Shape optimization under uncertainty

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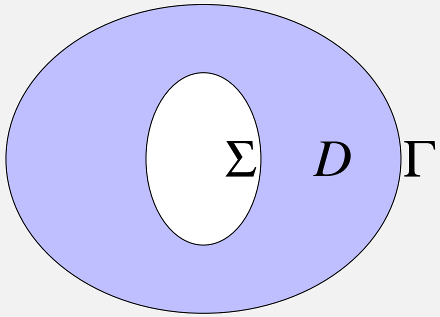
Overview

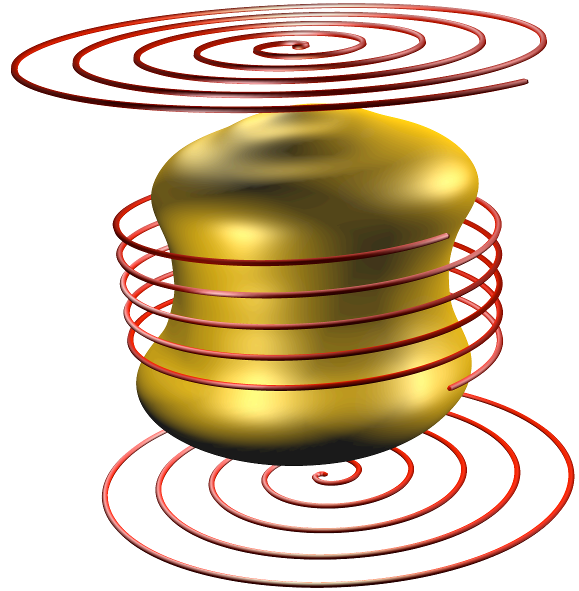
- Shape optimization in case of geometric uncertainty
- Shape optimization in case of random diffusion
- Shape optimization in case of random right-hand sides

Free boundary problems

Problem. Seek the free boundary Γ such that u satisfies

$$\begin{aligned} -\Delta u &= f && \text{in } D \\ u &= g && \text{on } \Sigma \\ u = 0,\ -\frac{\partial u}{\partial \mathbf{n}} &= h && \text{on } \Gamma \end{aligned}$$





- **Growth of anodes.** $f \equiv 0$, $g \equiv 1$, $h \equiv const$
  $\rightsquigarrow$ Bernoulli's free boundary problem
- **Electromagnetic shaping.** Exterior boundary value problem, uniqueness ensured by volume constraint.

Different formulations as shape optimization problem.

$$\left.\begin{aligned} J_1(D) &= \int_D \{\|\nabla v\|^2 - 2fv + h^2\}\,\mathrm{d}\mathbf{x} \to \inf \\ J_2(D) &= \int_D \|\nabla(v-w)\|^2\,\mathrm{d}\mathbf{x} \to \inf \\ J_3(D) &= \int_\Gamma \left(\frac{\partial v}{\partial \mathbf{n}} + h\right)^2 \mathrm{d}\mathbf{x} \to \inf \\ J_4(D) &= \int_\Gamma w^2\,\mathrm{d}\mathbf{x} \to \inf \end{aligned}\right\} \text{ where } \begin{cases} -\Delta v = f & -\Delta w = f \quad \text{in } D \\ v = g & w = g \quad \text{on } \Sigma \\ v = 0 & -\dfrac{\partial w}{\partial \mathbf{n}} = h \quad \text{on } \Gamma \end{cases}$$

Free boundary problems

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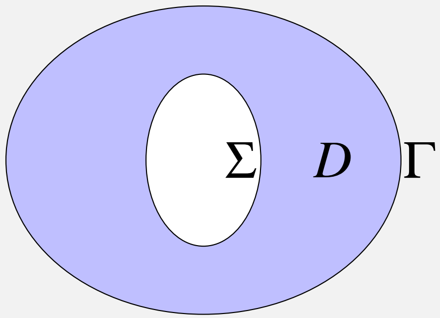

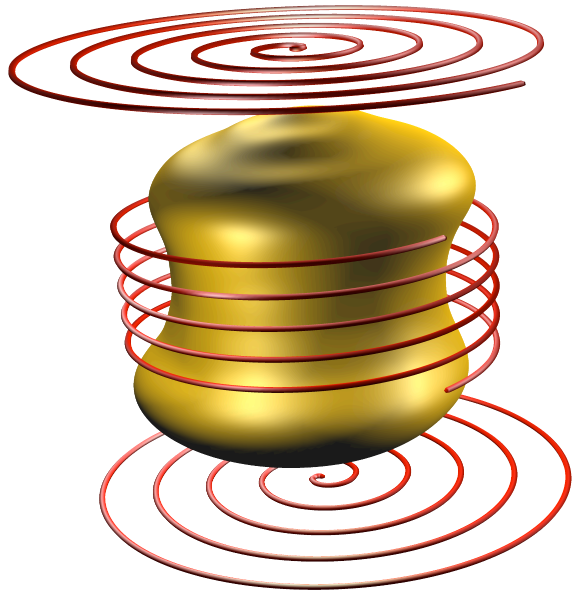

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J_2(D) &= \int_D \|\nabla(v - w)\|^2 \mathrm{d}\mathbf{x} \to \inf \\
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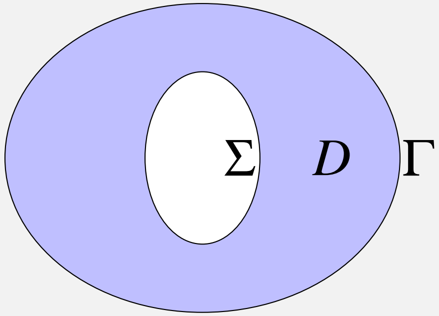
\end{aligned}\right\} \text{ where } \begin{cases} -\Delta v = f & -\Delta w = f & \text{in } D \\ v = g & w = g & \text{on } \Sigma \\ v = 0 & -\frac{\partial w}{\partial \mathbf{n}} = h & \text{on } \Gamma \end{cases}$$

Free boundary problem with geometric uncertainty

Problem. Seek the free boundary $\Gamma(\omega)$ such that $u(\omega)$ satisfies

$$\begin{aligned} \Delta u(\omega) &= 0 && \text{in } D(\omega) \\ u(\omega) &= 1 && \text{on } \Sigma(\omega) \\ u(\omega) = 0,\ -\frac{\partial u}{\partial \mathbf{n}}(\omega) &= h && \text{on } \Gamma(\omega) \end{aligned}$$



for all $\omega \in \Omega$.

The questions to be addressed in the following are

- How to model the random domain $D(\omega)$? Is the problem well-posed in the sense of $D(\omega)$ being almost surely well-defined?
- Since it is a free boundary problem, we are looking for a free boundary.
- Indeed, we are looking for the statistics of the domain itself. But how to define the expectation of a random domain?
- How to compute the solution to the random free boundary problem numerically?

Statistical quantities

- **Expectation** or **mean**.

$$\mathbb{E}[v](\mathbf{x}) := \int_\Omega v(\mathbf{x},\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

- **Correlation**.

$$\mathrm{Cor}[v](\mathbf{x},\mathbf{y}) := \int_\Omega v(\mathbf{x},\omega)v(\mathbf{y},\omega)\,\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[v(\mathbf{x})v(\mathbf{y})]$$

- **Covariance**.

$$\begin{aligned}\mathrm{Cov}[v](\mathbf{x},\mathbf{y}) &:= \int_\Omega \big(v(\mathbf{x},\omega) - \mathbb{E}[v](\mathbf{x})\big)\big(v(\mathbf{y},\omega) - \mathbb{E}[v](\mathbf{y})\big)\,\mathrm{d}\mathbb{P}(\omega)\\ &= \mathrm{Cor}[v](\mathbf{x},\mathbf{y}) - \mathbb{E}[v](\mathbf{x})\mathbb{E}[v](\mathbf{y})\end{aligned}$$

- **Variance**.

$$\begin{aligned}\mathbb{V}[v](\mathbf{x}) &:= \int_\Omega \big(v(\mathbf{x},\omega) - \mathbb{E}[v](\mathbf{x})\big)^2\,\mathrm{d}\mathbb{P}(\omega)\\ &= \mathrm{Cor}[v](\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}} - \mathbb{E}[v]^2(\mathbf{x}) = \mathrm{Cov}[v](\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\end{aligned}$$

- **k-th moment**.

$$\mathcal{M}[v](\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k) := \int_\Omega v(\mathbf{x}_1,\omega)v(\mathbf{x}_2,\omega)\cdots v(\mathbf{x}_k,\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

Existence and uniqueness of solutions

Remarks.

- The solution Γ to the free boundary problem exists if $h > 0$ is sufficiently large.
- If the interior boundary Σ is convex, then the solution is unique.
- If the interior boundary Σ is not convex, multiple solutions might exist.
- In case of a starshaped boundary Σ, the solution is unique and also starshaped.

Parametrization. Assume that $\Sigma(\omega)$ is $\mathbb{P}$-almost surely starlike. Then, we can parametrize

$$\Sigma(\omega) = \big\{\mathbf{x} = \boldsymbol{\sigma}(\phi,\omega) \in \mathbb{R}^2 : \boldsymbol{\sigma}(\phi,\omega) = q(\phi,\omega)\mathbf{e}_r(\phi),\ \phi \in [0,2\pi]\big\},$$
$$\Gamma(\omega) = \big\{\mathbf{x} = \gamma(\phi,\omega) \in \mathbb{R}^2 : \gamma(\phi,\omega) = r(\phi,\omega)\mathbf{e}_r(\phi),\ \phi \in [0,2\pi]\big\}.$$

Theorem (H/Peters [2015]). Assume that $q(\phi,\omega)$ satisfies

$$0 < \underline{r} \leq q(\phi,\omega) \leq \underline{R} \quad \text{for all } \phi \in [0,2\pi] \text{ and } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

Then, there exists a unique free boundary $\Gamma(\omega)$, for almost every $\omega \in \Omega$. Especially, with some constant $\overline{R} > \underline{R}$, the radial function $r(\phi,\omega)$ of the associated free boundary satisfies

$$q(\phi,\omega) < r(\phi,\omega) \leq \overline{R} \quad \text{for all } \phi \in [0,2\pi] \text{ and } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

Expectation and variance

Definition (Parametrization based expectation). The parametrization based expectation $\mathbb{E}_{\mathcal{P}}[D]$ of the boundaries $\Sigma(\omega)$ and $\Gamma(\omega)$ is given by

$$\mathbb{E}_{\mathcal{P}}[\Sigma] = \big\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{E}[q(\phi,\cdot)]\mathbf{e}_r(\phi),\ \phi \in [0,2\pi]\big\},$$
$$\mathbb{E}_{\mathcal{P}}[\Gamma] = \big\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{E}[r(\phi,\cdot)]\mathbf{e}_r(\phi),\ \phi \in [0,2\pi]\big\}.$$

Remark. The expected domain $\mathbb{E}_{\mathcal{P}}[D]$ is thus given by

$$\mathbb{E}_{\mathcal{P}}[D] = \big\{\mathbf{x} = (\rho,\phi) \in \mathbb{R}^2 : \mathbb{E}[q(\phi,\cdot)] \le \rho \le \mathbb{E}[r(\phi,\cdot)]\big\}.$$

This is also called the **radius-vector expectation**.

Theorem (H/Peters [2015]). The variance of the domain $D(\omega)$ in the radial direction is given via the variances of its boundaries parameterizations in accordance with

$$\mathbb{V}_{\mathcal{P}}[\Sigma(\omega)] = \big\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{V}[q(\phi,\cdot)]\mathbf{e}_r(\phi),\ \phi \in [0,2\pi]\big\},$$
$$\mathbb{V}_{\mathcal{P}}[\Gamma(\omega)] = \big\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{V}[r(\phi,\cdot)]\mathbf{e}_r(\phi),\ \phi \in [0,2\pi]\big\}.$$

$\leadsto$ The parametrization based expectation depends on the particular parametrization!

Stochastic quadrature method

► **Random parametrization of the interior boundary.**

$$q(\phi,\mathbf{y}) = \mathbb{E}[q](\phi) + \sum_{k=1}^{N} q_k(\phi) y_k \quad \text{for } \mathbf{y} = [y_1,\ldots,y_N]^{\mathsf{T}} \in \square := [-1/2, 1/2]^N.$$

It then holds

$$\mathbb{E}[q](\phi) = \int_\Omega q(\phi,\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_\square q(\phi,\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

$$\mathbb{V}[q](\phi) = \int_\Omega \big(q(\phi,\omega)\big)^2\,\mathrm{d}\mathbb{P}(\omega) - \big(\mathbb{E}[q](\phi)\big)^2 = \int_\square \big(q(\phi,\mathbf{y})\big)^2\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} - \big(\mathbb{E}[q](\phi)\big)^2.$$

► **Solution map.** Let

$$F : L^\infty\big(\Omega; C_{\text{per}}(0,2\pi)\big) \to L^\infty\big(\Omega; C_{\text{per}}(0,2\pi)\big),\ q(\phi,\omega) \mapsto r(\phi,\omega)$$

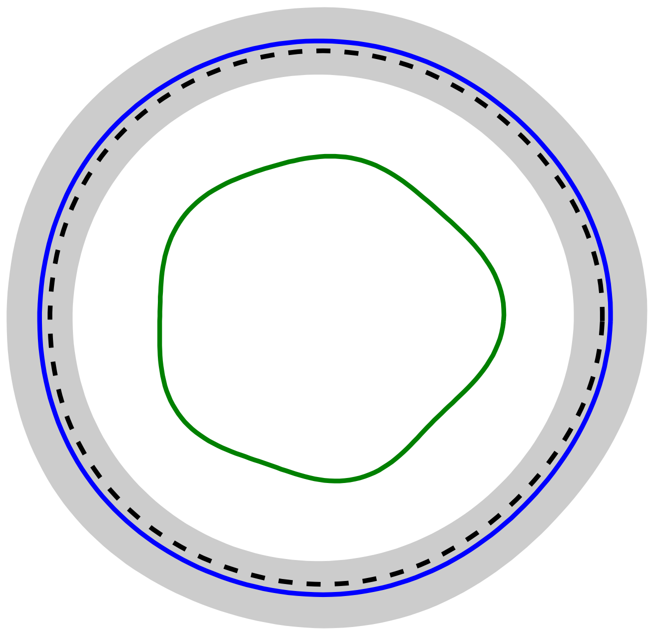
denote the solution map. Then, the expectation and the variance of $r(\phi,\omega)$ are given by

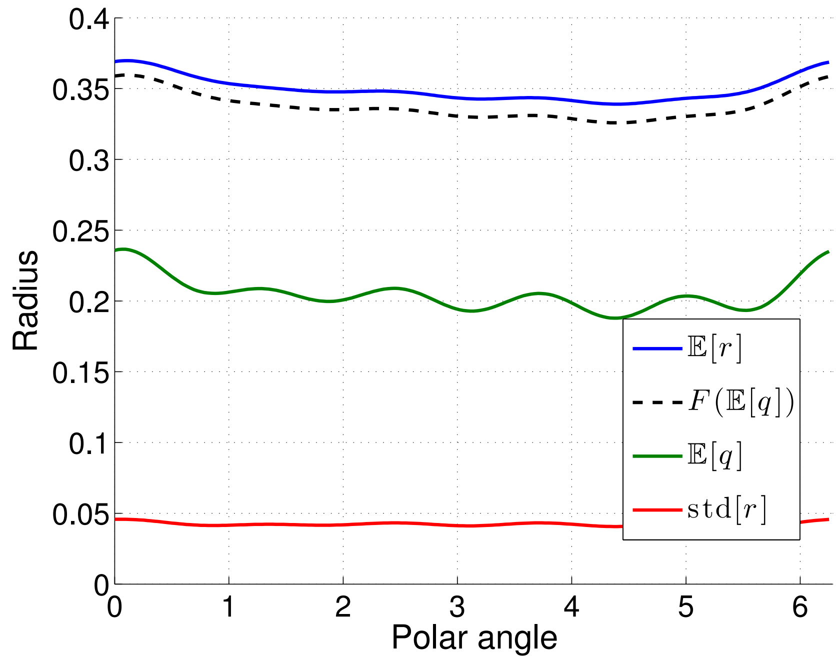
$$\mathbb{E}[r](\phi) = \mathbb{E}[F(q)](\phi) \quad \text{and} \quad \mathbb{V}[r](\phi) = \mathbb{V}[F(q)](\phi).$$

► **(Quasi-) Monte Carlo quadrature.** The high-dimensional integrals are approximated by means of a sampling method.

Numerical example

$$q(\phi,\omega) = \overline{q}(\phi,\omega) + \sum_{k=1}^{10} \frac{\sqrt{2}}{k}\{\sin(k\phi)Y_{2k-1}(\omega) + \cos(k\phi)Y_{2k}(\omega)\}$$





Vorob'ev expectation

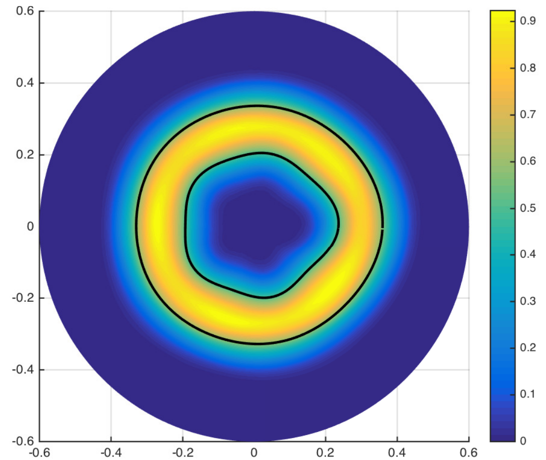
► **Leading idea.** Identify the random set $D(\omega)$ with its characteristic function

$$\mathbb{1}_{D(\omega)}(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in D(\omega), \\ 0, & \text{otherwise.} \end{cases}$$

This embeds the problem into the linear space $L^\infty(\mathbb{R}^2)$.

► **Coverage function.** The average of characteristic functions is not a characteristic function anymore but belongs to the cone $\{q \in L^\infty(\mathbb{R}^2) : 0 \leq q \leq 1\}$. The limit object is the so-called coverage function

$$p(\mathbf{x}) = \mathbb{P}\big(\mathbf{x} \in D(\omega)\big).$$

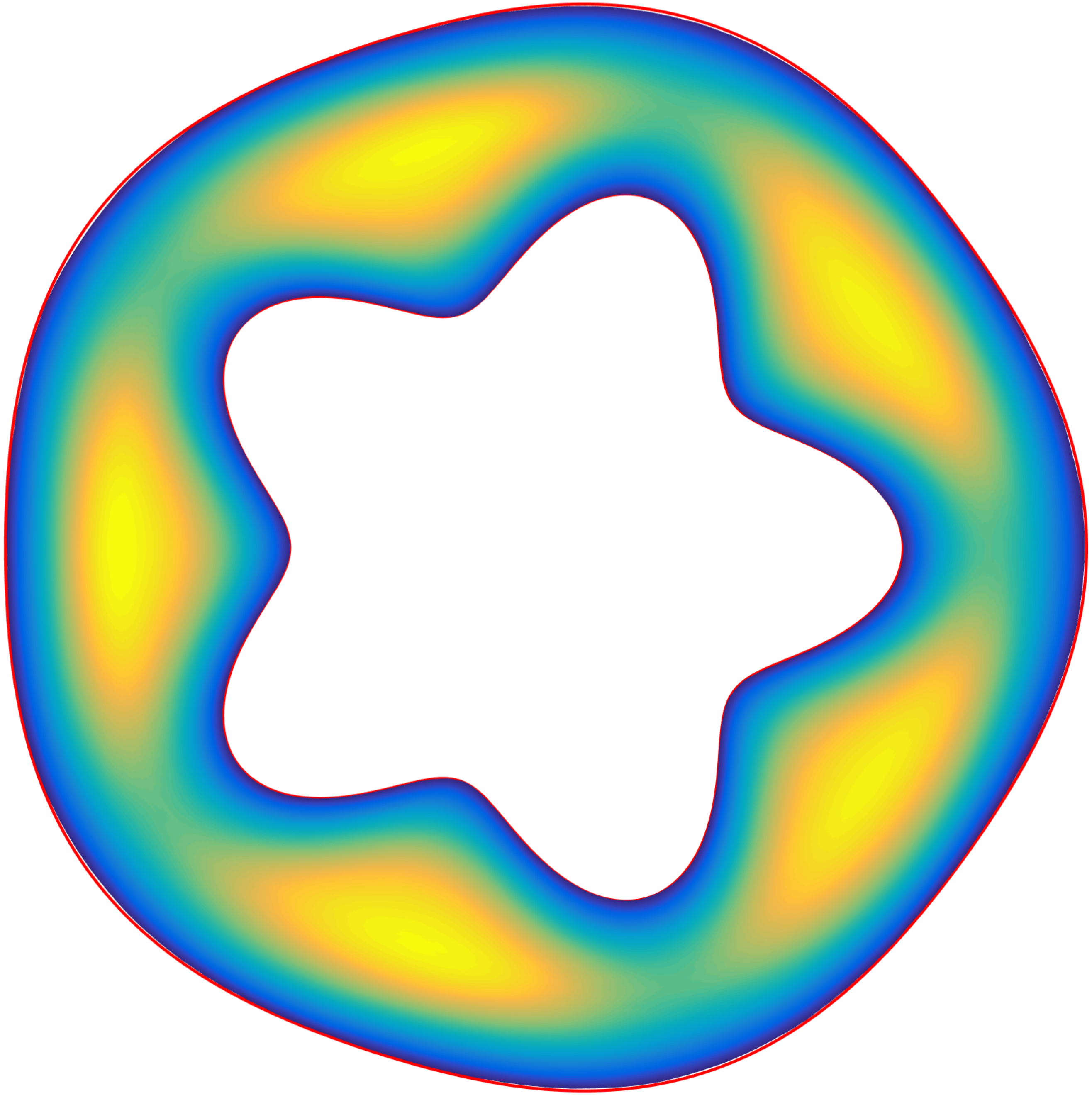


Definition (Vorob'ev expectation). The Vorob'ev expectation $\mathbb{E}_{\mathcal{V}}[D]$ of $D(\omega)$ is defined as the set $\{\mathbf{x} \in \mathbb{R}^2 : p(\mathbf{x}) \geq \mu\}$ for $\mu \in [0,1]$ which is determined from the condition

$$\mathcal{L}(\{\mathbf{x} \in \mathbb{R}^2 : p(\mathbf{x}) \geq \lambda\}) \leq \int_{\mathbb{R}^2} p(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \mathcal{L}(\{\mathbf{x} \in \mathbb{R}^2 : p(\mathbf{x}) \geq \mu\})$$

for all $\lambda > \mu$.

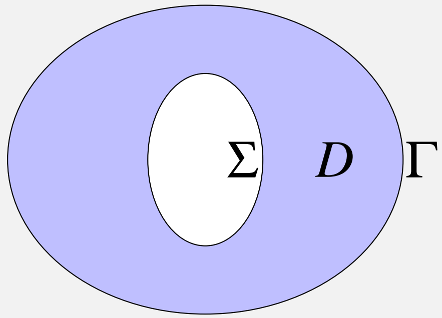
Numerical example



Free boundary problem with random diffusion

Problem. Seek the free boundary $\Gamma(\omega)$ such that $u(\omega)$ satisfies

$$\begin{aligned} \operatorname{div}\big(\alpha(\omega)\nabla u(\omega)\big) &= 0 && \text{in } D(\omega) \\ u(\omega) &= 1 && \text{on } \Sigma \\ u(\omega) = 0,\ -\alpha(\omega)\frac{\partial u}{\partial \mathbf{n}}(\omega) &= h && \text{on } \Gamma(\omega) \end{aligned}$$



for all $\omega \in \Omega$, where

$$0 < \underline{\alpha} \leq \alpha(\omega) \leq \overline{\alpha} < \infty.$$

Theorem (Brügger/Croce/H [2018]). For $\omega \in \Omega$, the solution $\big(u(\omega), \Gamma(\omega)\big)$ is given by the shape optimization problem

$$J(D,\omega) = \int_D \left\{ \alpha(\omega)\|\nabla u(\omega)\|^2 + \frac{h^2}{\alpha(\omega)} \right\} \mathrm{d}\mathbf{x} \to \inf$$

subject to

$$\begin{aligned} \operatorname{div}\big(\alpha(\omega)\nabla u(\omega)\big) &= 0 \text{ in } D \\ u(\omega) &= 1 \text{ on } \Sigma \\ u(\omega) &= 0 \text{ on } \Gamma \end{aligned}$$

Free boundary problem with random diffusion

- We shall minimize

$$\mathbb{E}\big[J(D,\omega)\big] = \int_D \int_\Omega \left\{ \alpha(\omega)\|\nabla u(\omega)\|^2 + \frac{h^2}{\alpha(\omega)} \right\} \mathrm{d}\mathbb{P}(\omega)\, \mathrm{d}\mathbf{x} \to \min .$$

- A minimizer exists since we have an energy type shape functional.
- The shape gradient reads

$$\delta\mathbb{E}\big[J(D,\omega)\big][\mathbf{V}] = \int_\Gamma \langle \mathbf{V},\mathbf{n}\rangle \int_\Omega \left\{ \alpha(\omega)\|\nabla u(\omega)\|^2 + \frac{h^2}{\alpha(\omega)} \right\} \mathrm{d}\mathbb{P}(\omega)\, \mathrm{d}\sigma.$$

- Compute the Karhunen-Loève expansion of the diffusion coefficient

$$\alpha(\mathbf{x},\omega) = \mathbb{E}[\alpha](\mathbf{x}) + \sum_{k=1}^{M} \alpha_k(\mathbf{x}) Y_k(\omega),$$

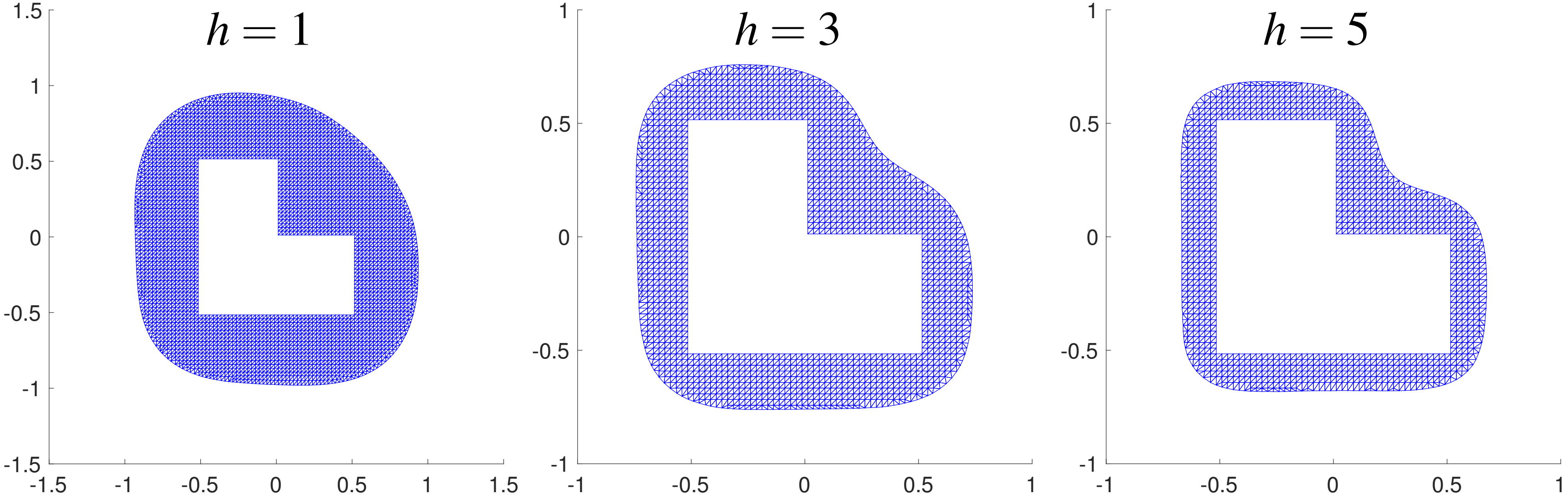
  where the coefficient functions $\{\alpha_k(\mathbf{x})\}_k$ are elements of $C^1(D)$ and the random variables $\{Y_k(\omega)\}_k$ are independently and uniformly distributed in $[-1/2, 1/2]$

  $\rightsquigarrow$ yields a parametric problem on $\square = [-1/2, 1/2]^M$

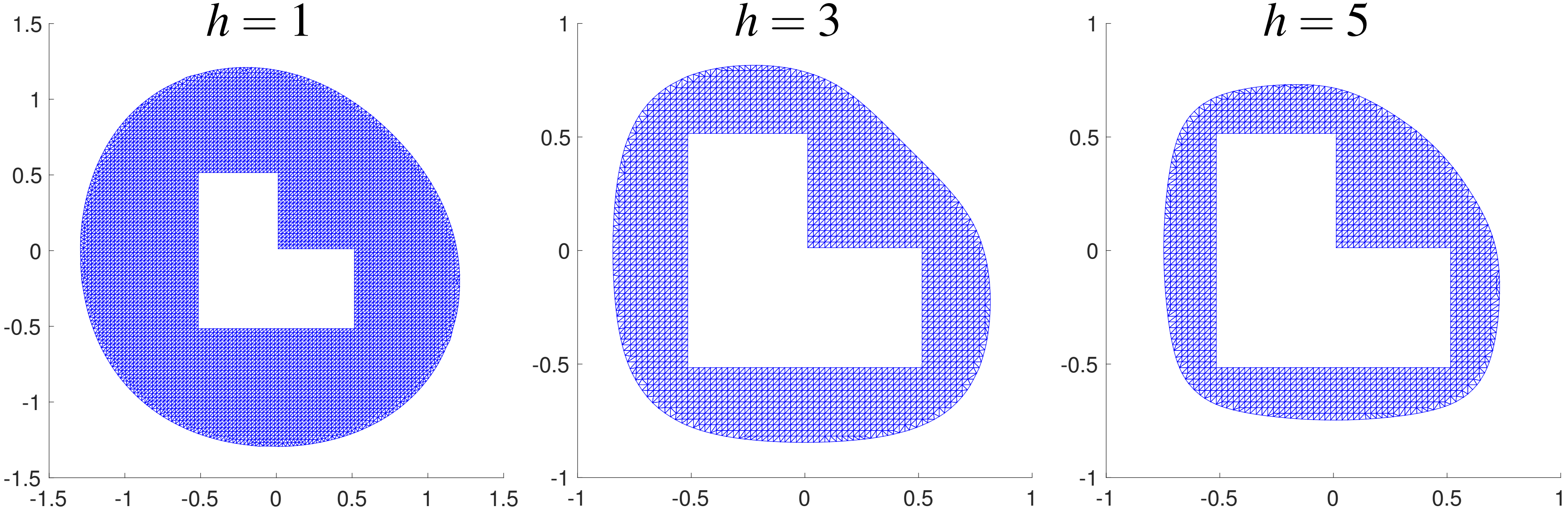
- Use a quasi Monte-Carlo method to approximate the integral over Ω by an integral over over $\square$.

Numerical results

random diffusion with $\mathbb{E}[\alpha] = 1$ and $\mathrm{Cov}[\alpha](\mathbf{x}, \mathbf{x}') = 0.2\exp(-\|\mathbf{x} - \mathbf{x}'\|^2)$

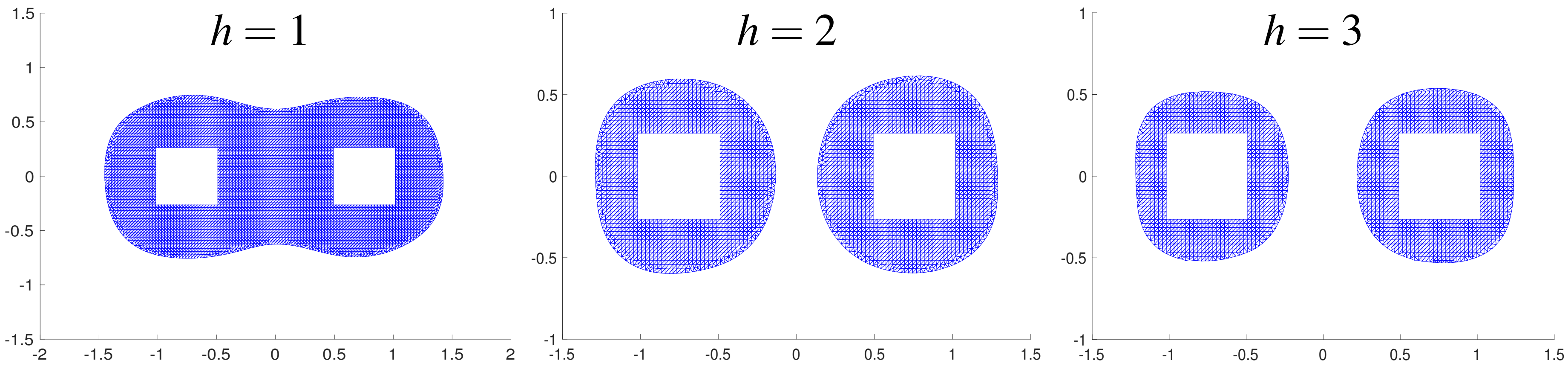


deterministic diffusion ($\alpha = 1$)



Numerical results

random diffusion with $\mathbb{E}[\alpha] = 1$ and $\mathrm{Cov}[\alpha](\mathbf{x}, \mathbf{x}') = 0.15 \exp(-\|\mathbf{x} - \mathbf{x}'\|^2)$



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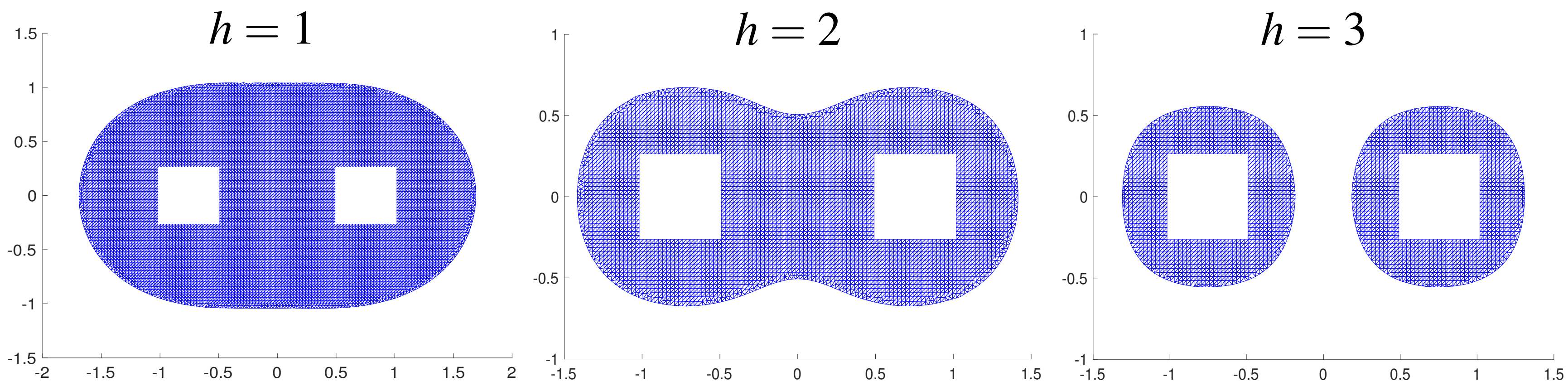


Shape optimization for random right-hand sides

- Consider an elliptic state equation with random right-hand side, for example, the equations of linear elasticity with random forcing:

$$\begin{aligned} -\operatorname{div}\left[\mathbf{A}e\big(\mathbf{u}(\omega)\big)\right] &= \mathbf{f}(\omega) && \text{in } D,\\ \mathbf{A}e\big(\mathbf{u}(\omega)\big)\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N^{\text{free}},\\ \mathbf{A}e\big(\mathbf{u}(\omega)\big)\mathbf{n} &= \mathbf{g}(\omega) && \text{on } \Gamma_N^{\text{fix}},\\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D. \end{aligned}$$

  where $e(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathsf{T}})/2$ stands for the linearized strain tensor and $\mathbf{A}$ is given by

$$\mathbf{AB} = 2\mu\mathbf{B} + \lambda\operatorname{tr}(\mathbf{B})\mathbf{I} \text{ for all } \mathbf{B} \in \mathbb{R}^{d\times d}$$

  with the Lamé coefficients λ and μ satisfying $\mu > 0$ and $\lambda + 2\mu/d > 0$.

- Consider a quadratic shape functional, for example, the compliance of shapes:

$$\begin{aligned} \mathcal{C}(D,\omega) &= \int_D \mathbf{A}e\big(\mathbf{u}(\mathbf{x},\omega)\big) : e\big(\mathbf{u}(\mathbf{x},\omega)\big)\,\mathrm{d}\mathbf{x}\\ &= \int_D \langle \mathbf{f}(\omega), \mathbf{u}(\omega)\rangle\,\mathrm{d}\mathbf{x} + \int_{\Gamma_N^{\text{fix}}} \langle \mathbf{g}(\mathbf{x},\omega), \mathbf{u}(\mathbf{x},\omega)\rangle\,\mathrm{d}\sigma_{\mathbf{x}}, \end{aligned}$$

- We aim at minimizing the expectation $\mathbb{E}[\mathcal{C}(D,\omega)]$ of the quadratic shape functional.

PDEs with random right-hand side

Random boundary value problem:

$$-\operatorname{div}\left[\alpha\nabla u(\omega)\right] = f(\omega) \text{ in } D, \quad u(\omega) = 0 \text{ on } \partial D$$

$\longrightarrow$ the random solution depends linearly on the random input parameter

Theorem (Schwab/Todor [2003]): It holds

$$-\operatorname{div}\left[\alpha\nabla\mathbb{E}[u]\right] = \mathbb{E}[f] \text{ in } D, \quad \mathbb{E}[u] = \mathbb{E}[g] \text{ on } \partial D$$

and

$$\begin{aligned}(\operatorname{div}\otimes\operatorname{div})\left[(\alpha\otimes\alpha)(\nabla\otimes\nabla)\operatorname{Cor}[u]\right] &= \operatorname{Cor}[f] && \text{in } D\times D,\\ \operatorname{Cor}[u] &= 0 && \text{on } \partial(D\times D).\end{aligned}$$

Numerical solution of the correlation equation:

- sparse grid approximation by the combination technique

H. Harbrecht, M. Peters, and M. Siebenmorgen. Combination technique based k-th moment analysis of elliptic problems with random diffusion. *J. Comput. Phys.*, 252:128–141, 2013.

- low-rank approximation by the pivoted Cholesky decomposition

H. Harbrecht, M. Peters, and R. Schneider. On the low-rank approximation by the pivoted Cholesky decomposition. *Appl. Numer. Math.*, 62:428–440, 2012.

- adaptive low-rank approximation by means of $\mathcal{H}$-matrices

J. Dölz, H. Harbrecht, and C. Schwab. Covariance regularity and $\mathcal{H}$-matrix approximation for rough random fields. *Numer. Math.*, 135(4):1045–1071, 2017.

Deterministic reformulation of the shape functional

Theorem (Dambrine/Dapogny/H [2015]). The expectation of the quadratic shape functional can be rewritten by

$$\mathbb{E}[\mathcal{C}(D,\omega)] = \int_D \big((\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}})\,\mathrm{Cor}[\mathbf{u}]\big)(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\,\mathrm{d}\mathbf{x},$$

where

$$(\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}}) : \big[H^1_{\Gamma_D}(D)\big]^d \otimes \big[H^1_{\Gamma_D}(D)\big]^d \to L^2(D)\otimes L^2(D)$$

is the linear operator induced from the bilinear mapping

$$\mathbf{u}\mathbf{v}^{\mathsf{T}} \mapsto \mathbf{A}e(\mathbf{u}) : e(\mathbf{v}).$$

Proof. The assertion follows from

$$\begin{aligned}
\mathbb{E}[\mathcal{C}(D,\omega)] &= \int_\Omega \int_D \mathbf{A}e\big(\mathbf{u}(\mathbf{x},\omega)\big) : e\big(\mathbf{u}(\mathbf{x},\omega)\big)\,\mathrm{d}\mathbf{x} \\
&= \int_D \left[(\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}}) \int_\Omega \mathbf{u}(\mathbf{x},\omega)\mathbf{u}(\mathbf{y},\omega)^{\mathsf{T}}\,\mathrm{d}\mathbb{P}(\omega)\right]\Bigg|_{\mathbf{x}=\mathbf{y}} \mathrm{d}\mathbf{x} \\
&= \int_D \big((\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}})\,\mathrm{Cor}[\mathbf{u}]\big)(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\,\mathrm{d}\mathbf{x}. \qquad \square
\end{aligned}$$

How to compute the correlation?

Theorem (Dambrine/Dapogny/H [2015]). The two-point correlation function

$$\operatorname{Cor}[\mathbf{u}] \in [H^1_{\Gamma_D}(D)]^d \otimes [H^1_{\Gamma_D}(D)]^d$$

is the unique solution to the following tensor-product boundary value problem:

$$\begin{aligned}
(\operatorname{div}_{\mathbf{x}} \otimes \operatorname{div}_{\mathbf{y}})\big[(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}]\big] &= \operatorname{Cor}[\mathbf{f}] && \text{in } D \times D,\\
(\operatorname{div}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}})(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{I}_{\mathbf{x}} \otimes \mathbf{n}_{\mathbf{y}}) &= \mathbf{0} && \text{on } D \times \Gamma_N^{\text{fix}\cup\text{free}},\\
(\mathbf{I}_{\mathbf{x}} \otimes \operatorname{div}_{\mathbf{y}})(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{n}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}) &= \mathbf{0} && \text{on } \Gamma_N^{\text{fix}\cup\text{free}} \times D,\\
(\operatorname{div}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}})(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}] &= \mathbf{0} && \text{on } D \times \Gamma_D,\\
(\mathbf{I}_{\mathbf{x}} \otimes \operatorname{div}_{\mathbf{y}})(\mathbf{I}_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}] &= \mathbf{0} && \text{on } \Gamma_D \times D,\\
(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{n}_{\mathbf{x}} \otimes \mathbf{n}_{\mathbf{y}}) &= \mathbf{0} && \text{on } \big(\Gamma_N^{\text{fix}\cup\text{free}} \times \Gamma_N^{\text{fix}\cup\text{free}}\big)\\
&&& \quad \setminus (\Gamma_N^{\text{fix}} \times \Gamma_N^{\text{fix}}),\\
(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{n}_{\mathbf{x}} \otimes \mathbf{n}_{\mathbf{y}}) &= \operatorname{Cor}[\mathbf{g}] && \text{on } \Gamma_N^{\text{fix}} \times \Gamma_N^{\text{fix}},\\
(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{n}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}) &= \mathbf{0} && \text{on } \Gamma_N^{\text{fix}\cup\text{free}} \times \Gamma_D,\\
(\mathbf{I}_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{I}_{\mathbf{x}} \otimes \mathbf{n}_{\mathbf{y}}) &= \mathbf{0} && \text{on } \Gamma_D \times \Gamma_N^{\text{fix}\cup\text{free}},\\
\operatorname{Cor}[\mathbf{u}] &= \mathbf{0} && \text{on } \Gamma_D \times \Gamma_D.
\end{aligned}$$

Proof. The assertion follows by tensorizing the state equation and the exploiting the linearity when taking the expectation. □

Computing the shape gradient

Theorem (Dambrine/Dapogny/H [2015]). The functional $\mathbb{E}[J(D,\omega)]$ is shape differentiable at any shape $D \in \mathcal{U}_{ad}$ and its derivative reads

$$\delta\mathbb{E}\big[\mathcal{C}(D,\omega)\big][\mathbf{V}] = \int_{\Gamma_N^{\text{free}}} \langle \mathbf{V}, \mathbf{n}\rangle \big((\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}]\big)(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\, \mathrm{d}\sigma_{\mathbf{x}}.$$

Proof. The assertion follows from

$$\begin{aligned}
\delta\mathbb{E}\big[\mathcal{C}(D,\omega)\big][\mathbf{V}] &= \mathbb{E}\big[\delta\mathcal{C}(D,\omega)[\mathbf{V}]\big] \\
&= \int_\Omega \int_{\Gamma_N^{\text{free}}} \langle \mathbf{V}, \mathbf{n}\rangle \big(\mathbf{A}e\big(\mathbf{u}(\mathbf{x},\omega)\big) : e\big(\mathbf{u}(\mathbf{x},\omega)\big)\ \mathrm{d}\sigma_{\mathbf{x}} \\
&= \int_{\Gamma_N^{\text{free}}} \langle \mathbf{V}, \mathbf{n}\rangle \left[(\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}}) \int_\Omega \mathbf{u}(\mathbf{x},\omega)\mathbf{u}(\mathbf{y},\omega)^{\mathsf{T}} \mathrm{d}\mathbb{P}(\omega)\right]\Bigg|_{\mathbf{x}=\mathbf{y}} \mathrm{d}\sigma_{\mathbf{x}} \\
&= \int_{\Gamma_N^{\text{free}}} \langle \mathbf{V}, \mathbf{n}\rangle \big((\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}]\big)(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\, \mathrm{d}\sigma_{\mathbf{x}}.
\end{aligned}$$

□

Low-rank approximation

- **Approximation of the input correlation.** Assume low-rank approximations

$$\mathrm{Cor}[\mathbf{f}] \approx \sum_i \mathbf{f}_i \mathbf{f}_i^\mathsf{T}, \quad \mathrm{Cor}[\mathbf{g}] \approx \sum_j \mathbf{g}_j \mathbf{g}_j^\mathsf{T}.$$

  Such expansions can efficiently be computed by e.g. a pivoted Cholesky decomposition.

- **Approximation of the shape functional.** The shape functional is simply given by

$$\mathbb{E}[\mathcal{C}(D,\omega)] = \int_D \sum_{i,j} \mathbf{A}e(\mathbf{u}_{i,j}) : e(\mathbf{u}_{i,j}) \,\mathrm{d}\mathbf{x},$$

  where

$$\begin{aligned} -\operatorname{div}\left[\mathbf{A}e(\mathbf{u}_{i,j})\right] &= \mathbf{f}_i && \text{in } D, \\ \mathbf{A}e(\mathbf{u}_{i,j})\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N^{\mathrm{free}}, \\ \mathbf{A}e(\mathbf{u}_{i,j})\mathbf{n} &= \mathbf{g}_j && \text{on } \Gamma_N^{\mathrm{fix}}, \\ \mathbf{u}_{i,j} &= \mathbf{0} && \text{on } \Gamma_D. \end{aligned}$$

- **Approximation of the shape gradient.** The shape gradient is given by

$$\delta\mathbb{E}\left[\mathcal{C}(D,\omega)\right][\mathbf{V}] = \int_{\Gamma_N^{\mathrm{free}}} \langle \mathbf{V}, \mathbf{n} \rangle \sum_{i,j} \mathbf{A}e(\mathbf{u}_{i,j}) : e(\mathbf{u}_{i,j}) \,\mathrm{d}\sigma_{\mathbf{x}}.$$

- **Alternative approach.** A direct discretization of $\mathrm{Cor}[\mathbf{u}]$ in a sparse grid space is possible as well.

First example

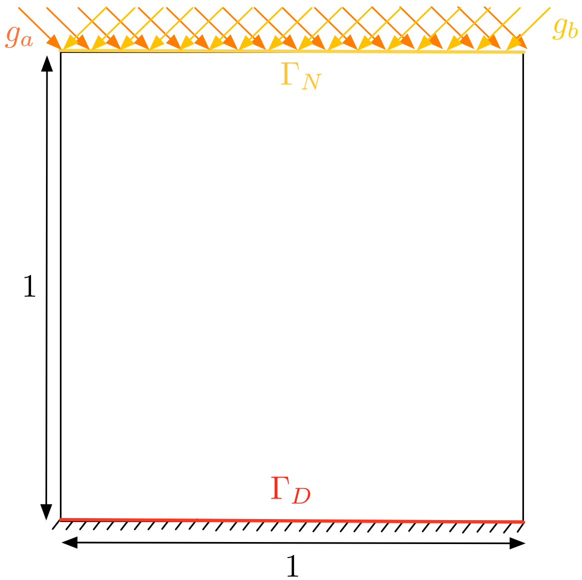
Problem. A bridge is clamped on its lower part two sets of loads $\mathbf{g}_a = (1, -1)$ and $\mathbf{g}_b = (-1, 1)$ are applied on its top, i.e.,

$$\mathbf{g}(\mathbf{x}, \omega) = \xi_1(\omega)\mathbf{g}_a(\mathbf{x}) + \xi_2(\omega)\mathbf{g}_b(\mathbf{x}).$$

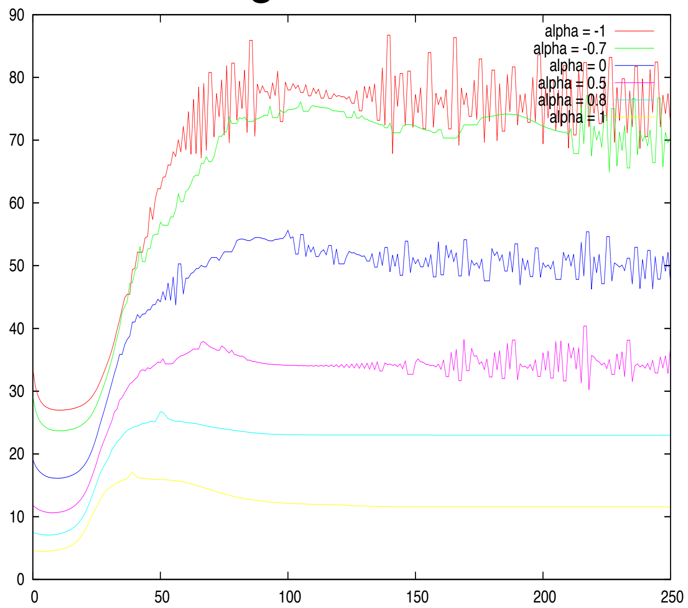
The choice $\mathbb{E}[\xi_i] = 0$, $\mathbb{V}[\xi_i] = 1$, $\operatorname{Cor}[\xi_1, \xi_2] = \alpha$ implies

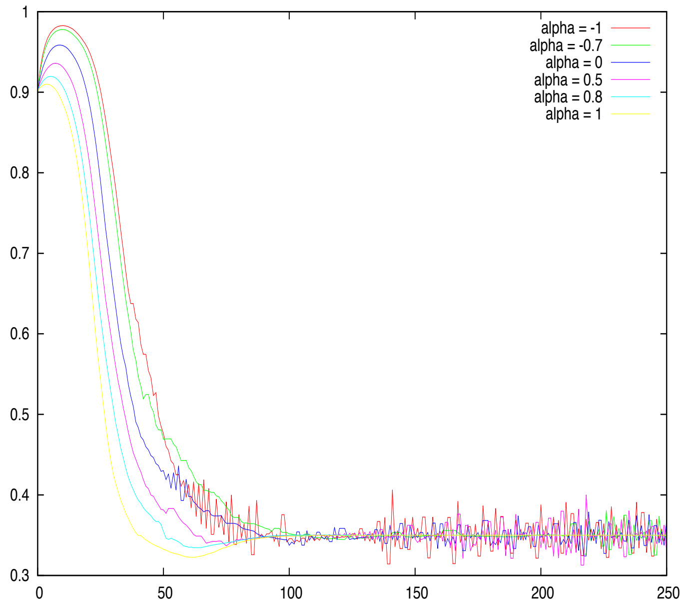
$$\operatorname{Cor}[\mathbf{g}] = \mathbf{g}_a\mathbf{g}_a^{\mathsf{T}} + \mathbf{g}_b\mathbf{g}_b^{\mathsf{T}} + \alpha\left(\mathbf{g}_a\mathbf{g}_b^{\mathsf{T}} + \mathbf{g}_b\mathbf{g}_a^{\mathsf{T}}\right).$$

Sketch:

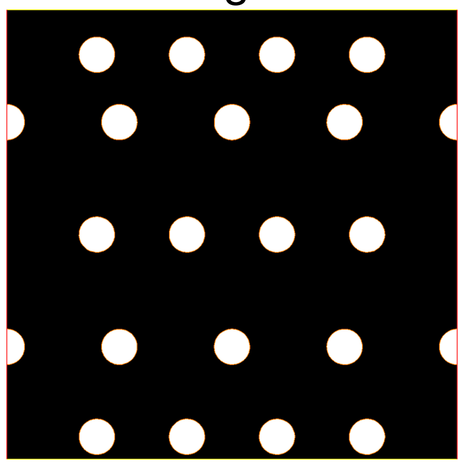


Convergence histories for the mean value and the volume:

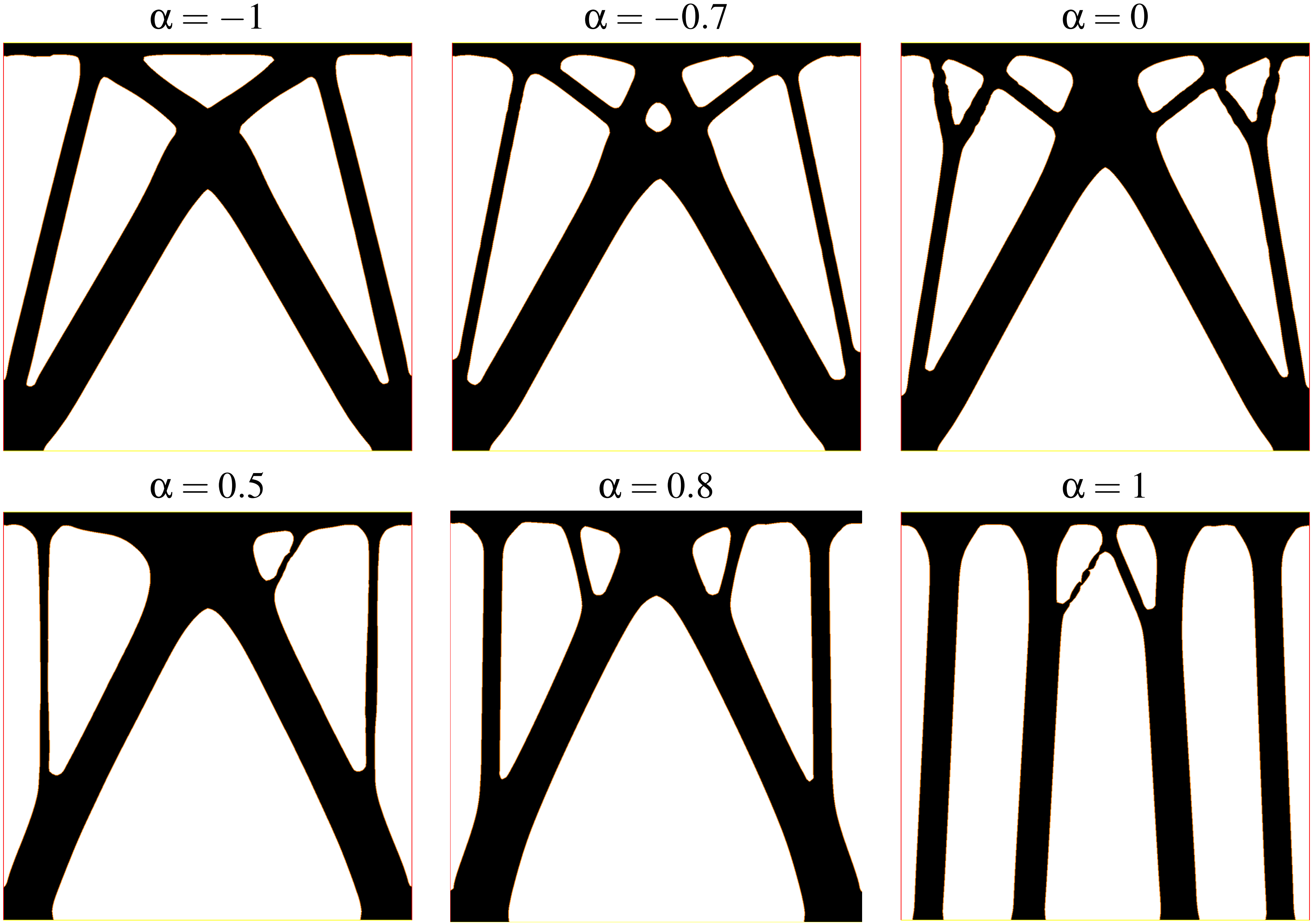




Initial guess:



First example



Second example

Problem. A bridge is clamped on its lower part two sets of loads $\mathbf{g}^i = (g_1^i, g_2^i)$, $i = 1,2,3$, are applied on its top such that

$$\mathrm{Cor}[g_1^i](\mathbf{x},\mathbf{y}) = 10^5 h_i^+\left(\frac{x_1+y_1}{2}\right)e^{-10|x_1-y_1|},$$

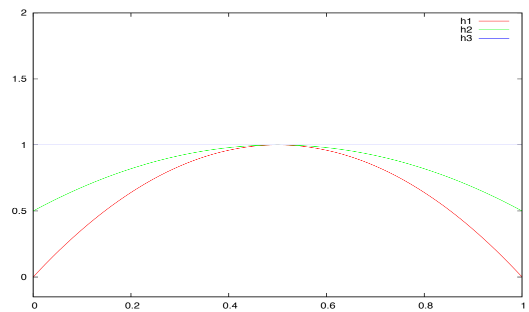
$$\mathrm{Cor}[g_2^i](\mathbf{x},\mathbf{y}) = 10^6 k_i^+\left(\frac{x_1+y_1}{2}\right)e^{-10|x_1-y_1|},$$

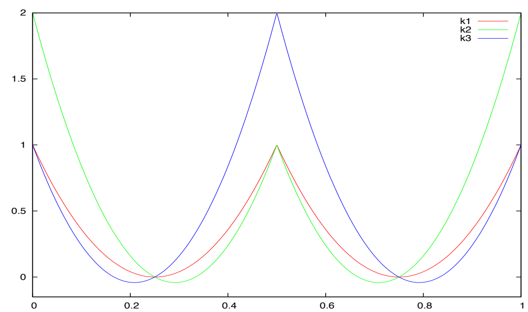
where

$$h_1(t) = 1-4\left(t-\frac{1}{2}\right)^2, \quad k_1(t) = \begin{cases}(4t-1)^2, & \text{if } t \le \frac{1}{2},\\ (4t-3)^2, & \text{else,}\end{cases}$$

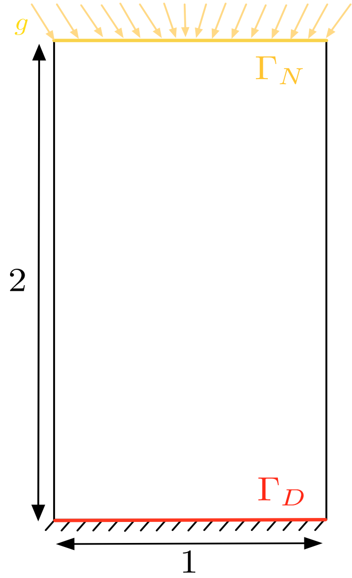
$$h_2(t) = 2t(1-t)+\frac{1}{2}, \quad k_2(t) = \begin{cases}(4t-1)(6t-2), & \text{if } t \le \frac{1}{2},\\ (4t-3)(6t-4), & \text{else,}\end{cases}$$

$$h_3(t) = 1, \quad k_3(t) = \begin{cases}(4t-1)(6t-1), & \text{if } t \le \frac{1}{2},\\ (4t-3)(6t-5), & \text{else.}\end{cases}$$

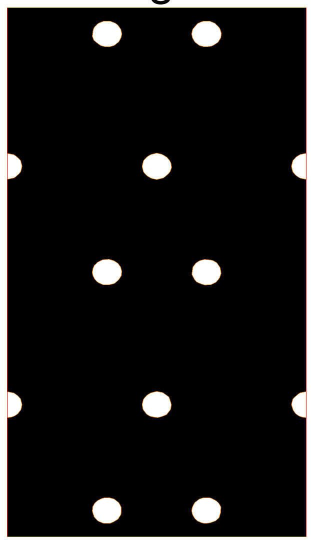




Sketch:

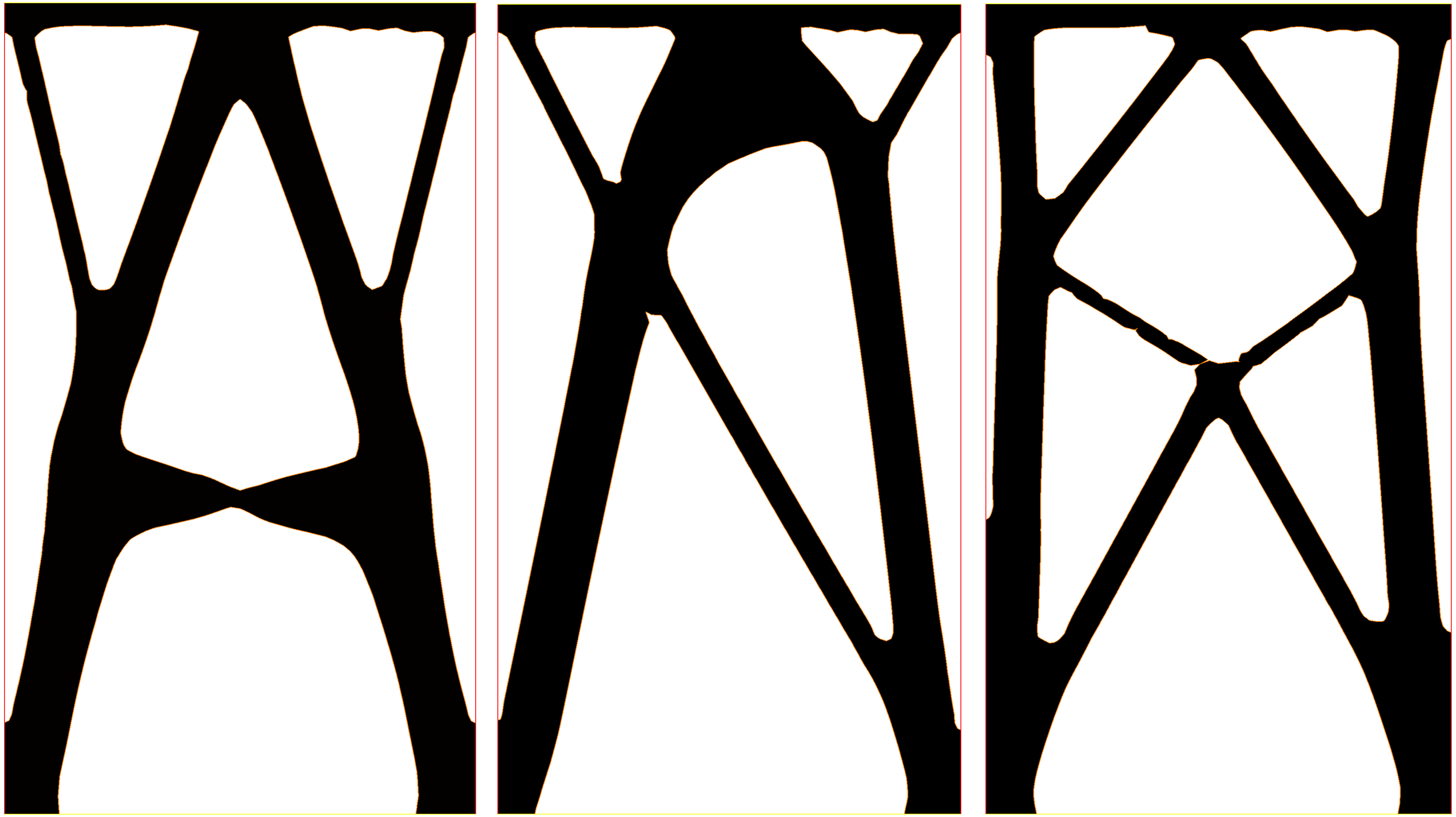


Initial guess:



Second example

surface load $\mathbf{g}^1(\omega)$ | surface load $\mathbf{g}^2(\omega)$ | surface load $\mathbf{g}^3(\omega)$



About measurement noise in EIT

Problem. Minimize

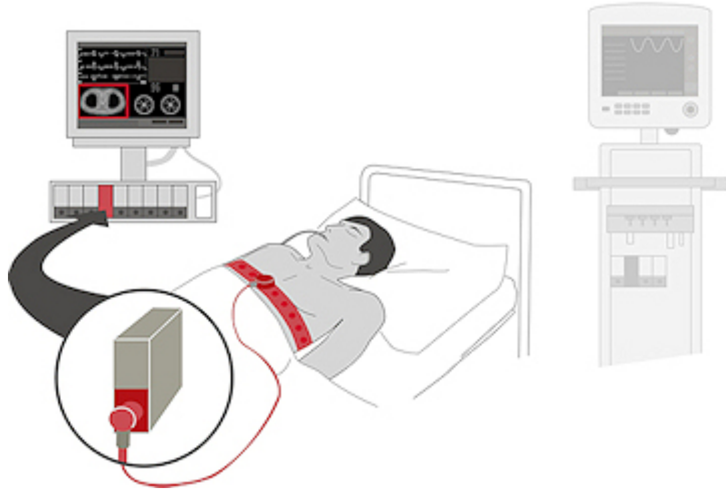
$$F(D) = (1-\alpha)\mathbb{E}\big[J(D,\omega)\big] + \alpha\sqrt{\mathbb{V}\big[J(D,\omega)\big]} \to \inf,$$

where the random shape functional reads as

$$J(D,\omega) = \int_D \big\|\nabla\big(v(\omega)-w\big)\big\|^2 \mathrm{d}\mathbf{x} \to \inf$$

and the states read as

$$\begin{aligned} \Delta v(\omega) &= 0 & \Delta w &= 0 & &\text{in } D, \\ v(\omega) &= 0 & w &= 0 & &\text{on } \Gamma, \\ \frac{\partial v}{\partial \mathbf{n}}(\omega) &= g(\omega) & w &= f & &\text{on } \Sigma. \end{aligned}$$



We assume that the Neumann data g are given as a Gaussian random field

$$g(\mathbf{x},\omega) = g_0(\mathbf{x}) + \sum_{i=1}^{M} g_i(\mathbf{x}) Y_i(\omega),$$

where the random variables are independent, satisfying $Y_i \sim \mathcal{N}(0,1)$.

Taking measurement noise in EIT into account

It holds for the shape functional

$$\mathbb{E}\big[J(D,\omega)\big] = \sum_{i=1}^{M} \int_{\Sigma} v_i g_i \,\mathrm{d}\sigma + \int_{\Sigma} \left(g_0 - \frac{\partial w}{\partial \mathbf{n}}\right)(v_0 - f)\,\mathrm{d}\sigma,$$

$$\mathbb{V}\big[J(D,\omega)\big] = 2\sum_{i,j=1}^{M} \left(\int_{\Sigma} v_i g_j \,\mathrm{d}\sigma\right)^2 + 4\sum_{i=1}^{M} \left(\int_{\Sigma} g_i (v_0 - f)\,\mathrm{d}\sigma\right)^2$$

and for the shape gradient

$$\delta\mathbb{E}\big[J(D,\omega)\big][\mathbf{V}] = \int_{\Gamma} \langle \mathbf{V}, \mathbf{n}\rangle \left[\sum_{i=0}^{M} \left(\frac{\partial v_i}{\partial \mathbf{n}}\right)^2 - \left(\frac{\partial w}{\partial \mathbf{n}}\right)^2\right] \mathrm{d}\sigma,$$

$$\delta\mathbb{V}\big[J(D,\omega)\big][\mathbf{V}] = 4\sum_{i,j=1}^{M} \left(\int_{\Sigma} v_i g_j \,\mathrm{d}\sigma\right)\left(\int_{\Gamma} \langle \mathbf{V}, \mathbf{n}\rangle \frac{\partial v_i}{\partial \mathbf{n}} \frac{\partial v_j}{\partial \mathbf{n}} \,\mathrm{d}\sigma\right)$$

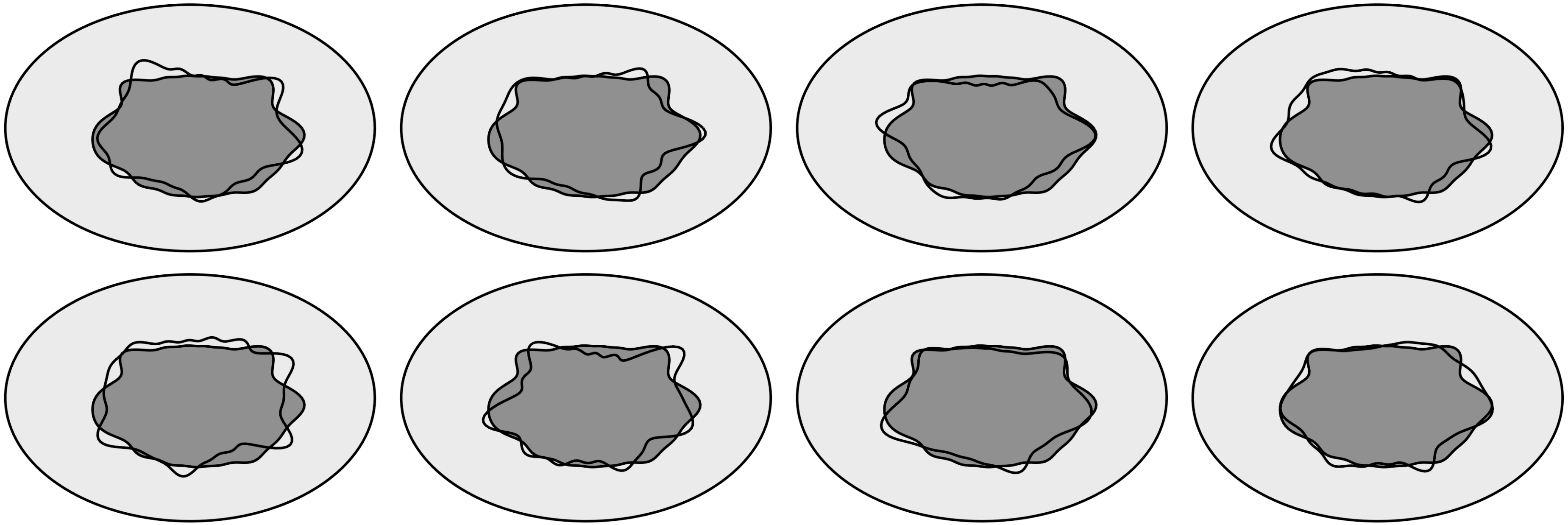
$$+ 8\sum_{i=1}^{M} \left(\int_{\Sigma} g_i (v_0 - f)\,\mathrm{d}\sigma\right)\left(\int_{\Gamma} \langle \mathbf{V}, \mathbf{n}\rangle \frac{\partial v_i}{\partial \mathbf{n}} \frac{\partial v_0}{\partial \mathbf{n}} \,\mathrm{d}\sigma\right).$$

where

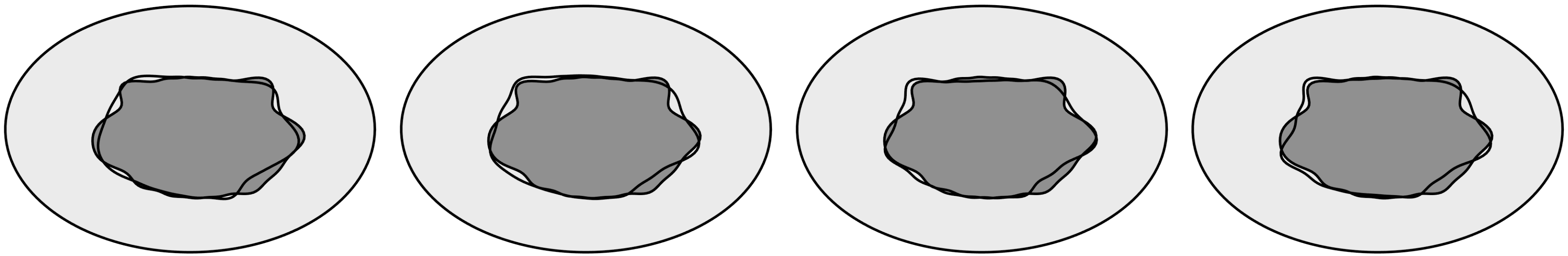
$$\Delta v_i = 0 \text{ in D}, \quad v_i = 0 \text{ on } \Gamma, \quad \frac{\partial v_i}{\partial \mathbf{n}} = g_i \text{ on } \Sigma.$$

Numerical results (5% noise, 10 samples)

Reconstructions for different realizations of the measurement:



Reconstructions for $\alpha = 0$, $\alpha = 0.5$, $\alpha = 0.75$, $\alpha = 0.875$



Conclusion

- We considered several sources of uncertainty in shape optimization.
- We discussed the notion of expected domains and introduced the parametrization based expectation as well as the Vorob'ev expectation. The computations require a huge number of solutions of the shape optimization problem under consideration.
- A free boundary problem with random diffusion has been treated by minimizing a mean energy functional. This results in a high-dimensional state equation.
- Shape optimization of the expectation and/or the variance of a polynomial shape functional and a state with random right-hand side is a deterministic problem. The mean of quadratic shape functionals can be even computed without assuming a specific model for the randomness.
- Numerical results have been presented to illustrate the results.

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