Weierstrass Institute for
Applied Analysis and Stochastics

JOHANN · RADON · INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Probabilistic Constraints in Optimization with PDEs

R. Henrion (Weierstrass Institute Berlin)

Based on joint work with H. Farshbaf Shaker, H. Heitsch, D. Hömberg (WIAS, Berlin), M. Gugat (FAU Erlangen), W.V. Ackooij (EDF R&D, Paris)

A probabilistic program - standard setting

Optimization problem with probabilistic constraint:

minimize $f(x)$

subject to

$$\mathbb{P}(g_i(x,\xi) \leq 0 \;\; (i = 1, \dots, m)) \geq p\}$$

$$x \in X \subseteq \mathbb{R}^n$$

ξ: s-dimensional random vector (continuously distributed)

chronology: $x \curvearrowright \xi$

A probabilistic program - standard setting

Optimization problem with probabilistic constraint:

$$\text{minimize } f(x)$$

$$\text{subject to}$$

$$\mathbb{P}(g_i(x,\xi) \leq 0 \ (i = 1,\ldots,m)) \geq p\}$$

$$x \in X \subseteq \mathbb{R}^n$$

ξ: s-dimensional random vector (continuously distributed)

chronology: $x \curvearrowright \xi$

Perspectives:

A probabilistic program - standard setting

Optimization problem with probabilistic constraint:

$$\text{minimize } f(x)$$

$$\text{subject to}$$

$$\mathbb{P}(g(x,\xi,t) \leq 0 \ \forall t \in T) \geq p\}$$

$$x \in X \subseteq \mathbb{R}^n$$

ξ: s-dimensional random vector (continuously distributed)

chronology: $x \curvearrowright \xi$

Perspectives:

- infinite inequality system

A probabilistic program - standard setting

Optimization problem with probabilistic constraint:

minimize $f(x)$

subject to

$$\mathbb{P}(g(x,\xi,t) \leq 0 \ \forall t \in T) \geq p\}$$

$x \in X \subseteq$ Banach space

ξ: s-dimensional random vector (continuously distributed)

chronology: $x \curvearrowright \xi$

Perspectives:

- infinite inequality system
- infinite dimensional decisions

A probabilistic program - standard setting

Optimization problem with probabilistic constraint:

$$\text{minimize } f(x)$$

$$\text{subject to}$$

$$\mathbb{P}(g(x,\xi,t) \leq 0 \ \forall t \in T) \geq p\}$$

$$x \in X \subseteq \text{Banach space}$$

ξ: s-dimensional random vector (continuously distributed)

chronology: $x_1 \curvearrowright \xi_1 \curvearrowright x_2 \curvearrowright \xi_2 \curvearrowright \cdots \curvearrowright x_s \curvearrowright \xi_s$

Perspectives:

- infinite inequality system
- infinite dimensional decisions

A probabilistic program - standard setting

Optimization problem with probabilistic constraint:

minimize $f(x)$

subject to

$$\mathbb{P}(g(x_1, x_2(\xi_1), \dots, x_s(\xi_{s-1}), \xi, t) \leq 0 \ \forall t \in T) \geq p\}$$

$x \in X \subseteq$ Banach space

ξ: s-dimensional random vector (continuously distributed)

chronology: $x_1 \curvearrowright \xi_1 \curvearrowright x_2 \curvearrowright \xi_2 \curvearrowright \cdots \curvearrowright x_s \curvearrowright \xi_s$

Perspectives:

- infinite inequality system
- infinite dimensional decisions
- dynamic decisions

Probabilistic state constraints in PDE constrained optimization[1]

Simple PDE arising in shape optimization of mechanical structures or crystal growth:

$$
\begin{aligned}
-\nabla_x \cdot (\kappa(x)\, \nabla_x y(x)) &= r(x,\xi), \quad & x \in D \\
n \cdot (\kappa(x)\, \nabla_x y(x)) + \alpha\, y(x) &= u(x) \quad & x \in \partial D,
\end{aligned}
$$

Probabilistic state constraint:

$$\mathbb{P}(y(x,\xi) \le \bar{y} \quad \forall x \in C \subseteq D) \ge p$$

Using control to state operator $Y(r,u)$, probabilistic state constraint turns into a probabilistic constraint on the decision (control) variable:

$$\mathbb{P}(\bar{y} - Y(r(x,\xi), u(x)) \ge 0 \quad \forall x \in C) \ge p$$

Motivates to investigate optimization problems

$$\min\{f(x) \mid \underbrace{\mathbb{P}(g(x,\xi,t) \ge 0 \quad \forall t \in T)}_{\varphi(x)} \ge p\}$$

with X Banach space and T arbitrary (maybe compact) index set.

[1] Farshbaf-Shaker, H.. D. Hömberg 2018

Proposition

Let X be a Banach space. Assume that g is weakly sequentially upper semicontinuous (w.s.u.s.) in the first two arguments. Then, $\varphi : X \to \mathbb{R}$ is w.s.u.s.
In particular, the probabilistic constraint $M := \{x \in X \mid \varphi(x) \geq p\}$ is weakly sequentially closed.

Proposition

Assume that

1. *g is weakly sequentially lower semicontinuous (w.s.l.s.).*
2. *T is compact.*
3. *Let $x \in X$ be such that* $\mathbb{P}(\inf_{t\in T} g(x,\xi,t) = 0) = 0$

Then, φ is w.s.l.s. at x.

The technical condition 3. may be replaced by the easier to verify conditions

- ξ has a density.
- g is concave in the second argument.
- There exists $\bar{z} \in \mathbb{R}^m$ with $g(x,\bar{z},t) > 0$ for all $t \in T$.

Convexity of the probabilistic constraint $M := \{x \in X \mid \varphi(x) \geq p\}$

As before, let $\varphi(x) := \mathbb{P}(g(x, \xi, t) \geq 0 \quad \forall t \in T)$.

Theorem (Prekopa)

Assume that

- g *is quasiconcave in the first two variables simultaneously.*
- ξ *has a log-concave density (e.g. Gaussian etc.)*

Then, the probabilistic constraint defines a convex set M *for all* $p \in [0, 1]$.

All these properties may be verified by imposing standard assumptions for the simple PDE displayed before.

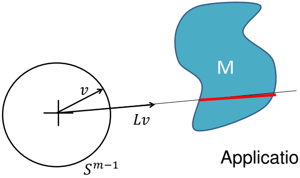
$\Longrightarrow$ existence of solutions, convex optimization problem (along with convex objective)

Spheric-radial decomposition of a Gaussian random vector in $\mathbb{R}^m$

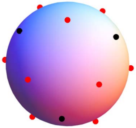
Let $\xi \sim \mathcal{N}(0,R)$ with $R = LL^T$. Then,

$$\mathbb{P}\left(\xi \in M\right) = \int\limits_{v \in \mathbb{S}^{m-1}} \mu_\eta\left(\{r \geq 0 : rLv \in M\}\right) d\mu_\zeta(v),$$

where μ_η, μ_ζ are the laws of $\eta \sim \chi(m)$ and of the uniform distribution on $\mathbb{S}^{m-1}$.



Sampling of uniform distribution on the sphere



Application to parameter-dependent inequality systems: [2],

$$\varphi(x) := \mathbb{P}(g(x,\xi,t) \leq 0\ \forall t \in T) = \int\limits_{v \in \mathbb{S}^{m-1}} \mu_\eta\left(\{r \geq 0 : g(x, rLv, t) \leq 0\ \forall t \in T\}\right) d\mu_\zeta(v)$$

Obtain $\nabla\varphi$ as another spherical integral by differentiating under the integral (if allowed!)

Apply nonlinear programming method to solve optimization problem.

[2] Deák (1980,2000), Royset/Polak (2004,2007), W.v. Ackooij, H. (2014,2017)

Theorem

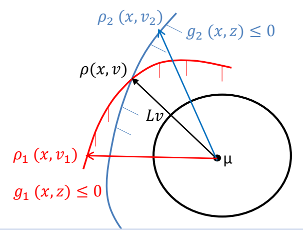
Assume that

- X *- ref.+sep. B-space,* $g \in \mathcal{C}^1(X \times \mathbb{R}^m, \mathbb{R}^p)$ *and* $g_i(x,\cdot)$ *convex.*
- $\varphi(\bar{x}) > 0.5$ *at a point of interest* $\bar{x}$*.*
- $\exists l > 0 : \|\nabla_x g_i(x,z)\| \le l e^{\|z\|} \quad \forall x \in \mathbb{B}_{1/l}(\bar{x}) \ \forall z : \|z\| \ge l \ \forall i = 1,\dots,m.$
- $\operatorname{rank}\{\nabla_z g_i(\bar{x},z), \nabla_z g_j(\bar{x},z)\} = 2 \quad \forall i \ne j \in \mathcal{I}(z) \ \forall z : g(\bar{x},z) \le 0,$
  where, $\mathcal{I}(z) := \{i \mid g_i(\bar{x},z) = 0\}$*.*

Then, φ *is strictly differentiable at* $\bar{x}$ *and the gradient formula*

$$\nabla\varphi(\bar{x}) = -\int\limits_{v \in \mathbb{S}^{m-1}} \frac{\chi(\rho(\bar{x},v))}{\langle \nabla_z g_{i^*(v)}(\bar{x}, \rho(\bar{x},v)Lv), Lv\rangle} \nabla_x g_{i^*(v)}(\bar{x}, \rho(\bar{x},v)Lv)\, d\mu_\zeta(v)$$

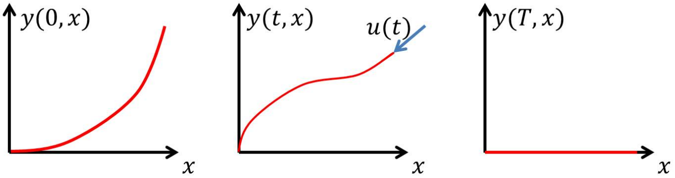
holds true. Here, $i^*(v) := \{i | \rho(\bar{x},v) = \rho_i(\bar{x},v)\}$*.*



Optimal Neumann boundary control for vibrating string[3]

For given initial conditions $y_0 \in H^1(0,1), y_1 \in L^2(0,1)$, solve

$\min \|u\|^2_{L^2(0,T)}$ subject to	cost function
$y(0,x) = y_0(x),\ y_t(0,x) = y_1(x),\ x \in (0,1)$	initial conditions
$y(t,0) = 0,\ y_x(t,1) = u(t), t \in (0,T)$	boundary conditions
$y_{tt}(t,x) = c^2\, y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1)$	wave equation
$y(T,x) = 0,\ y_t(T,x) = 0,\ x \in (0,1)$	terminal conditions



[3] Farshbaf-Shaker, Gugat, Heitsch, H. 2019

Analytical solutions

Control-to-state operator $u \mapsto y$ given analytically by $y(t,x) = \sum_{n=0}^{\infty} \alpha_n(t)\,\varphi_n(x)$, where

$$\begin{aligned}
\varphi_n(x) &:= \frac{\sqrt{2}}{\sqrt{L}}\,\sin\left(\left(\frac{\pi}{2} + n\pi\right)\frac{x}{L}\right)\\
\alpha_n(t) &:= \alpha_n^0 \cos\left(\sqrt{\lambda_n}\,c\,t\right) + \alpha_n^1\,\frac{1}{\sqrt{\lambda_n}\,c}\,\sin\left(\sqrt{\lambda_n}\,c\,t\right)\\
&\quad + c^2\,\varphi_n(L)\,\frac{1}{\sqrt{\lambda_n}\,c}\int_0^t u(s)\sin\left(\sqrt{\lambda_n}\,c\,(t-s)\right)ds
\end{aligned}$$

Theorem (Gugat 2015)

Let $T \geq 2$, $k := \max\{n \in \mathbb{N} : 2n \leq T\}$ and $\Delta := T - 2k$.... For $t \in [0,2)$, let

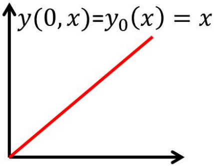
$$d(t) := \begin{cases} k+1, & t \in (0,\Delta], \\ k, & t \in (\Delta, 2). \end{cases}$$

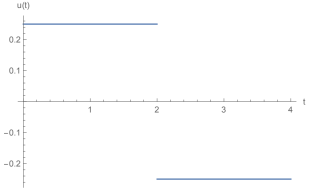
*Then the optimal control u_0 that solves **(NEC)** is 4–periodic, with*

$$u_0(t) = \begin{cases} \frac{1}{2d(t)}\left[y_0'(1-t) - y_1(1-t)\right], & t \in (0,1), \\ \\ \frac{1}{2d(t)}\left[y_0'(t-1) + y_1(t-1)\right], & t \in (1,2). \end{cases}$$

Solution of the deterministic problem

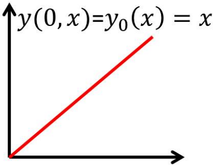
For initial conditions $y_0(x) = x,\ y_1(x) = 0$ one gets a bang-bang solution:

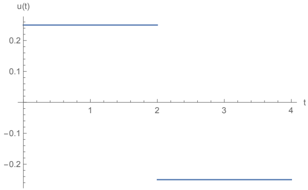




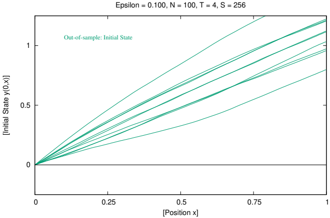
Solution of the deterministic problem

For initial conditions $y_0(x) = x,\ y_1(x) = 0$ one gets a bang-bang solution:



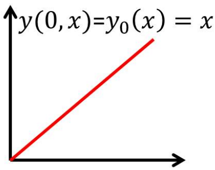


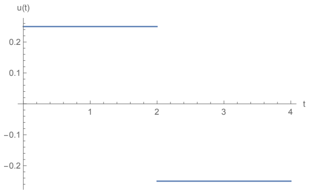
How to adapt the problem when initial conditions are random?



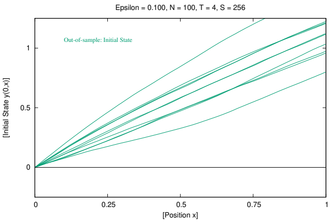
Solution of the deterministic problem

For initial conditions $y_0(x) = x,\ y_1(x) = 0$ one gets a bang-bang solution:





How to adapt the problem when initial conditions are random?



Terminal conditions of deterministic problem

$$y(T, x) = 0,\ y_t(T, x) = 0,\ x \in (0, 1)$$

equivalent to "Terminal energy = zero":

$$E(u) := \int_0^1 y_x(T, x)^2 + \frac{1}{c^2} y_t(T, x)^2\, dx = 0$$

Relaxation: Probability $(E(u) \leq \varepsilon) \geq p$.

Probabilistic problem

$\min \|u\|^2_{L^2(0,T)}$ subject to — cost function

$y(0,x) = y_0^\omega(x),\ y_t(0,x) = 0,\ x \in (0,1)$ — initial conditions

$y(t,0) = 0,\ y_x(t,1) = u(t),\ t \in (0,T)$ — boundary conditions

$y_{tt}(t,x) = c^2\, y_{xx}(t,x),\ (t,x) \in (0,T) \times (0,1)$ — wave equation

$\mathbb{P}(E^\omega(u)) = \mathbb{P}\left(\int_0^1 y_x^\omega(T,x)^2 + \frac{1}{c^2} y_t^\omega(T,x)^2\, dx \le \varepsilon\right) \ge p$ — Terminal conditions

Modeling random initial data

Deterministic initial data (representation as Fourier series):

$$y_0 = \sum_{n=0}^{\infty} \alpha_n^0 \varphi_n; \quad y_1 = \sum_{n=0}^{\infty} \alpha_n^1 \varphi_n$$

Random initial data by multiplicative noise (representation as Fourier series):

$$y_0^\omega = \sum_{n=0}^{\infty} a_n^\omega \alpha_n^0 \varphi_n; \quad y_1^\omega = \sum_{n=0}^{\infty} b_n^\omega \alpha_n^1 \varphi_n$$

Series converge a.s., e.g., if all random coefficients are identically distributed with finite variance.

Random control-to-state operator $(u, \omega) \mapsto y$ can be analytically described similar as in the deterministic case. This allows us to shortly write our control problem as

$$\min \ \|u\|^2_{L^2(0,T)} \quad \text{subject to} \quad \varphi(u) := \mathbb{P}(g(u, (a_n^\omega)_{n=0}^\infty, (b_n^\omega)_{n=0}^\infty) \leq \varepsilon) \geq p \qquad (P).$$

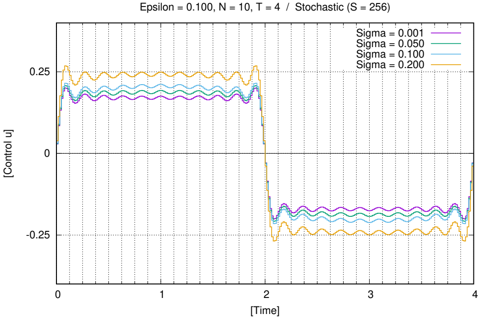
with an analytically given function g.

Theorem

The approximating problem with finite number of Fourier coefficients

$$\min \ \|u\|^2_{L^2(0,T)} \quad \textit{subject to} \quad \varphi_N(u) \geq p \qquad (P_N)$$

is convex and - if the feasible set is nonempty - has a unique solution u^*_N. *Moreover,* $u^*_N \to u^*$ *in the* L^2 *norm, where* u^* *is a solution of the true problem (P).*



Algorithmic approach

In order to solve our problem

$$\min \|u\|^2_{L^2(0,T)} \quad \text{subject to} \quad \varphi_N(u) := \mathbb{P}(g_N(u, a_n^\omega, b_n^\omega) \leq 0) \geq p,$$

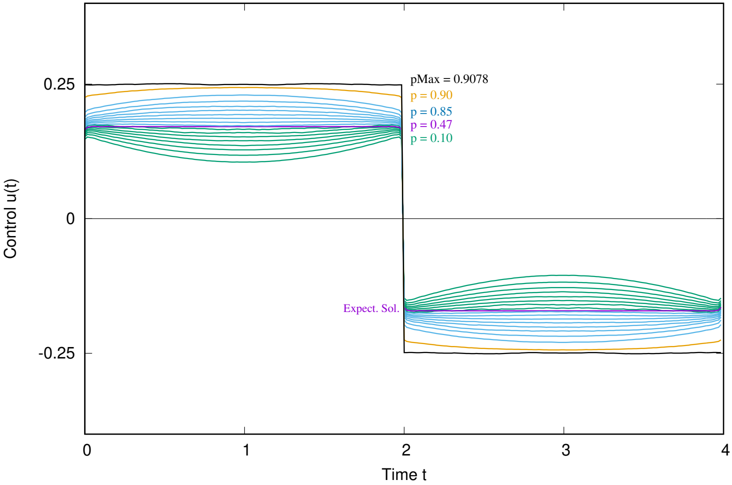
we

- assume a joint multivariate distribution of (a_n^ω, b_n^ω) with identical marginals $\mathcal{N}(1, 0.2)$
- develop analytical formulae for $\varphi_N, \nabla\varphi_N$ using spheric-radial decomposition of Gaussian random vectors
- assume piecewise constant controls on a mesh of size M
- apply a projected gradient algorithm for the numerical solution

In our examples, we put $N = 100$ (number of Fourier coefficients = dimension of multivariate Gaussian distribution) and $M = 256$ (grid of time interval)

Solution of the probabilistic problem (Example 1)

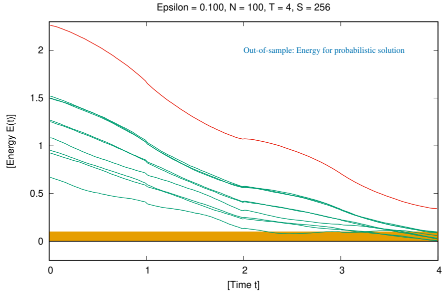
Optimal control for $y_0(x) = x, y_1(x) = 0, \varepsilon = 0.1$ and $p = 0.10, 0.15, \ldots, 0.85, 0.9,$



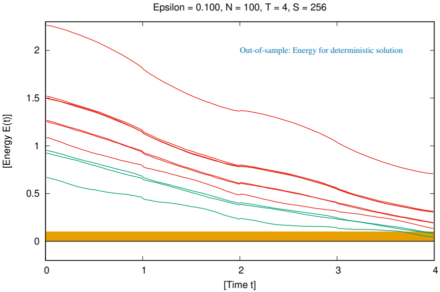
Energy of scenarios as function of time

$$E^\omega(u,t) := \int_0^1 y_x^\omega(t,x)^2 + \frac{1}{c^2} y_t^\omega(t,x)^2 \, dx$$

Optimal probabilistic control for $\varepsilon = 0.1$ and $p = 0.9$

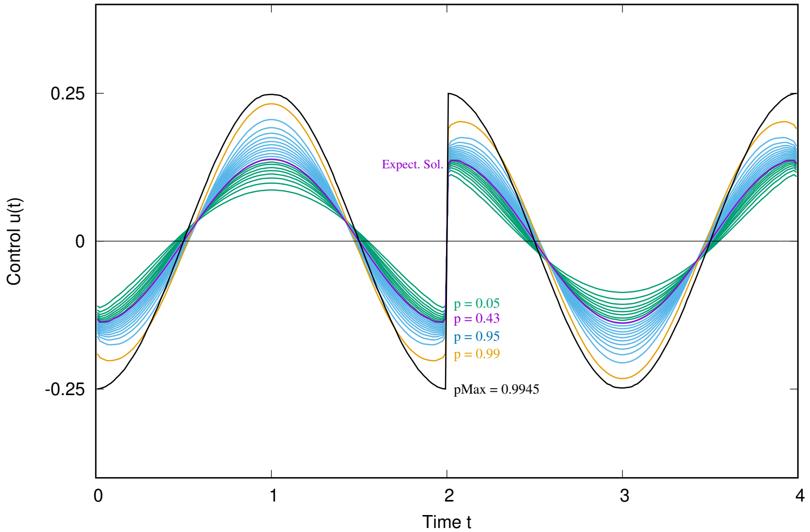


Optimal deterministic control (expected initial condition) for $\varepsilon = 0.1$

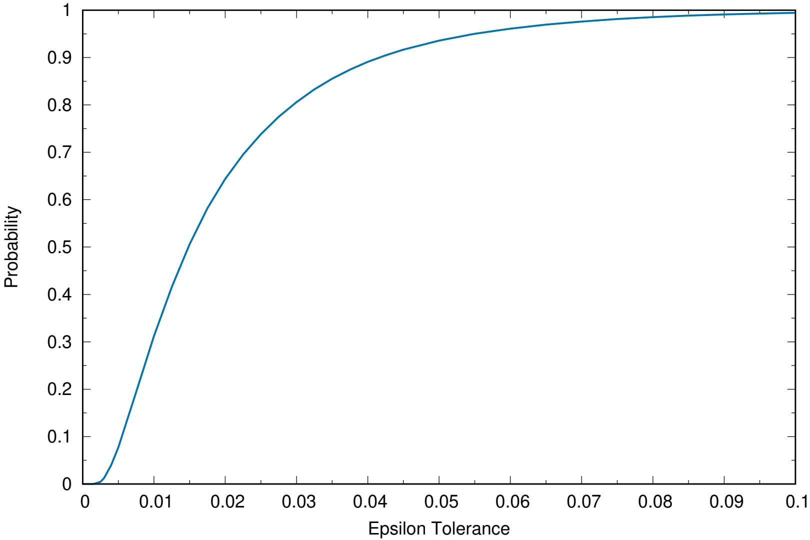


Solution of the probabilistic problem (Example 2)

Optimal control for $y_0(x) = \pi^{-1} sin(\pi x), y_1(x) = 0, \varepsilon = 0.1$ and $p = 0.10, 0.15, \ldots, 0.85, 0.9,$



Maximum probability as function of tolerance



Cost as function of probability