Discontinuous Galerkin Methods for Variational Inequalities Arising in Elastoplasticity

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Abstract
Discontinuous Galerkin (DG) methods have achieved some measure of popularity in recent years, and have been applied successfully to a wide range of problems in mechanics and other areas (see, for example, [1]). They are particularly attractive when applied to problems in which derivatives of high order appear, and carry significant advantages in situations involving adaptive mesh refinement.

The aim of this presentation is to discuss a DG method for a time-dependent variational inequality of the second kind, that is, one in which the inequality arises as a result of the presence of a non-differentiable functional. The problem takes the form of finding $w(\cdot,t) \in Z$ which satisfies

$$a(w, z - \dot{w}) + j(z) - j(\dot{w}) - \ell(z - \dot{w}) \geq 0 \quad \text{for all } z \in Z \ . \tag{1}$$

Here $a(\cdot,\cdot)$ is a bilinear form on a Banach space Z, $\ell(\cdot)$ is a bounded linear functional, and $j(\cdot)$ is a convex, positively homogenous non-differentiable functional. A superposed dot denotes a time derivative, in a suitable weak sense.

In previous work [2] this problem has been studied in detail, and results on well-posedness and on convergence of finite element approximations have been obtained, both for the abstract problem and for a problem of elastoplasticity, which is a special case of (1).

Here a fully discrete approximation of (1) is studied, in which time discretization is carried out using a conventional finite difference approximation and spatial approximation is achieved through an interior penalty DG finite element formulation. The emphasis is on the problem in classical elastoplasticity. Well-posedness of the discrete problem is established, and it is shown that approximations converge at the optimal rate. It is also shown that the conventional predictor-corrector algorithms converge for suitable choices of predictor.

In order to model behaviour appropriately at microscopic scales it is essential that the classical theory of plasticity be extended to include strain gradients [3]. The presence of these high-order gradients again suggests DG methods as an appropriate vehicle through which to construct finite element approximations. It is shown here in the context of a simple gradient plasticity formulation how the DG formulation developed for the classical theory may be extended to incorporate this more complex physical theory. A selection of numerical results is presented to illustrate the theoretical results obtained.

References

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