Risk averse dynamic optimization

Progress in continuous time
Linz

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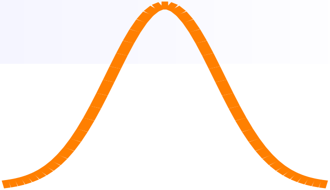
[1] Mathematik! TU Chemnitz

Stochastic optimization
[Markowitz, 1952]



Primal

$$\begin{aligned} \text{minimize} \quad & \operatorname{var}\left(x^\top \xi\right) \\ \text{subject to} \quad & x \in \mathbb{R}^d, \\ & \mathbb{E} x^\top \xi \geq \mu, \\ & \sum_{i=1}^d x_i = 1 \\ & (x_i \geq 0) \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & \mathbb{E} x^\top \xi \\ \text{subject to} \quad & x \in \mathbb{R}^d, \\ & \operatorname{var}\left(x^\top \xi\right) \leq q, \\ & \sum_{i=1}^d x_i = 1 \\ & (x_i \geq 0) \end{aligned}$$

Assessment of risk

Proposition (Axioms, cf. [Deprez and Gerber, 1985], [Artzner et al., 1999])

$\mathcal{R}\colon \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$

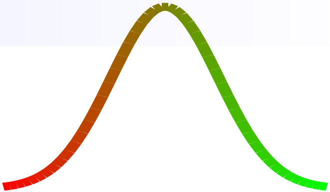
1. *Monotonicity: if* $Y \le Y'$, *then* $\mathcal{R}(Y) \le \mathcal{R}(Y')$,
2. *Subadditivity:* $\mathcal{R}(Y + Y') \le \mathcal{R}(Y) + \mathcal{R}(Y')$,
3. *Translation equivariance:* $\mathcal{R}(Y + c) = \mathcal{R}(Y) + c$ *for* $Y \in \mathcal{Y}$ *and* $c \in \mathbb{R}$,
4. *Positive homogeneity,* $\mathcal{R}(\lambda Y) = \lambda \cdot \mathcal{R}(Y)$ *for* $\lambda > 0$.

Equivalence principle

$$\mathcal{R}(Y) := \mathbb{E}\, Y \qquad \text{most fair, risk neutral}$$
$$\mathcal{R}(Y) := \operatorname{ess\,sup} Y \qquad \text{most unfair, totally risk averse.}$$

Reformulation

[Markowitz, 1952]



Primal

$$\text{maximize} \quad \mathbb{E}x^\top\xi$$

$$\text{s.t.} \quad -\mathcal{R}\left(-x^\top\xi\right) \leq q,$$

$$\sum_{i=1}^{d} x_i = 1$$

$$(\text{and } x_i \geq 0).$$

Dual

$$\text{minimize} \quad \mathbb{E}x^\top\xi + \mathcal{R}\left(-x^\top\xi\right) =: \mathcal{D}\left(-x^\top\xi\right)$$

$$\text{s.t.} \quad \mathbb{E}x^\top\xi \geq \mu,$$

$$\sum_{i=1}^{d} x_i = 1$$

$$(\text{and } x_i \geq 0).$$

Outline

1. **The discrete setting**
   - The general multistage problem
   - Dynamic programming
2. **Continuous time**
   - Generators
   - Risk generator
3. **Spanning horizons**
   - Nested Expressions
   - Explicit definition
4. **Hamilton Jacobi Bellman**
   - Hamilton Jacobi
   - Further assessments of risk
   - Applications
5. **References**

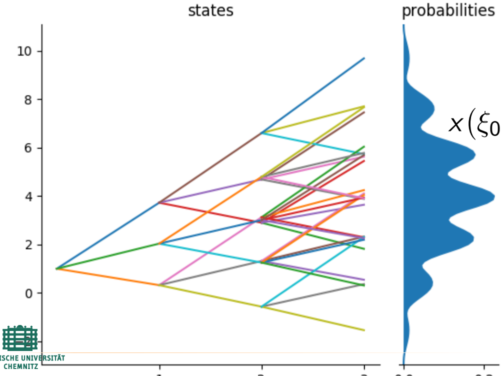
Outline

Multistage problem
Non-Markovian difficulties

Multistage optimization

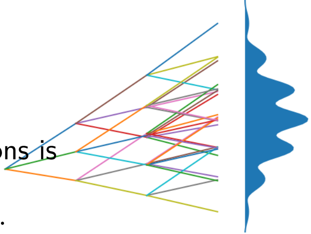
$$\begin{aligned} \text{minimize } & \mathbb{E}\, c(\xi, x(\xi)) \\ \text{subject to } & x \in \mathbb{X}, \\ & x \text{ nonanticipative} \end{aligned}$$

$x(\cdot)$ is adapted (nonanticipative) iff



$$x(\xi_0, \dots, \xi_T) = \begin{pmatrix} x_0(\xi_0) \\ x_1(\xi_0, \xi_1) \\ \vdots \\ x_t(\xi_0, \dots, \xi_t) \\ \vdots \\ x_T(\xi_0, \xi_1, \dots, \xi_T) \end{pmatrix}$$

Problem description
Discrete time



In a discrete framework, the sequence of decisions is

$$x_0 \rightsquigarrow \xi_1 \rightsquigarrow x_1 \cdots \rightsquigarrow \xi_T \rightsquigarrow x_T.$$

$$\inf\left\{\mathbb{E}\, c\big(\xi, x(\xi)\big) : x(\cdot) \in \mathbb{X},\ x(\cdot) \text{ adapted}\right\}$$

Problem (Risk aversion)

The risk averse stochastic problem is

$$\text{minimize } \mathcal{R}\Big(c_0(x_0), c_1(\xi, x_1), \dots, c_T(\xi, x_T)\Big)$$
$$x_0 \in \mathcal{X}_0, \dots, x_t \in \mathcal{X}_t(x_{t-1}, \xi)$$

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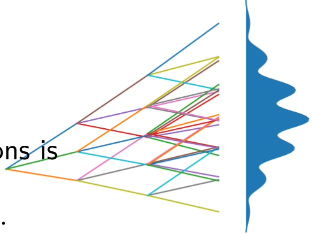

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Risk

Example

In the simplest case,

$$\mathcal{R}\Big(c_0(\xi, x_0(\xi)), \dots, c_T(\xi, x_T(\xi))\Big) = \mathbb{E} \sum_{t=0}^{T} c_t(\xi, x_t(\xi)).$$

Problem

$$\inf\Big\{\mathcal{R}\Big(c_0(\xi, x_0(\xi)), \dots, c_T(\xi, x_T(\xi))\Big) \colon x \in \mathbb{X},\ x(\cdot) \text{ adapted}\Big\}.$$

Outline

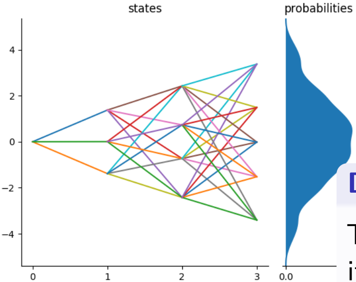


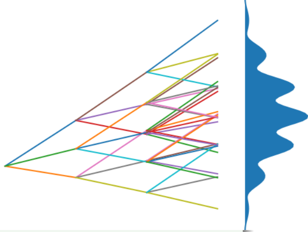
Figure: Lattice approximation

$$\begin{aligned}\min\quad & \mathcal{R}\big(c_0(x_0), c_1(\xi, x_1), \dots, c_T(\xi, x_T)\big),\\ \text{s.t.}\quad & x_0 \in \mathcal{X}_0,\ x_t \in \mathcal{X}_t(x_{t-1}, \xi),\\ & \qquad t = 1, \dots, T.\end{aligned}$$

Definition (Time consistent)

The *transition* functionals are *recursive,* if

$$\mathcal{R}_{t,u}(Y_t, \dots, Y_u) = \mathcal{R}_{t,v}\big(Y_t, \dots, Y_{v-1}, \mathcal{R}_{v,u}(Y_v, \dots, Y_u)\big).$$



Conditional risk functionals

- Semideviation $\quad \beta \lhd \mathcal{F}_t$

$$SD(Y \mid \mathcal{F}_t) := \mathbb{E}[Y \mid \mathcal{F}_t] + \beta \cdot \mathbb{E}\left[(Y - \mathbb{E}[Y \mid \mathcal{F}_t])_+ \mid \mathcal{F}_t \right],$$

- Average Value-at-Risk $\quad \alpha \lhd \mathcal{F}_t$

$$\mathrm{AV@R}_\alpha(Y \mid \mathcal{F}_t) := \operatorname*{ess\,inf}_{q \lhd \mathcal{F}_t} q + \frac{1}{1-\alpha} \mathbb{E}\left[(Y - q)_+ \mid \mathcal{F}_t \right],$$

- Entropic Value-at-Risk $\quad \alpha \lhd \mathcal{F}_t$

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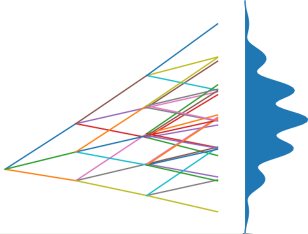
Towards dynamic programming
Examples



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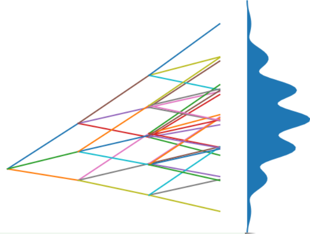

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Dynamic programming equations

Proposition (Bellman equations, recursive transitions [2018])

$$V_T(\xi, x_{T-1}) := \operatorname*{ess\,inf}_{x_T \in \mathcal{X}_Z(x_{T-1},\xi)} c_T(\xi, x_T),$$

$$V_t(\xi, x_{t-1}) := \operatorname*{ess\,inf}_{x_t \in \mathcal{X}_t(\xi, x_{t-1})} \mathcal{R}_{t:t+1}\big(c_t(\xi, x_t),\, V_{t+1}(\xi, x_t)\big).$$

V_0 solves the problem

$$\begin{aligned} \text{minimize} \quad & \mathcal{R}\Big(c_0(x_0), c_1(\xi, x_1), \dots, c_T(\xi, x_T)\Big), \\ \text{subject to} \quad & x_0 \in \mathcal{X}_0,\; x_t \in \mathcal{X}_t(x_{t-1}, \xi), \\ & t = 1, \dots, T. \end{aligned}$$

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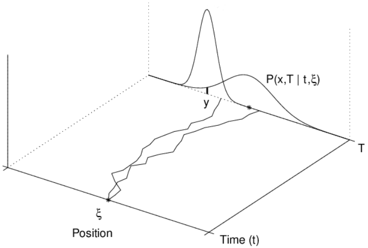
Outline



The motion is generated by

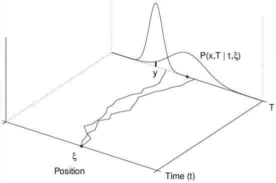
$$\mathrm{d}X_t = b\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

Definition (Generator)

For a smooth function ϕ,

$$\mathcal{G}\phi(t,\xi) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\left[\begin{array}{c} \phi(t+\Delta t, X_{t+\Delta t}) \\ -\phi(t,\xi) \end{array} \middle| X_t = \xi \right].$$

Ito's formula



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Lemma (Ito)

For $\mathrm{d}X_t = b\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$ *it holds that*

1. *For* $\phi = 1$, $\mathcal{G}\phi = 0$;
2. *for* $\phi(\xi) = \xi$, *then* $\mathcal{G}\phi = b$, *the drift;*
3. *for* $\phi(\xi) = \xi^2$, *then* $\mathcal{G}\phi = 2b\xi + \sigma^2$, *the volatility;*
4. *for general* $\phi(\xi)$,

$$\mathcal{G} = \frac{\partial}{\partial t} + b\frac{\partial}{\partial \xi} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial \xi^2}.$$

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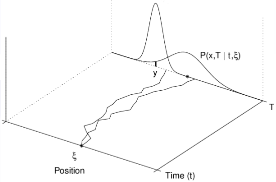

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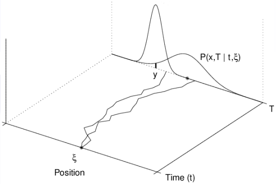

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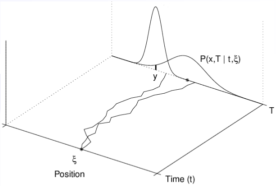

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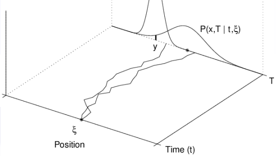
Outline

Problems under uncertainty



Problem

$$\mathcal{G}\phi(t,\xi) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathrm{AV@R}_\alpha \left[\begin{array}{c} \phi(t+\Delta t, X_{t+\Delta t}) \\ -\phi(t,\xi) \end{array} \middle| X_t = \xi \right]$$

$$= \infty \qquad : \text{ degenerate}$$

Remark

For $Y \sim \mathcal{N}(\mu, \Delta t)$,

$$\mathrm{AV@R}_\alpha(Y) = \mu + \sqrt{\Delta t} \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}.$$

Escape

$$\alpha(\Delta t) = \Phi\left(-\sqrt{-\log \alpha \cdot \Delta t}\right) \sim \sqrt{\frac{\Delta t}{\log \Delta t}}.$$

Definition (Semi-deviation)

Consider the semi-deviation

$$\mathrm{SD}_{p,\beta}(Y) := \mathbb{E}\, Y + \sqrt{\beta} \cdot \left\| (Y - \mathbb{E}\, Y)_+ \right\|_p$$

for $\beta \in (0,1)$, then, for $Y \sim \mathcal{N}(\mu, \sigma^2)$,

$$\mathrm{SD}_{p,\beta}(Y) = \mu + \sigma\sqrt{\beta} \cdot \frac{\sqrt{2}}{(2\sqrt{\pi})^{\frac{1}{p}}} \Gamma\left(\frac{1+p}{2}\right)^{\frac{1}{p}};$$

$$\mathrm{SD}_{1,\beta}(Y) = \mu + \sigma\sqrt{\beta} \cdot \frac{1}{\sqrt{2\pi}}.$$

TECHNISCHE UNIVERSITÄT CHEMNITZ

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Proposition

The semi-deviation generator is

$$\mathcal{G}\phi(t,\xi) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathsf{SD}_{\beta \cdot \Delta t} \left[\begin{array}{c} \phi(t+\Delta t, X_{t+\Delta t}) \\ -\phi(t,\xi) \end{array} \middle| X_t = \xi \right]$$

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$$\mathcal{G}\phi(t,\xi) = \frac{\partial}{\partial t}\phi + b\frac{\partial}{\partial \xi}\phi + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial \xi^2}\phi + \tilde{\beta} \cdot \left| \sigma \cdot \frac{\partial}{\partial \xi}\phi \right|,$$

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Definition (Risk generator)

The entropic generator is

$$\mathcal{G}\phi(t,\xi) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathsf{EV@R}_{\beta \cdot \Delta t}\left[\begin{array}{c} \phi(t+\Delta t, X_{t+\Delta t}) \\ -\phi(t,\xi) \end{array} \middle| X_t = \xi \right].$$

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is not linear any longer.

Entropic generator
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Outline

Problem (Journal of Indian Mathematical Society)

What is the nested expression

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\dots}}}}=?$$

Figure: Srinivasa Ramanujan, 1887–1920: the man who knew infinity

Dynamic nestings
Risk averse

Definition (by recursivity)

$$\mathcal{R}^{t_i:t_n}\left(Y \,\middle|\, \mathcal{F}_{t_i}\right) := \mathcal{R}^{t_i}\left(\mathcal{R}^{t_{i+1}} \cdots \left(\mathcal{R}^{t_{n-1}}\left(Y \,\middle|\, \mathcal{F}_{t_{n-1}}\right) \cdots \middle|\, \mathcal{F}_{t_{i+1}}\right) \middle|\, \mathcal{F}_{t_i}\right).$$

Tower property of the Expectation:

$$\mathbb{E}\, Y = \mathbb{E}\left[\mathbb{E}[Y \mid \mathcal{F}_t]\right].$$

Proposition

For $Y_T = Y_{t_i} + \sum_{j=i}^{n-1} \Delta Y_{t_j}$ and $\Delta Y_t \lhd \mathcal{F}_t$ it holds that

$$\begin{aligned}\mathcal{R}^{t_0:T}&\left(Y_T \,\middle|\, \mathcal{F}_{t_i}\right)\\ &:= Y_{t_0} + \mathcal{R}^{t_0}\left(\Delta Y_{t_0} \cdots + \mathcal{R}^{t_{n-1}}\left(\Delta Y_{t_{n-1}} \,\middle|\, \mathcal{F}_{t_{n-1}}\right) \cdots \middle|\, \mathcal{F}_{t_1} \,\middle|\, \mathcal{F}_{t_0}\right)\end{aligned}$$

Risk martingales

Lemma (Dual representation involves stochastic processes)

$$\mathsf{nEV@R}_\beta^{0:T}(Y) = \\ = \sup\left\{ \mathbb{E}[Y\,\mathbf{Z}_T] \;\middle|\; \begin{array}{l} \mathbb{E}\left[\mathbf{Z}_{t_i}\log \mathbf{Z}_{t_i} \,\middle|\, \mathcal{F}_{t_{i-1}}\right] \\ \qquad \leq \beta_{i-1}\cdot \Delta t_{i-1}\mathbf{Z}_{t_{i-1}} + \mathbf{Z}_{t_{i-1}}\log \mathbf{Z}_{t_{i-1}}, \\ \mathbb{E}\left[\mathbf{Z}_{t_i} \,\middle|\, \mathcal{F}_{t_{i-1}}\right] = \mathbf{Z}_{t_{i-1}},\; 0 \leq \mathbf{Z}_{t_i} \lhd \mathcal{F}_{t_i} \textit{ for all } i \end{array} \right\}$$

Proposition (Tower properties)

$$Y_t := \mathsf{nEV@R}_\beta(Y \mid \mathcal{F}_t)$$

is a risk martingale

$$Y_t = \mathsf{nEV@R}_\beta(Y_{t+1} \mid \mathcal{F}_t)$$

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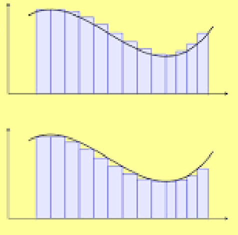
Outline

Extension to continuous time
Nested risk functionals



Definition

For $\beta(\cdot)$ Riemann integrable,

$$\mathrm{nEV@R}^{t:T}_{\beta(\cdot)}(Y \mid \mathcal{F}_t) := \operatorname*{ess\,inf}_{\tilde\beta(\cdot)\geq\beta(\cdot)} \mathrm{nEV@R}_{\widetilde{\beta}(\cdot)}(Y \mid \mathcal{F}_t),$$

where the infimum is among simple functions $\tilde\beta(\cdot) \geq \beta(\cdot)$.

Remark

For $Y_T = Y_{t_i} + \sum_{j=i}^{n-1} \Delta Y_{t_j}$ with $\Delta Y_{t_j} := \int_{t_{j-1}}^{t_j} c(\cdot)\mathrm{d}t$ it holds that

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Extension to continuous time
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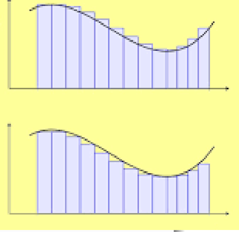

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Explicit evaluations for nestings

Proposition (Wiener process)

For the Wiener process, W_t,

$$\mathrm{nEV@R}_\beta(W_T) = T\sqrt{2\beta}, \text{ or}$$

$$\mathrm{nEV@R}_{\beta(\cdot)}(W_T) = \int_0^T \sqrt{2\beta(t)}\,\mathrm{d}t.$$

Proposition (Ornstein–Uhlenbeck)

The process

$$\mathrm{d}X_t = \theta(\mu - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t,$$

has

$$\mathrm{nEV@R}_\beta(X_T) = e^{-T\theta}x_0 + \mu(1-e^{-\theta T}) + \frac{\sigma\sqrt{2\beta}}{\theta}\left(1-e^{-\theta T}\right)$$

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Continuous time martingales

Dual involves stochastic processes in continuous time

Lemma (Dual representation involves stochastic processes)

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$$= \sup\left\{ \mathbb{E}[Y\,\mathbf{Z}_T] \;\middle|\; \begin{array}{l} \mathbb{E}[\mathbf{Z}_s \log \mathbf{Z}_s \,|\mathcal{F}_u] \\ \qquad \le \mathbf{Z}_u \int_u^s \beta(r)\,\mathrm{d}r + \mathbf{Z}_u \log \mathbf{Z}_u \\ \mathbb{E}[\mathbf{Z}_s \mid \mathcal{F}_u] = \mathbf{Z}_u, \\ \textit{for } t \le u \le s \le T \textit{ and } \mathbf{Z}_s \lhd \mathcal{F}_s \end{array} \right\}$$

Outline

Hamilton Jacobi Bellman equation
Optimal decisions

(a) William Rowan Hamilton, 1805 – 1865

(b) Carl Gustav Jacob Jacobi, 1804 – 1851

(c) Richard Bellman, 1920 – 1984

Hamilton Jacobi Bellman equation

Risk neutral — the classical situation

Proposition (Risk neutral)

The risk neutral value function

$$V(t,\xi) := \inf_{x(\cdot) \text{ adapted}} \mathbb{E}\left[\int_t^\infty c(s, X_s, x(s, X_s))\, \mathrm{d}s \,\middle|\, X_t = \xi\right]$$

satisfies the HJB equations

$$\frac{\partial}{\partial t} V = \mathcal{H}\left(t, \xi, \nabla_\xi V, \nabla_\xi^2 V\right)$$

with Hamiltonian

$$\mathcal{H}(t, \xi; g, H) := \sup_{x \in \mathbb{X}} \Big\{ \underbrace{-b(x) \cdot g - \frac{1}{2}\sigma(x)^2 \cdot H - c(x)}_{-\mathcal{G}} \Big\}.$$

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Hamilton Jacobi Bellman equation
Risk averse

Proposition (Risk averse)

The value function

$$V(t,\xi) := \inf_{x(\cdot)\ \text{adapted}} n\mathcal{R}\left(\int_t^\infty c(s, X_s, x(s,X_s))\,\mathrm{d}s \,\middle|\, X_t = \xi\right)$$

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Outline

TECHNISCHE UNIVERSITÄT CHEMNITZ

Consider

$$\mathcal{R}(Y) := u^{-1}\Big(\mathbb{E}\, u(Y)\Big).$$

Proposition

The generator is

$$\begin{aligned}\mathcal{G}\phi(t,\xi) &:= \lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\mathcal{R}\left(\begin{array}{c}\phi(t+\Delta t, X_{t+\Delta t})\\ -\phi(t,\xi)\end{array}\middle| X_t=\xi\right)\\ &= \left(\frac{\partial\Phi}{\partial t}+b\,\frac{\partial\Phi}{\partial\xi}+\frac{1}{2}\sigma^2\,\frac{\partial^2\Phi}{\partial\xi^2}\right)(t,\xi)\\ &\qquad +\frac{1}{2}\frac{u''(\Phi(t,\xi))}{u'(\Phi(t,\xi))}\left(\sigma(t,\xi)\frac{\partial\Phi}{\partial\xi}(t,\xi)\right)^2.\end{aligned}$$

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Special cases: exponential utility

The special case $u(x) = e^{\lambda x}$

The generator for

$$\mathcal{R}(Y) := \frac{1}{\lambda} \log \mathbb{E}\, e^{\lambda Y}$$

is

$$\mathcal{G}\phi(t,\xi) = \left(\frac{\partial \Phi}{\partial t} + b\, \frac{\partial \Phi}{\partial \xi} + \frac{1}{2}\sigma^2\, \frac{\partial^2 \Phi}{\partial \xi^2} \right)(t,\xi) + \frac{1}{2}\lambda \left| \sigma(t,\xi) \frac{\partial \Phi}{\partial \xi}(t,\xi) \right|^2 .$$

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Special cases: power utility

The special case $u(x) = x^\kappa$

The generator for

$$\mathcal{R}(Y) := \left(\mathbb{E}\, Y^\kappa\right)^{1/\kappa}$$

is

$$\mathcal{G}\phi(t,\xi) = \left(\frac{\partial \Phi}{\partial t} + b\,\frac{\partial \Phi}{\partial \xi} + \frac{1}{2}\sigma^2\,\frac{\partial^2 \Phi}{\partial \xi^2}\right)(t,\xi) + \frac{1}{2}\,\frac{(\kappa-1)\Phi^{\kappa-2}(t,\xi)}{\Phi^{\kappa-1}(t,\xi)} \cdot \left|\sigma(t,\xi)\,\frac{\partial \Phi}{\partial \xi}(t,\xi)\right|^2.$$

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Outline

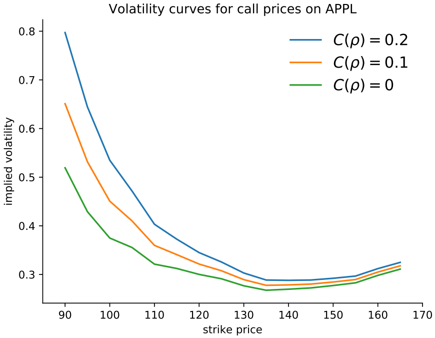
Differential equation

Optimal control

Lemma (Black and Scholes)

$$0 = \partial_t V + \frac{\sigma^2}{2}\partial_{xx} V + b\,\partial_x V - \beta\,|\sigma\cdot\partial_x V| - r\,V$$

$$V(T,x) = p(x)$$



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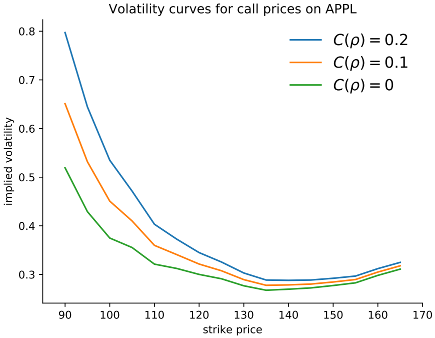

Surprise:
Explicit expression

The wealth process is

$$w_t := (1-\pi_t)B_t + \pi_t S_t,$$

with

$$V(t,w) := \max_{\pi_t, c_t} n\mathcal{R}\left[\int_t^T e^{-\rho s} u(c_s)\,\mathrm{d}s + e^{-\rho T} p(T) u(w_T) \,\middle|\, w_t = w\right].$$

Proposition (Merton's fraction)

Therefore is an explicit expression for Merton's fraction π under risk,

$$\pi = \frac{\mu - \sigma\sqrt{2\beta} - r}{\sigma^2}.$$

Figure: Robert Merton, 1944. Nobel Memorial Prize in Economic Sciences (1997)

Summary

- $$\mathrm{nEV@R}^{t:T}_{\beta(\cdot)}(Y \mid \mathcal{F}_t) := \operatorname*{ess\,inf}_{\tilde\beta(\cdot)\geq\beta(\cdot)} \mathrm{nEV@R}_{\widetilde{\beta}(\cdot)}(Y \mid \mathcal{F}_t),$$
- $$\mathcal{G}\phi = \frac{\partial}{\partial t}\phi + b\frac{\partial}{\partial \xi}\phi + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial \xi^2}\phi + \sqrt{2\beta}\left|\sigma\cdot\frac{\partial}{\partial\xi}\phi\right|$$
- $$V(t,\xi) := \inf_{x(\cdot)\text{ adapted}} n\mathcal{R}\left(\int_t^\infty c(s,X_s,x(s,X_s))\,\mathrm{d}s \,\middle|\, X_t=\xi\right)$$
- $$\mathcal{H}(t,\xi,g,H) := \sup_{x\in\mathbb{X}}\left\{-b(x)\cdot g - \frac{1}{2}\sigma(x)^2\cdot H - c(x) - \sqrt{2\beta}\cdot|\sigma\cdot g|\right\}.$$

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