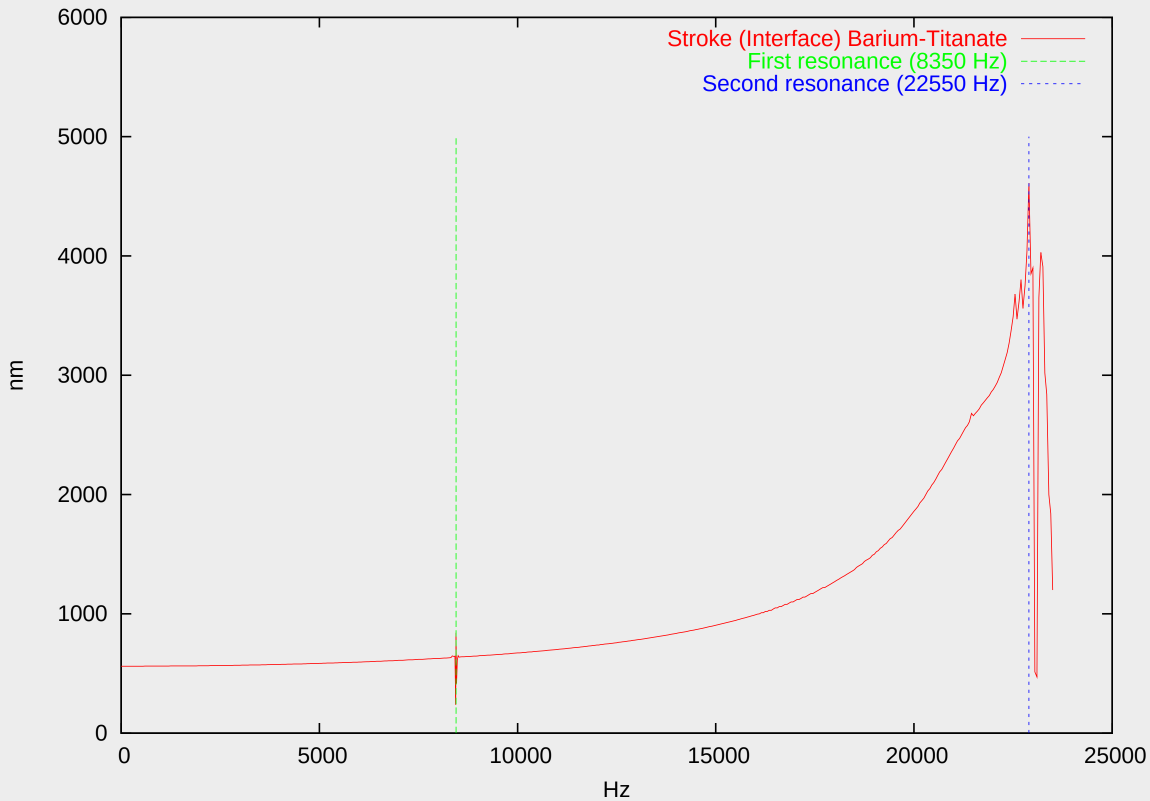
What are "small frequencies"?

Steady oscillation case

- ➜ For small frequencies, we can neglect the term $\rho\omega^2\underline{\boldsymbol{u}}$:
- ⇒ Stationary boundary-transmission-problem
- ➜ For large frequencies, the term $\rho\omega^2\underline{\boldsymbol{u}}$ should be taken into account.

What are "small frequencies"?

Barium-Titanate



Winfried Geis

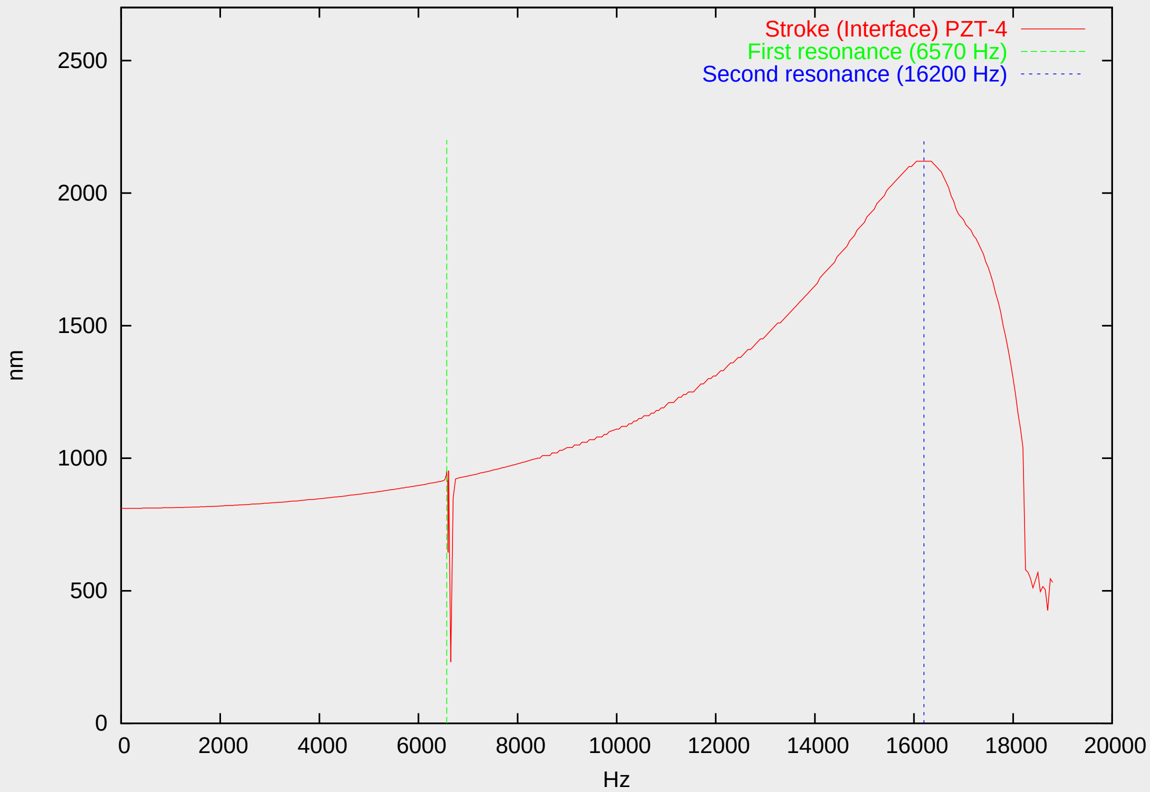
Steady oscillation case

BOSCH

- ➜ For small frequencies, we can neglect the term $\rho\omega^2\underline{\boldsymbol{u}}$:
- $\Rightarrow$ Stationary boundary-transmission-problem
- ➜ For large frequencies, the term $\rho\omega^2\underline{\boldsymbol{u}}$ should be taken into account.

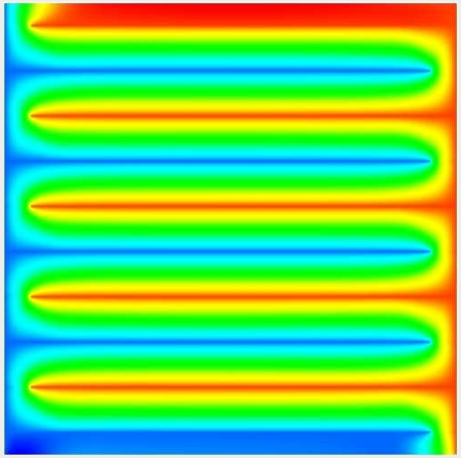
What are "small frequencies"?

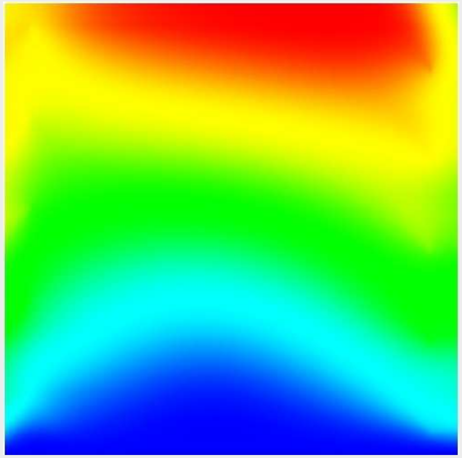
PZT-4



Static case, reference temperature 20°C

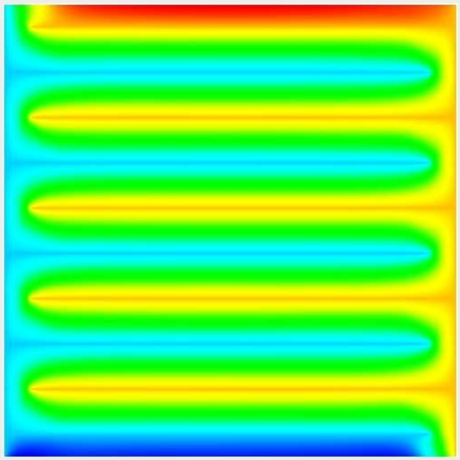
Stack at 20°C

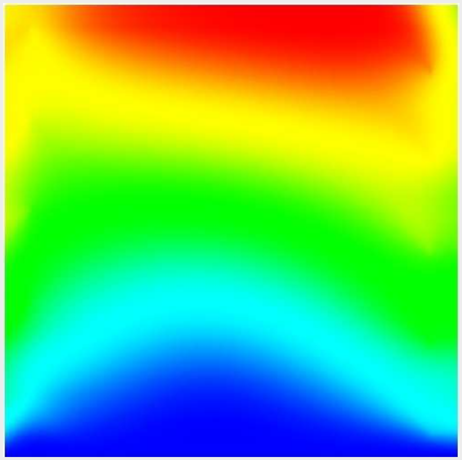




PZT 4, 20°C	
potential [kV]	[-0.256,0.243]
stroke [mm]	0.000828

Stack after heating at 30°C



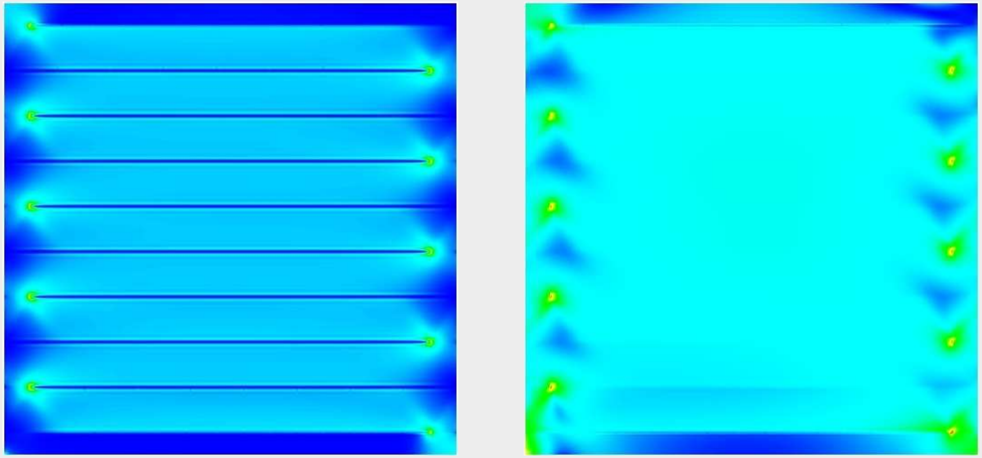


PZT 4, 30°C	
potential [kV]	[-0.311,0.303]
stroke [mm]	0.000893

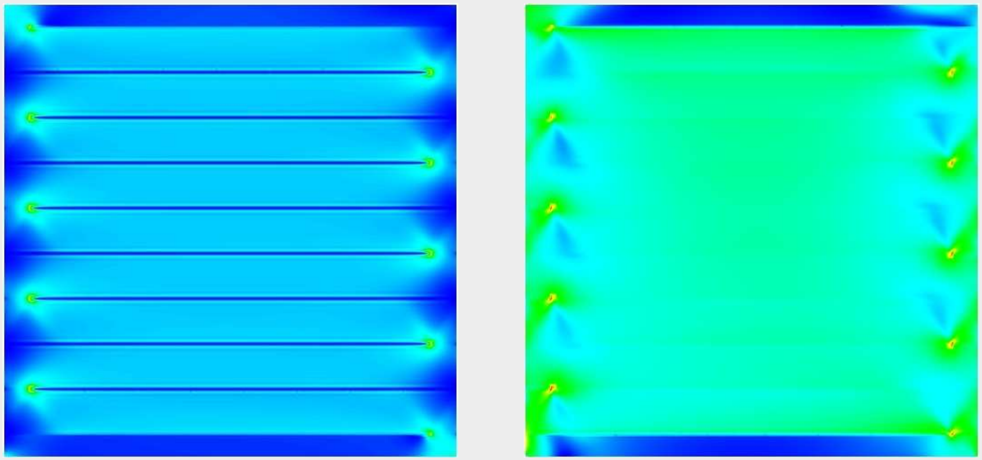
Winfried Geis

Static case, reference temperature $20°$C

Stack at $20°$C



Stack after heating at $30°$C



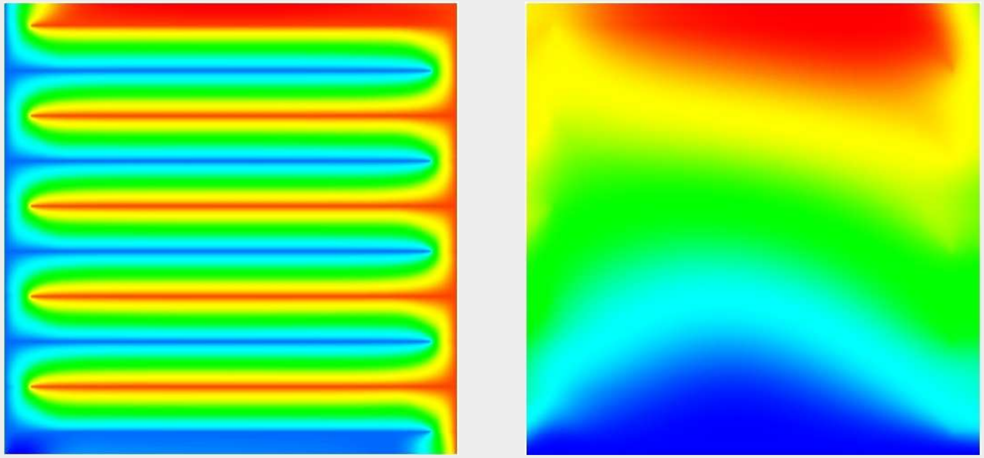
PZT 4, $20°$C	
iterations	435
nodes	29857

PZT 4, $30°$C	
iterations	429
nodes	29857

Static case, reference temperature $200°$C

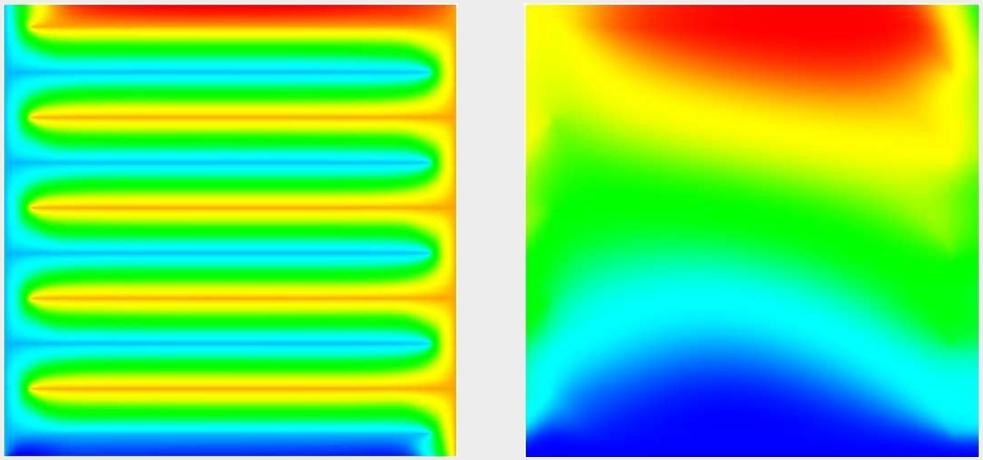
BOSCH

Stack at $200°$C



PZT 4, $200°$C	
potential [kV]	[-0.256,0.243]
stroke [mm]	0.000828

Stack after heating at $210°$C

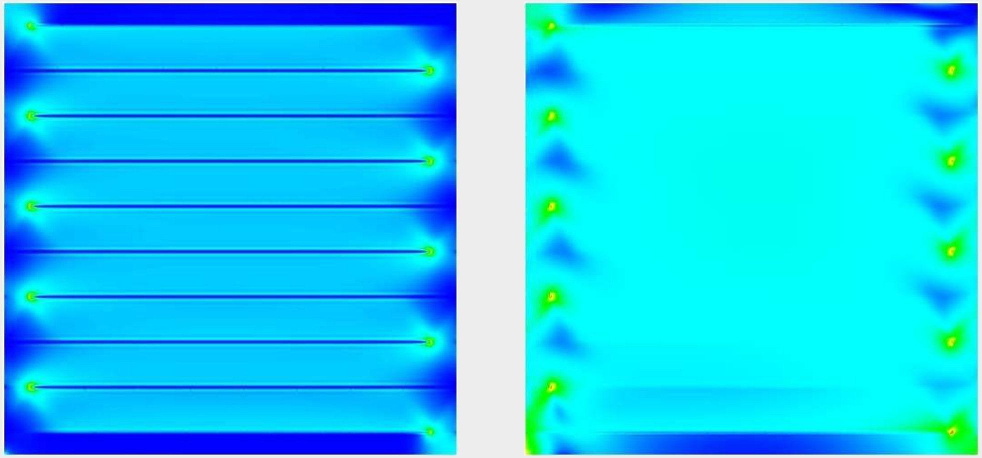


PZT 4, $210°$C	
potential [kV]	[-0.296,0.287]
stroke [mm]	0.000713

Winfried Geis

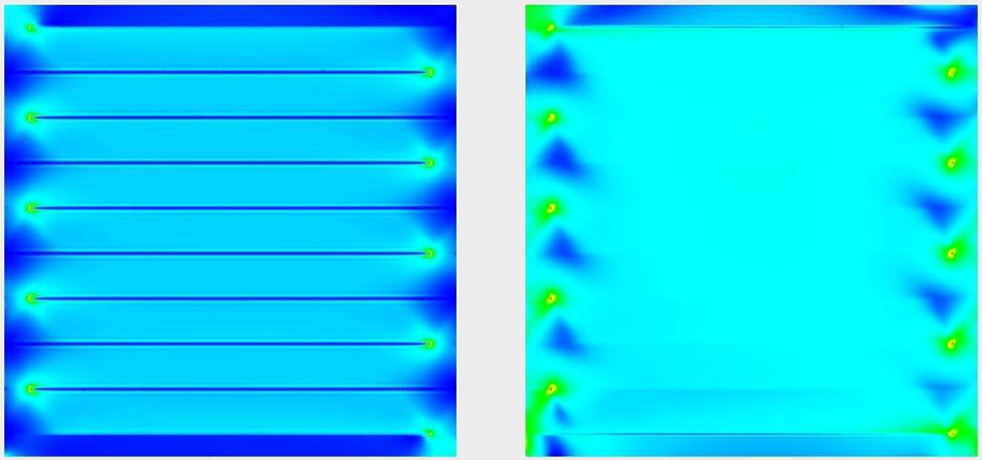
Static case, reference temperature $200°$C

Stack at 200°C



PZT 4, 200°C	
iterations	435
nodes	29857

Stack after heating at 210°C



PZT 4, 210°C	
iterations	434
nodes	29857

Conclusion and future prospects

Conclusion

1. The linear Voigt model for the composite has a uniquely defined weak solution $\underline{\boldsymbol{U}} \in \mathcal{V}$
2. The linear Voigt model for the simplified asymptotic problem has a uniquely defined weak solution $\underline{\boldsymbol{U}} \in \tilde{\mathcal{V}}$.
3. The 2D mechanical and electric fields can be computed by FEM with a Bramble Pasciak Conjugated Gradient (BPCG) solver.
4. Numerical experiments confirm, that the simplified model gives a sufficiently exact solution. It can be calculated more efficient than the full problem (factor 10).
5. The static model is applicable for "small exciting frequencies".
6. The given temperature field has a great influence on the expansion of the stack actuator.

Future Prospects

1. Computation of stress singularities in the electrode tips of the stack actuator.
2. Derivation and computation of a local failure criterion to reflect the damage.