Optimal Control of Hyperbolic Conservation Laws with State Constraints and Convergent Numerical Schemes for Adjoints

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Stefan Ulbrich
Department of Mathematics
TU Darmstadt

Joint work with Paloma Schäfer Aguilar, Johann M. Schmitt and Michael Moos

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Nonlinear Optimization

TRR 154

SPP 1962

Support by DFG within SPP 1962 and Project A02 in CRC TRR 154.

Outline

Motivation

Initial-boundary control problem for a balance law

Optimality conditions for the problem with state constraints

Moreau-Yosida type regularization

Convergence of numerical discretization

Summary

Outline

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Optimal control of networks for nonlinear hyperbolic conservation laws

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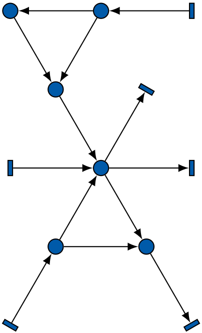
Setting

- directed graph $G = (V, E)$
- edges correspond to real intervals
- state $y = (y^i)_{e_i \in E}$



Optimal control of networks for nonlinear hyperbolic conservation laws

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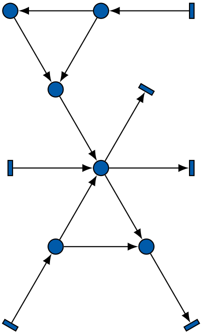
Every y^i has to satisfy...

- conservation law on I_i
- initial conditions
- node conditions
- boundary conditions



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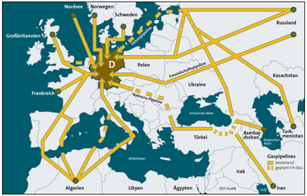

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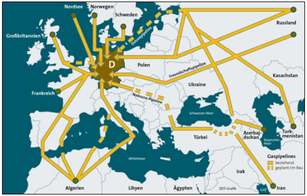

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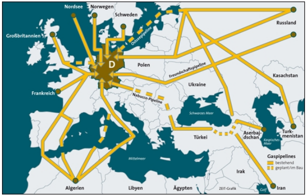
Objective Functional

$$J(y(T,\cdot)) = \sum_{e_i \in E} \int_{a_i}^{b_i} \psi_i(y_i(T,x), y_{d,i}(x))\, dx$$

Covers usual tracking-type functionals

Optimization w.r.t.

- initial value
- control of the source term
- **boundary data**
- **node conditions**
- **switching times**



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Optimal boundary control problem for conservation laws

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Optimal Control Problem

$$\min J(y(T,\cdot),u)$$

$$\text{s.t. } u=(u_0,u_B,u_1)\in U_{ad},\quad y(T,\cdot)\le\bar y,\quad y=y(u)\text{ solves}$$

$$\begin{aligned} y_t+(f(y))_x &= g(\cdot,y,u_1) && \text{on } (0,T)\times\mathbb{R}^+ =: \Omega_T,\\ y(0,\cdot) &= u_0 && \text{on } \mathbb{R}^+ =: \Omega,\\ \text{"}y(\cdot,0) &= u_B\text{" in the BLN-sense} && \text{on } (0,T). \end{aligned}$$

Assumptions:

- Source term: $g\in C\left([0,T];C^1_{\text{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^m)\right)$
- Flux: $f\in C^2_{\text{loc}}(\mathbb{R}),\quad f''\ge m_f>0$
- More details later.

Nonlinear Optimization

Applications

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Optimal control and sensitivity analysis for conservation laws is relevant, e.g., for

- Optimal control of / games on traffic networks (Bressan, Gugat, Herty, Klar, Leugering, S.U. at al.)
- Optimal control of gas and water networks (Colombo, Gugat, Herty, Leugering at al.)
- Turbomachinery aeroelastic analysis (Giles et al.)
- Optimization/optimal control of discontinuous flows (Bardos, Bressan, Gugat, Gunzburger, Heinkenschloss, Herty, Homescu, Ghattas, Giles, Leugering, Klar, Navon, Pironneau, Sager, S.U., Zuazua ...)

State constraints (pressure or velocity bounds etc.) and switching (valves, traffic lights etc.) play a role.

Related work

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- **Differentiability w.r.t. initial and boundary data:** Bressan, Guerra 97; Bouchut, James 99; S.U. 02; Colombo, Groli 02; S.U. 03; Giles 03; Bardos, Pironneau 05; Paff, S.U. 15, Pfaff, S.U. 16
- **Variational calculus for piecewise Lipschitz solutions of systems:** Bressan, Marson 95; Bressan, Shen 07
- **Convergence of discrete sensitivities and adjoints:** Gosse, James 00; S.U. 02; Giles 03; Giles, S.U. 11; Herty, Steffensen 11; Hajian, Hintermüller, S.U. 17; Schäfer Aguilar, Schmitt, S.U., Moos 19
- **Alternating descent method for optimal control of conservation laws:** Castro, Zuazua 09, 10; Lecaros, Zuazua 16
- **Networks in case of strong solutions:** Dick, Gugat, Herty, Leugering, S.U. et al.
- **Modal switchings in networks:** Hante, Leugering, Seidman 09
- **Methods for PDE-constrained optimization with state constraints:** Bergounioux, Casas, Ito, Kunisch, Tröltzsch, Hinze, Hintermüller, Rösch, M. Ulbrich, Meyer, De Los Reyes, Yousept, Krumbiegel, Neitzel, Schiela, Wollner, ...

Entropy solutions for the initial boundary value problem

Conservation Law

$$y_t + (f(y))_x = g(\cdot, y, u_1) \qquad \text{on } \Omega_T$$

Initial Value

$$y(0, \cdot) = u_0 \qquad \text{on } \mathbb{R}^+$$

Boundary Condition

$$\text{"}y(\cdot, 0) = u_B\text{"} \qquad \text{on } [0, T]$$

Entropy solutions for the initial boundary value problem

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Entropy Condition

For every convex entropy η and entropy-flux q satisfying $q' = \eta' f'$ the following inequality holds in the sense of distributions:

$$\eta(y)_t + q(y)_x \leq \eta'(y) g(t, x, y, u_1) \qquad \text{in } \mathcal{D}'(\Omega_T).$$

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Initial Value

For every $R > 0$ it holds $\lim_{t \to 0+} \|y(t, \cdot) - u_0\|_{1,(0,R)} = 0$.

Boundary Condition

$$\text{"}y(\cdot, 0) = u_B\text{"} \qquad \text{on } [0, T]$$

Nonlinear Optimization

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Boundary Condition (Bardos, LeRoux, Nédélec 1979, c.f. Le Floch 1988 and Otto 1996)

For almost all $t \in (0, T)$ it holds

$$\min_{k \in I(y(t,0+), u_B)(t)} \operatorname{sgn}(u_B(t) - y(t, 0+))(f(y(t, 0+)) - f(k)) = 0.$$

Entropy solutions for the initial boundary value problem

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$$\min_{k \in I(y(t,0+), u_B)(t)} \operatorname{sgn}(u_B(t) - y(t, 0+))(f(y(t, 0+)) - f(k)) = 0.$$

$\Rightarrow$ Existence, uniqueness, stability of solutions $y \in L^\infty(\Omega_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^+))$

An optimal control problem for IBVP with switching times

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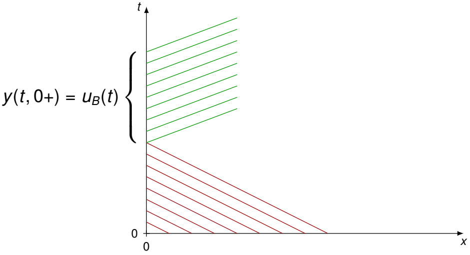
$$
\begin{aligned}
y_t + f(y)_x &= g(\cdot, y, u_1), && \text{on } \Omega_T := (0, T) \times (0, \infty),\\
y(0, \cdot) &= u_0(\cdot; w), && \text{on } \Omega := (0, \infty),\\
y(\cdot, 0+) &= u_B(\cdot; w), && \text{in the sense of Bardos, LeRoux, Nédélec (BLN)}
\end{aligned}
$$



See: [Bardos, LeRoux and Nédélec, 1979]

An optimal control problem for IBVP with switching times

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$$\begin{aligned} y_t + f(y)_x &= g(\cdot, y, u_1), && \text{on } \Omega_T := (0,T) \times (0,\infty),\\ y(0,\cdot) &= u_0(\cdot; w), && \text{on } \Omega := (0,\infty),\\ y(\cdot, 0+) &= u_B(\cdot; w), && \text{in the sense of Bardos, LeRoux, Nédélec (BLN)} \end{aligned}$$

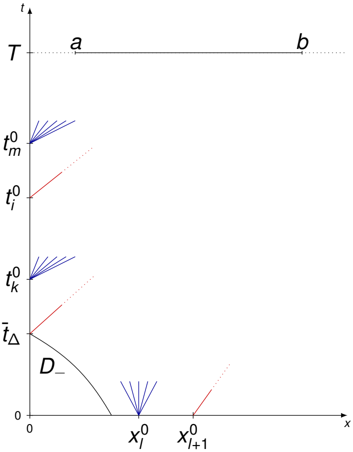
- Associate with control $w = (u^0, u^B, x^0, t^0, u_1) \in W_{ad}$ piecewise C^1 initial and boundary data

$$u_0(x; w) = \begin{cases} u_1^0(x) & \text{if } x \in [0, x_1^0],\\ u_j^0(x) & \text{if } x \in (x_{j-1}^0, x_j^0], \quad 2 \le j \le n_x,\\ u_{n_x+1}^0(x) & \text{if } x \in (x_{n_x}^0, \infty) \end{cases}$$

$$u_B(t; w) = \begin{cases} u_1^B(t) & \text{if } t \in [0, t_1^0],\\ u_j^B(t) & \text{if } t \in (t_{j-1}^0, t_j^0], \quad 2 \le j \le n_t,\\ u_{n_t+1}^B(t) & \text{if } t \in (t_{n_t}^0, T] \end{cases}$$

$$0 < x_1^0 < \ldots < x_{n_x}^0, \quad 0 < t_1^0 < \ldots < t_{n_t}^0 < T.$$

Illustration



$$I_{s,0}(w) := \{j \in \{1, \dots, n_x\} : [u_0(x_j^0)] > 0\}$$
$$I_{r,0}(w) := \{j \in \{1, \dots, n_x\} : [u_0(x_j^0)] < 0\}$$
$$I_{s,B}(w) := \{j \in \{1, \dots, n_t\} : [u_{B,0}(t_j^0)] < 0\}$$
$$I_{r,B}(w) := \{j \in \{1, \dots, n_t\} : [u_{B,0}(t_j^0)] > 0\}$$

$$[\psi(x)] := \psi(x-) - \psi(x+)$$

Assumptions

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Assumption A1:

- $f \in C^2_{\text{loc}}(\mathbb{R}),\ \exists\, m_{f''} > 0:\ f'' \geq m_{f''}$
- $g \in C\left([0, T]; C^1_{\text{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^m)\right)$ and for every $M_u > 0$ there exist $C_1, C_2 > 0$ such that

  $$g(t, x, y, u_1)\text{sgn}(y) \leq C_1 + C_2|y|$$

  for all $(t, x, , y, u_1) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times [-M_u, M_u]^m$.
- W_{ad} is nonempty and bounded in

  $$W := \{(u^0, u^B, x^0, t^0, u_1) \in C^1(\overline{\Omega})^{n_x+1} \times C^1([0, T])^{n_t+1} \times \mathcal{X} \times \mathcal{T} \times C([0, T]; C^1(\overline{\Omega})^m)\}$$

  with

  $$\mathcal{X} := \{x^0 \in \Omega^{n_x} : 0 < x^0_1 < ... < x^0_{n_x} < \infty\},$$
  $$\mathcal{T} := \{t^0 \in [0, T]^{n_t} : 0 < t^0_1 < ... < t^0_{n_t} < T\}$$

Well-posedness of the IBVP

Theorem: Let Assumption A1 hold. **Then:**

- For all $w \in W_{ad}$ the IBVP has a unique entropy solution $y(w) \in C([0,T]; L^1_{loc}(\mathbb{R}_+))$ with $y(t,.;w) \in L^\infty(\mathbb{R}_+) \cap BV_{loc}(\mathbb{R}_+)$ for all $t \in [0,T]$.
- The mapping $w \in W_{ad} \mapsto y(w) \in C([0,T]; L^1_{loc}(\mathbb{R}_+))$ is Lipschitz continuous.

See [Bardos, LeRoux, Nédélec 1979], [Le Floch 1988], [Otto 1996].

Optimal control of the IBVP with state constraints

$$
\min_{w \in W} J(y(w), w) = \int_a^b \psi(y(T, x; w), y_d(x))\, \mathrm{d}x + R(w)
$$

$$
\begin{aligned}
\text{where} \quad & y(w) \text{ solves IBVP} \\
& w \in W_{ad} \\
& y(T, \cdot; w) \leq \bar{y}(x) \quad \forall x \in [a, b]
\end{aligned}
\qquad (P)
$$

- Prove existence of an optimal solution $\bar{w} \in W_{ad}$.
- Derive necessary optimality conditions for (P).
- Analyze convergence of Moreau-Yosida type regularization.
- Convergence of numerical discretizations.

Optimal control of the IBVP with state constraints

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$$\begin{aligned} \text{where} \quad & y(w) \text{ solves IBVP} \\ & w \in W_{ad} \\ & y(T, \cdot; w) \le \bar{y}(x) \quad \forall x \in [a, b] \end{aligned} \qquad (P)$$

Difficulty to derive necessary optimality conditions:

- The mapping $w \in W_{\text{ad}} \mapsto y(T, \cdot; w) \in L^1([a, b])$ is Lipschitz continuous, but not differentiable.
- State constraints require $y(T, \cdot; w) \in L^\infty([a, b])$ for a constraint qualification
- Well known: $y(\cdot; w)$ can develop shocks after finite time

Consequence: $w \mapsto y(T, \cdot; w) \in L^\infty([a, b])$ not even continuous.

Outline

Assumptions for the optimal control problem

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Assumption A2:

- $f \in C^3_{\text{loc}}(\mathbb{R})$ and $f'^{-1} \in C^{2,\beta}_{\text{loc}}(\mathbb{R})$ for some $\beta \in (0,1]$
- $g \in C\left([0,T]; C^1_{\text{loc}}\left(\Omega \times \mathbb{R} \times \mathbb{R}^m\right)\right)$
- g is Lipschitz w.r.t. x and affine linear w.r.t. y.
- There exists $\varepsilon_g > 0$ such that

$$g(t,x,y,u_1) = 0 \quad \text{if } x \in [0, \varepsilon_g] \,.$$

- $\psi \in C^{1,1}_{\text{loc}}(\mathbb{R}^2)$, $y_d \in C(\overline{\Omega})$ and $\bar{y} \in C^1(\overline{\Omega})$

Assumptions for the optimal control problem

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Assumption A2:

- W_{ad} is convex and compact in

$$W := \{(u^0, u^B, x^0, t^0, u_1) \in C^1(\overline{\Omega})^{n_x+1} \times C^1([0,T])^{n_t+1} \times \mathcal{X} \times \mathcal{T} \times C([0,T]; C^1(\overline{\Omega})^m)\}$$

with
$$\mathcal{X} := \{x^0 \in \Omega^{n_x} : 0 < x_1^0 < \ldots < x_{n_x}^0 < \infty\},$$
$$\mathcal{T} := \{t^0 \in [0,T]^{n_t} : 0 < t_1^0 < \ldots < t_{n_t}^0 < T\}$$

- $f'(u_j^B) \geq \alpha > 0,\ j = 1, \ldots, n_t + 1$ holds for all $w = (u^0, u^B, x^0, t^0, u_1) \in W_{\mathrm{ad}}$.
- There exists $\tilde{w} \in W_{ad}$ such that $y(T, x, \tilde{w}) \leq \bar{y}(x)$ for all $x \in [a, b]$.
- $R : W \to \mathbb{R}$ is continuously Fréchet-differentiable.

$\Rightarrow$ There exists a global solution for (P).

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Nondegeneracy-condition

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Definition: $w \in W_{ad}$ satisfies the **nondegeneracy-condition (ND)** if

1. Points where $f'(y(\cdot, 0+; w))$ changes sign (inflow-outflow change) are nondegenerated and $y(\cdot, 0+; w)$ satisfies

$$\operatorname*{essinf}_{t\,:\,u_B(t,w)\neq y(t,0+;w)} \left| f(u_B(t,w)) - f(y(t,0+;w)) \right| > 0.$$

2. $y(T, \cdot; w)$ has no shock generation points on $[a, b]$,
3. $y(T, \cdot; w)$ has a finite number of nondegenerated shocks

$$a < x_1(w) < \cdots < x_K(w) < b$$

that are no shock interaction points.

Remark: One can show that 2. and 3. hold for a.a. T. (ND) is generically satisfied.

Structure of the state

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Theorem: (Pfaff 15, Pfaff, S.U. SICON 15, Schmitt, S.U. 18)
Let Assumption A2 hold and let $\bar{w} \in W_{\text{ad}}$ satisfy (ND).
Then: There exist a neighborhood $U(\bar{w}) \subset W$ of $\bar{w}, \varepsilon > 0$ and continuously F-differentiable mappings

$$U(\bar{w}) \ni w \mapsto x_k(w) \in (x_k(\bar{w}) - \frac{\varepsilon}{2}, x_k(\bar{w}) + \frac{\varepsilon}{2}), \qquad k \in \{1, \dots, K\}$$

$$U(\bar{w}) \ni w \mapsto Y_k(T, \cdot; w) \in C(x_k(\bar{w}) - \varepsilon, x_{k+1}(\bar{w}) + \varepsilon), \qquad k \in \{0, \dots, K\}$$

$$x_0 := a, \ x_{K+1} := b,$$

such that

$$y(T, x; w) \big|_{(x_k(w), x_{k+1}(w))} = Y_k(T, x; w), \quad \forall w \in U(\bar{w}), \quad k = 0, \dots, K$$

Furthermore:

- $Y_k(T, \cdot; w) \in C^1(x_k(\bar{w}) - \varepsilon, x_{k+1}(\bar{w}) + \varepsilon) \quad \forall w \in U(\bar{w}), \quad k = 0, \dots, K$
- $U(\bar{w}) \ni w \mapsto J(y(w), w) \in \mathbb{R}$ is continuously F-differentiable in $\bar{w}$

Remark: Involved result, allows for arbitrary shock structures (uses gen. charact.).

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Characteristics for an example

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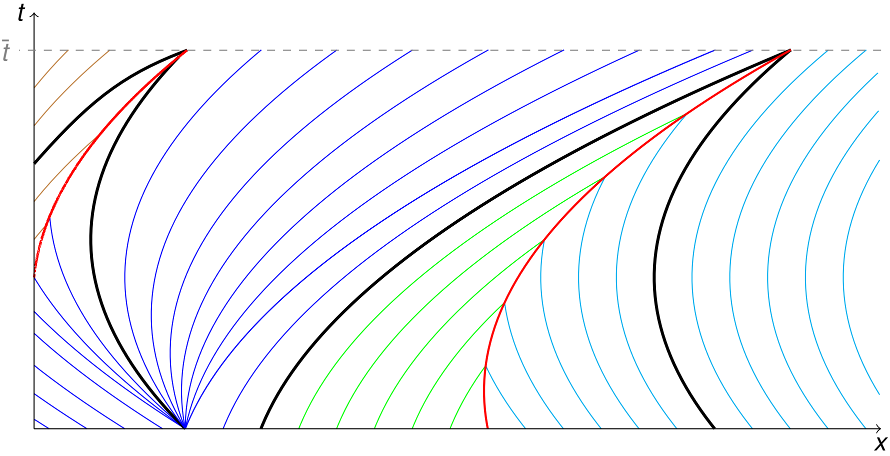

Nonlinear Optimization

Adjoint representation of the derivative of the reduced cost functional

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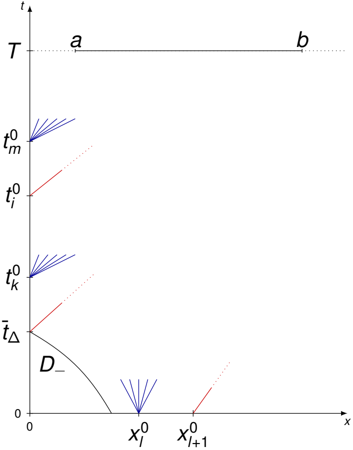

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$$[\psi(x)] := \psi(x-) - \psi(x+)$$

Adjoint representation of the derivative of the reduced cost functional

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$$\frac{d}{dw}J(y(w),w)\cdot\delta w = R'(w)\delta w + \big(p, g_{u_1}(\cdot,y,u_1)\delta u_1\big)_{2,(0,T)\times\Omega} + \sum_{j=1}^{n_x+1}(p(0,\cdot),\delta u_j^0)_{2,(x_{j-1}^0,x_j^0)}$$
$$+\sum_{j=1}^{n_t+1}(p(\cdot,0),f'(u_j^B)\delta u_j^B)_{2,(t_{j-1}^0,t_j^0)} + \sum_{j\in I_{s,0}(w)} p(0,x_j^0)[u_0(x_j)]\delta x_j$$
$$+\sum_{j\in I_{s,B}(w)} p(t_j^0,0)[f(y(t_j^0,0+;w))]\delta t_j^0 - \sum_{j\in I_{r,0}(w)} p_j^{r,0}\delta x_j^0 + \sum_{j\in I_{r,B}(w)} p_j^{r,B}\delta t_j^0,$$

where p denotes the **reversible solution** of the adjoint equation

$$p_t + f'(y)p_x = -g_y(\cdot,y,u_1)p, \qquad \text{on } \Omega_T\setminus D_-,$$
$$p(T,x) = \begin{cases} \mathbb{1}_{[a,b]}(x)\psi_y\left(y(T,x;w),y_d(x)\right) & \text{if } x \text{ is continuity point}\\ \mathbb{1}_{[a,b]}(x)\frac{[\psi(y(T,x;w),y_d(x))]}{[y(T,x;w)]} & \text{if } x \text{ is discontinuity point}\end{cases}$$

and is **equal to zero** on D_- (transport equation with OSLC coefficient).

Reversible solution: Internal boundary condition along shocks.

Adjoint representation of the derivative of the reduced cost functional

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$$\frac{d}{dw}J(y(w),w)\cdot\delta w = R'(w)\delta w + \big(p, g_{u_1}(\cdot,y,u_1)\delta u_1\big)_{2,(0,T)\times\Omega} + \sum_{j=1}^{n_x+1}(p(0,\cdot),\delta u_j^0)_{2,(x_{j-1}^0,x_j^0)}$$
$$+\sum_{j=1}^{n_t+1}(p(\cdot,0),f'(u_j^B)\delta u_j^B)_{2,(t_{j-1}^0,t_j^0)} + \sum_{j\in I_{\mathrm{s},0}(w)} p(0,x_j^0)[u_0(x_j)]\delta x_j$$
$$+\sum_{j\in I_{\mathrm{s},B}(w)} p(t_j^0,0)[f(y(t_j^0,0+;w))]\delta t_j^0 - \sum_{j\in I_{\mathrm{r},0}(w)} p_j^{\mathrm{r},0}\delta x_j^0 + \sum_{j\in I_{\mathrm{r},B}(w)} p_j^{\mathrm{r},B}\delta t_j^0,$$

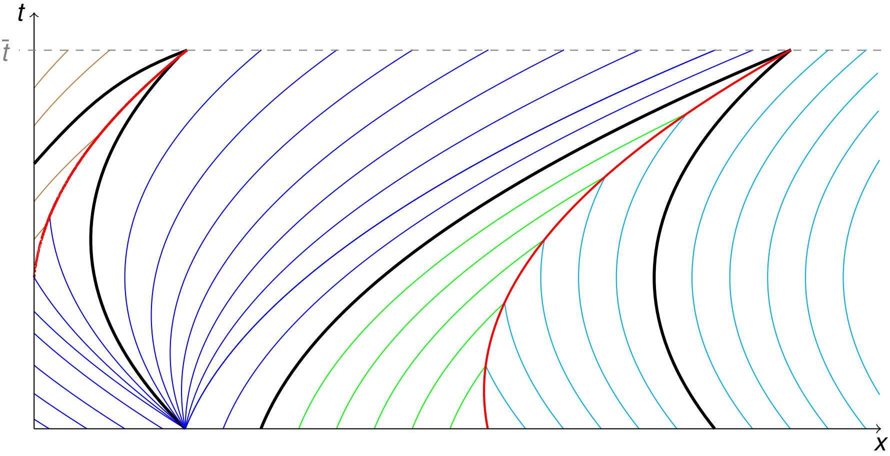
$$p_j^{\mathrm{r},0} := \int_{f'(u_j^0(x_j^0))}^{f'(u_{j+1}^0(x_j^0))} \lim_{t\searrow 0} p(t, zt + x_j^0)\frac{z}{f''(f'^{-1}(z))}\mathrm{d}z, \quad j\in I_{\mathrm{r},0},$$

$$p_j^{\mathrm{r},B} := \int_{f'(u_{j+1}^B(t_j^0))}^{f'(u_j^B(t_j^0))} \lim_{t\searrow t_j^0} p(t, z(t-t_j^0))\frac{z}{f''(f'^{-1}(z))}\mathrm{d}z, \quad j\in I_{\mathrm{r},B}.$$

See also [Pfaff, S.U., 2015], [S.U., 2003] and [Bouchut, James, 1998].

Nonlinear Optimization

Characteristics for an example



Technical tool: Reformulation of the state

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Introduce new state variables $(y_0, \ldots, y_K, x_1, \ldots, x_K)$:

$$y_k(\lambda, w) := Y_k(T, x_k(w) + \lambda(x_{k+1}(w) - x_k(w)), w), \quad \lambda \in [0,1], \; w \in U(\bar{w})$$

- The mappings

$$U(\bar{w}) \ni w \mapsto (y_0(\lambda, w), \ldots, y_K(\lambda, w), x_1(w), \ldots, x_K(w)) \in C([0,1])^{K+1} \times \mathbb{R}^K$$

  are continuously Fréchet-differentiable.

- Reformulated upper bounds $(\bar{y}_0, \ldots, \bar{y}_K)(\lambda, w)$:

$$y_k(\lambda, w) \leq \bar{y}(a + \lambda(x_{k+1}(w) - x_k(w))) =: \bar{y}_k(\lambda, w) \quad \forall k \in \{0, \ldots, K\}$$

Reformulation of optimal control problem

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$$\min J((y_0,\dots,y_K,x_1,\dots,x_K)(w),w) \quad \text{s.t. } G((y_0,\dots,y_K,x_1,\dots,x_K)(w)) \in \mathcal{K}, \quad w \in W_{\text{ad}},$$

where

$$G(y_0,\dots,y_K,x_1,\dots,x_K) = \begin{pmatrix} y_0 - \bar{y}_0 \\ \vdots \\ y_K - \bar{y}_K \\ x_1 \\ \vdots \\ x_K \end{pmatrix} \in C([0,1])^{K+1} \times \mathbb{R}^K, \quad \mathcal{K} = \begin{pmatrix} C_{\le 0}([0,1]) \\ \vdots \\ C_{\le 0}([0,1]) \\ \mathbb{R} \\ \vdots \\ \mathbb{R} \end{pmatrix}$$

Robinson's Constraint Qualification: Holds at $\bar{w} \in W_{ad}$ with $G((y,x)(\bar{w})) \in \mathcal{K}$ if

$$0 \in \operatorname{int}\left(G((y,x)(\bar{w})) + \frac{d}{dw} G((y,x)(\bar{w}))(W_{ad} - \bar{w}) - \mathcal{K} \right).$$

Optimality conditions for IBVP with state constraints

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Theorem: (Karush-Kuhn-Tucker conditions, [Schmitt, S.U. 2018])

- Assume that (A2) holds and
- $\bar{w} \in W_{ad}$ is a local solution of (P) that satisfies Robinson's CQ and (ND).

Then: $\exists$ nonneg. regular Borel measures $\mu_0, \dots, \mu_K \in \mathcal{M}([0,1])$: such that:

$$y_k(\lambda, \bar{w}) \leq \bar{y}_k(\lambda, \bar{w}) \quad \forall \lambda \in [0,1], \ \forall k \in \{0, \dots, K\} \tag{F}$$

$$\sum_{k=0}^{K} \int_{[0,1]} \Big(\bar{y}_k(\lambda, \bar{w}) - y_k(\lambda, \bar{w}) \Big) \, d\mu_k(\lambda) = 0 \tag{C}$$

$$\frac{d}{dw} J(y(\bar{w}), \bar{w})(w - \bar{w}) + \sum_{k=0}^{K} \int_{[0,1]} \frac{d}{dw} \Big(y_k(\lambda, \bar{w}) - \bar{y}_k(\lambda, \bar{w}) \Big)(w - \bar{w}) \, d\mu_k(\lambda) \geq 0, \quad \forall\, w \in W_{ad} \tag{S}$$

Robinson's CQ can be shown to hold under suitable assumptions on W_{ad} and the source term.

Formulation in the original state

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Theorem: (Karush-Kuhn-Tucker conditions, [Schmitt, S.U. 2018])

- Assume that (A2) holds and
- $\bar{w} \in W_{ad}$ is a local solution of (P) that satisfies Robinson's CQ and (ND).

Then: $\exists$ nonneg. regular Borel measures $\mu_k \in \mathcal{M}\left([x_k(\bar{w}), x_{k+1}(\bar{w})]\right)$, $0 \le k \le K$:

$$y\,(T, x, \bar{w}) \le \bar{y}\,(x) \quad \forall x \in [a, b] \tag{F}$$

$$\sum_{k=0}^{K} \int_{x_k(\bar{w})}^{x_{k+1}(\bar{w})} \Big(y(T, x, \bar{w}) - \bar{y}(x)\Big)\, d\mu_k\,(x) = 0 \tag{C}$$

Formulation in the original state

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem: (Karush-Kuhn-Tucker conditions, [Schmitt, S.U. 2018])

- Assume that (A2) holds and
- $\bar{w} \in W_{ad}$ is a local solution of (P) that satisfies Robinson's CQ and (ND).

Then: $\exists$ nonneg. regular Borel measures $\mu_k \in \mathcal{M}\left([x_k(\bar{w}), x_{k+1}(\bar{w})]\right)$, $0 \leq k \leq K$:

$$\frac{d}{dw} J\left(y(\bar{w}), \bar{w}\right)(w - \bar{w})$$

$$+ \sum_{k=0}^{K} \Bigg[\int_{x_k(\bar{w})}^{x_{k+1}(\bar{w})} \frac{\partial}{\partial x}\left[y(T, x, \bar{w}) - \bar{y}(x)\right] \frac{x - x_k(\bar{w})}{x_{k+1}(\bar{w}) - x_k(\bar{w})} \, d\mu_k(x) \cdot \frac{d}{dw} x_{k+1}(\bar{w})(w - \bar{w})$$

$$+ \int_{x_k(\bar{w})}^{x_{k+1}(\bar{w})} \frac{\partial}{\partial x}\left[y(T, x, \bar{w}) - \bar{y}(x)\right] \frac{x_{k+1}(\bar{w}) - x}{x_{k+1}(\bar{w}) - x_k(\bar{w})} \, d\mu_k(x) \cdot \frac{d}{dw} x_k(\bar{w})(w - \bar{w})$$

$$+ \int_{x_k(\bar{w})}^{x_{k+1}(\bar{w})} \frac{d}{dw} y(T, x, \bar{w})(w - \bar{w}) \, d\mu_k(x) \Bigg] \geq 0 \quad , \forall w \in W_{\mathrm{ad}} \qquad \text{(S)}$$

Formulation in the original state

Theorem: (Karush-Kuhn-Tucker conditions, [Schmitt, S.U. 2018])

- Assume that (A2) holds and
- $\bar{w} \in W_{ad}$ is a local solution of (P) that satisfies Robinson's CQ and (ND).

Then: $\exists$ nonneg. regular Borel measures $\mu_k \in \mathcal{M}([x_k(\bar{w}), x_{k+1}(\bar{w})])$, $0 \le k \le K$:

$$
\begin{aligned}
&\frac{d}{dw} J(y(\bar{w}), \bar{w})(w - \bar{w}) \\
&+ \sum_{k=0}^{K} \Bigg[\frac{\partial}{\partial x}[y(T, x_{k+1}(\bar{w})-, \bar{w}) - \bar{y}(x_{k+1}(\bar{w}))] \cdot \mu_k\big(\{x_{k+1}(\bar{w})\}\big) \cdot \frac{d}{dw} x_{k+1}(\bar{w}) \cdot (w - \bar{w}) \\
&\qquad + \frac{\partial}{\partial x}[y(T, x_k(\bar{w})+, \bar{w}) - \bar{y}(x_k(\bar{w}))] \cdot \mu_k\big(\{x_k(\bar{w})\}\big) \cdot \frac{d}{dw} x_k(\bar{w}) \cdot (w - \bar{w}) \\
&\quad + \int_{x_k(\bar{w})}^{x_{k+1}(\bar{w})} \frac{d}{dw} y(T, x, \bar{w})(w - \bar{w})\, d\mu_k(x) \Bigg] \ge 0 \quad , \forall w \in W_{ad} \qquad \text{(S)}
\end{aligned}
$$

$\frac{d}{dw} J(y(\bar{w}), \bar{w})$ and $\frac{d}{dw} x_k(\bar{w})$ can be expressed by using an adjoint state.

Outline

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Moreau-Yosida type regularization

Approximate (P) by:

$$\min_{w \in W} J_\gamma(y(w), w) := J(y(w), w) + \frac{1}{2\gamma} \int_a^b (y(T, x; w) - \bar{y}(x))_+^2 \, \mathrm{d}x \qquad (P_\gamma)$$

where $y(w)$ solves IBVP

$w \in W_{ad}$

See, e.g., [Ito, Kunisch, 2003], [Hintermüller, Kunisch, 2005] [Hintermüller, Hinze, 2009], [Meyer, Yousept, 2009] ...

For alternative approaches, see for example: [Hinze, Meyer, 2008], [Krumbiegel, Neitzel, Rösch 2010], ...

Moreau-Yosida type regularization

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Approximate (P) by:

$$\min_{w \in W} J_\gamma(y(w), w) := J(y(w), w) + \frac{1}{2\gamma} \int_a^b (y(T, x; w) - \bar{y}(x))_+^2 \, \mathrm{d}x \qquad (P_\gamma)$$

where $y(w)$ solves IBVP

$w \in W_{ad}$

Let (A2) hold **then**

- For all $\gamma > 0$ there exists a global solution w_γ of P_γ.
- If $\bar{w} \in W_{ad}$ satisfies (ND), then $W_{ad} \ni w \mapsto J_\gamma(y(w))$ is continuously differentiable in $\bar{w}$ with the above adjoint representation of the gradient.

Moreau-Yosida type regularization: Convergence result

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Theorem:

- Assume that (A1) holds and
- $(w_{\gamma_l})_{l\in\mathbb{N}} \subset W_{ad}$ is a sequence of local solutions of (P_{γ_l}) with $\lim_{l\to\infty} \gamma_l = 0$.
- There exist $\varepsilon, \delta > 0$ such that for all $l \in \mathbb{N}$ and all $w \in W_{ad}$ with $\|w - w_{\gamma_l}\|_W < \varepsilon$ it holds

$$J_{\gamma_l}(y(w_{\gamma_l}), w_{\gamma_l}) + \frac{\delta}{2}\|w - w_{\gamma_l}\|_H^2 \leq J_{\gamma_l}(y(w), w), \qquad (QGC)$$

where H Hilbert space with $W \hookrightarrow H$.

Then: There exists a subsequence $(w_{\gamma_l})_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} w_{\gamma_l} = w^* \quad \text{and} \quad w^* \text{ is a local solution of } (P).$$

See: [Meyer, Yousept, 2009], [De Los Reyes, Yousept, 2009]

Moreau-Yosida type regularization: Convergence result

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem:

- Assume that (A1) holds and
- $(w_{\gamma_l})_{l\in\mathbb{N}} \subset W_{ad}$ is a sequence of local solutions of (P_{γ_l}) with $\lim_{l\to\infty} \gamma_l = 0$.
- There exist $\varepsilon, \delta > 0$ such that for all $l \in \mathbb{N}$ and all $w \in W_{ad}$ with $\|w - w_{\gamma_l}\|_W < \varepsilon$ it holds

$$J_{\gamma_l}(y(w_{\gamma_l}), w_{\gamma_l}) + \frac{\delta}{2}\|w - w_{\gamma_l}\|_H^2 \le J_{\gamma_l}(y(w), w), \qquad (QGC)$$

where H Hilbert space with $W \hookrightarrow H$.

Then: There exists a subsequence $(w_{\gamma_l})_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} w_{\gamma_l} = w^* \quad \text{and} \quad w^* \text{ is a local solution of } (P).$$

Remark: If $(w_{\gamma_l})_{l\in\mathbb{N}} \subset W_{ad}$ are global solutions, then (QGC) not necessary.

Necessary optimality conditions for (P_γ)

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Theorem:

- Assume that (A2) holds and
- $w_\gamma \in W_{ad}$ is a local solution for (P_γ) with $\gamma > 0$ satisfsfying (ND).

Then it holds

$$\frac{d}{dw} J_\gamma(y(w_\gamma)) \cdot \left(w - w_\gamma\right) \geq 0 \quad \forall w \in W_{ad}. \tag{1}$$

Define the Lagrange multiplier estimates:

$$\lambda_k(x; w_\gamma) = \begin{cases} \dfrac{\left(y(T, x; w_\gamma) - \bar{y}(x)\right)_+}{\gamma}, & \text{for } x_k(w_\gamma) \leq x \leq x_{k+1}(w_\gamma), \\ 0, & \text{else.} \end{cases}$$

Convergence of Lagrange multiplier estimates

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Theorem: [Schmitt, S.U. 2018]

- Assume that (A2) holds and
- $(w_{\gamma_l})_{l\in\mathbb{N}} \subset W_{ad}$ is sequence of local solutions of (P_{γ_l}) satisfying (ND) with

$$\lim_{l\to\infty} w_{\gamma_l} = \bar{w},$$

where $\bar{w}$ is a local solution for (P) such that Robinson's CQ is satisfied.

Then: There exists a subsequence $(\gamma_l)_{l\in\mathbb{N}}$, such that

$$y(\cdot; w_{\gamma_l}) \to y(\cdot; \bar{w}) \ \text{ in } \ C([0,T]; L^1_{\text{loc}}(\Omega)),$$

$$\lambda_k(\cdot, w_{\gamma_l}) \xrightarrow{w^*} \mu_k(\cdot) \ \text{ in } \ \mathcal{M}([a,b]), \quad \forall k = 0, \dots, K,$$

Furthermore: $(\bar{w}, \mu_0, \dots, \mu_K)$ satisfy the KKT-conditions for (P).

Nonlinear Optimization

Convergence of Lagrange multiplier estimates

Theorem: [Schmitt, S.U. 2018]

- Assume that (A2) holds and
- $(w_{\gamma_l})_{l\in\mathbb{N}} \subset W_{ad}$ is sequence of local solutions of (P_{γ_l}) satisfying (ND) with

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where $\bar{w}$ is a local solution for (P) such that Robinson's CQ is satisfied.

Then: There exists a subsequence $(\gamma_l)_{l\in\mathbb{N}}$, such that

$$y(\cdot; w_{\gamma_l}) \to y(\cdot; \bar{w}) \text{ in } C([0,T]; L^1_{\text{loc}}(\Omega)),$$

$$\lambda_k(\cdot, w_{\gamma_l}) \xrightarrow{\text{w}^*} \mu_k(\cdot) \text{ in } \mathcal{M}([a,b]), \quad \forall k = 0, \dots, K,$$

$$\lambda_k\left(x_k(w_{\gamma_l}) + (x - x_k(\bar{w}))\frac{x_{k+1}(w_{\gamma_l}) - x_k(w_{\gamma_l})}{x_{k+1}(\bar{w}) - x_k(\bar{w})}\right) \xrightarrow{\text{w}^*} \mu_k(\cdot) \text{ in } \mathcal{M}([x_k(\bar{w}), x_{k+1}(\bar{w})]).$$

Furthermore: $(\bar{w}, \mu_0, \dots, \mu_K)$ satisfy the KKT-conditions for (P).

Outline

Optimal control problem for IVP: Convergence of discretizations

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Objective function: $J(y(u), u) = \int_{\mathbb{R}} \omega(x)\psi(y(T, x; u), y_d(x))\, dx, \quad \omega \in C_c^1(\mathbb{R})$

State equation: $y_t + (f(y))_x = 0 \quad \text{on } (0, T) \times \mathbb{R}, \qquad y(0, \cdot) = u_0 \quad \text{on } \mathbb{R}.$

Adjoint equation: p reversible solution of

$$p_t + f'(y)p_x = 0, \quad \text{on } (0, T) \times \mathbb{R},$$

$$p(T, x) = \begin{cases} \omega(x)\psi_y\,(y(T, x; u), y_d(x)) & \text{if } x \text{ is continuity point} \\ \omega(x)\frac{[\psi(y(T,x;u),y_d(x))]}{[y(T,x;u)]} & \text{if } x \text{ is discontinuity point} \end{cases}$$

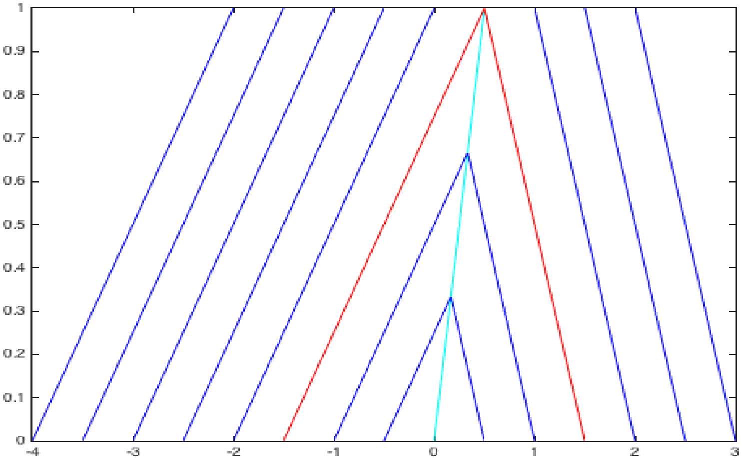
Reversible solution: Define the generalized forward characteristics

$$\frac{d}{ds}X(s; t, x) \in [f'(y(s, X(s; t, x)+)), f'(y(s, X(s; t, x)-))], \quad s \in [t, T], \quad X(t; t, x) = x.$$

Then the reversible solution is uniquely defined by

$$p(s, X(s; t, x) = p^T(T, X(T; t, x)), \quad s \in [t, T].$$

Characteristics for an example



Discrete approximation (1)

Let $\lambda > 0$ be fixed and set for a grid size $h > 0$

$$\Delta t = \lambda h, \quad t_n := n\Delta t, \quad x_j := jh.$$

Conservative finite difference scheme for the IVP:

$$y_j^{n+1} = y_j^n - \lambda \Delta^+ F_{j-\frac{1}{2}}^n =: H(y_{j-1}^n, y_j^n, y_{j+1}^n), \quad j \in \mathbb{Z}, \quad n = 0, \dots, N_T - 1,$$

$$y_j^0 = u_j, \quad j \in \mathbb{Z},$$

with a numerical flux $F_{j-\frac{1}{2}}^n := F(y_{j-1}^n, y_j^n)$, $F(y, y) = f(y)$, $\Delta^+ F_{j-\frac{1}{2}}^n := F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n$.

Engquist-Osher scheme: For $\bar{y} \in \mathbb{R}$ fixed set

$$F^{EO}(y_0, y_1) = f(\bar{y}) + \int_{\bar{y}}^{y_0} \max(0, f'(y))\, dy + \int_{\bar{y}}^{y_1} \min(0, f'(y))\, dy.$$

Modified Lax-Friedrichs scheme:

$$F^{LF}(y_0, y_1) = \frac{1}{2}\left(f(y_0) + f(y_1) - \frac{\gamma}{\lambda}(y_1 - y_0)\right), \quad \gamma \in [\lambda \max_{|y| \le M_y} |f'(y)|, 1).$$

Discrete approximation (2)

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Discrete state and control: With $R_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad Q_j^n := [t_n, t_{n+1}) \times R_j$ set

$$y_h(t,x) := \sum\nolimits_{n\geq 0,j} y_j^n 1_{Q_j^n}(t,x), \quad u_h(x) := \sum\nolimits_j u_j 1_{R_j}(x)$$

Discrete objective function:

$$u_h \mapsto J^h(y_h) := \sum\nolimits_j h\,\omega(x_j)\,\psi(y_j^{N_T}, y_{d,j}), \quad y_{d,j} := \frac{1}{h}\int_{R_j} y_d(x)\,dx.$$

Corresponding discrete adjoint scheme:

$$p_j^n = p_j^{n+1} + \lambda \sum_{k=0}^{1} (\partial_{y_j^n} F_{j-k+\frac{1}{2}}^n) \Delta^+ p_{j-k}^{n+1}, \quad j \in \mathbb{Z}, \quad n = N_T, \dots, 1,$$

$$p_j^{N_T} = \omega(x_j)\,\partial_{y_j^{N_T}} \psi(y_j^{N_T}, y_{d,j}).$$

Properties of the discrete state (1)

Finite difference scheme for the IVP:

$$y_j^{n+1} = y_j^n - \lambda \Delta^- F^n_{j+\frac{1}{2}} =: H(y^n_{j-1}, y^n_j, y^n_{j+1}), \quad j \in \mathbb{Z}, \quad n = 0, \dots, N_T - 1,$$

$$y_j^0 = u_j, \quad j \in \mathbb{Z}, \quad u_j = \frac{1}{h} \int_{R_j} u(x)\, dx.$$

Theorem SC. Consider a monotone scheme, i.e. $H(y^n_{j-1}, y^n_j, y^n_{j+1})$ is monotone increasing in each argument. Then for any $u, \hat{u} \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$

1. $\|y_h(t,\cdot;u)\|_\infty \le \|u_h\|_\infty \le \|u\|_\infty \quad \forall\, t \in [0,T]$
2. $\|y_h(t,\cdot;u_h) - y_h(t,\cdot;\hat{u}_h)\|_1 \le \|u_h - \hat{u}_h\|_1 \le \|u - \hat{u}\|_1 \quad \forall\, t \in [0,T]$
3. $|y_h(t,\cdot;u_h)|_{TV} \le |u_h|_{TV} \le |u|_{TV} \quad \forall\, t \in [0,T]$
4. $y_h \to y$ in $L^\infty(0,T;L^1_{loc}(\mathbb{R}))$ as $h \searrow 0$ with the entropy solution $y = y(u)$ of IVP.
5. There exists a constant $C(t) > 0$ such that

$$\|y^h(t,\cdot;u_h) - y(t,\cdot;u)\|_1 \le C(t)\, |u|_{TV} h^{1/2} \quad \forall\, t \in [0,T],\ 0 < h \le h_0.$$

Proof: See, e.g., Crandall, Majda 1980, Kuznecov 1976.

Properties of the discrete state (2)

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Discrete one-sided Lipschitz condition (DOSLC):
As the entropy solution $y = y(u)$, also y_h satisfies e.g. for modified Lax-Friedrichs and Enquist-Osher scheme, under a CFL-cond. ($\lambda = \frac{\Delta t}{h}$ small enough) for a $\beta > 0$

$$\frac{\Delta^+ y_j^n}{h} \leq \frac{1}{M_{u'}^{-1} + \beta n \Delta t} \quad \forall j \in \mathbb{Z},\ n = 0, \ldots, N_T - 1, \quad \text{where } u' \leq M_{u'} \in (0, \infty].$$

Interpolation between the OSLC and the L^1-norm yields

Theorem. Let the state scheme satisfy the DOSLC. Then for any $t > 0$ and $x \in \mathbb{R}$ there exists a constant $C(t) > 0$ such that

$$|y(t,x) - y_h(t,x)| \leq C(t) \left(1 + \max_{|\xi - x| \leq h^{1/3}} |y_x(t,\xi)| \right) h^{1/3}.$$

Proof: See Nessyahu, Tadmor 1992.

Nonlinear Optimization

Convergence of the discrete adjoint: Lipschitz end data

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Discrete adjoint scheme: Consider first Lipschitz end data $p^T \in C^{0,1}(\mathbb{R})$.

$$p_j^n = p_j^{n+1} + \lambda \sum_{k=0}^{1} (\partial_{y_j^n} F^n_{j-k+\frac{1}{2}}) \Delta^+ p^{n+1}_{j-k}, \quad j \in \mathbb{Z}, \; n = N_T, \dots, 1, \qquad p_j^{N_T} = \frac{1}{h} \int_{R_j} p^T(x)\, dx.$$

Theorem. Let $u \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Consider the EO-scheme with $1/2$-CFL condition or modified LF-scheme with $\min(\gamma, 1-\gamma)$-CFL condition. Then

1. $y_h \to y$ in the sense of Theorem SC and satisfies DOSLC.
2. $\|p_h(t, \cdot; u)\|_\infty \le \|p_h^T\|_\infty \le \|p^T\|_\infty \quad \forall\, t \in [0, T]$
3. $|p_h(t, \cdot)|_{TV} \le |p_h^T|_{TV} \le |p^T|_{TV} \quad \forall\, t \in [0, T]$
4. If $u' \le M_{u'} < \infty$ then there is $C > 0$ such that $|p_h(t, \cdot)|_{\mathrm{Lip}_h} \le C\|(p^T)'\|_\infty$ and

   $$p_h \to p \text{ uniformly on any compact subset of } [0, T] \times \mathbb{R}$$

   with the reversibel solution p of the adjoint equation.

   Else this holds outside of any neighborhood of the up-jumps of $y(0, \cdot) = u$.

Proof: See, e.g., S.U. 2001, Schäfer Aguilar, Schmitt, S.U., Moos 2019.

Convergence of the discrete adjoint: Discontinuous end data

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The end data in the adjoint equation are

$$p(T,x) = \begin{cases} \omega(x)\psi_y\,(y(T,x;u),y_d(x)) & \text{if } x \text{ is continuity point} \\ \omega(x)\frac{[\psi(y(T,x;u),y_d(x))]}{[y(T,x;u)]} & \text{if } x \text{ is discontinuity point} \end{cases}$$

The value at the discontinuity points is propagated within the whole shock funnel. The discrete adjoint scheme does usually not converge to the correct value.

Possible approaches to achieve convergence:

- Use modified LF-scheme with numerical viscosity $O(h^\alpha)$, $2/3 < \alpha < 1$, i.e., with $\lambda = \frac{\Delta t}{h} = O(h^{1-\alpha})$, see Giles, S.U. 2010.
- Use modified end data for the discrete adjoint scheme, Schäfer Aguilar, Schmitt, S.U., Moos 2019.

Convergence of the discrete adjoint: Discontinuous end data

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The end data in the adjoint equation are

$$p^T(x) = p(T,x) = \begin{cases} \omega(x)\,\psi_y\,(y(T,x;u), y_d(x)) & \text{if } x \text{ is continuity point} \\ \omega(x)\,\frac{[\psi(y(T,x;u),y_d(x))]}{[y(T,x;u)]} & \text{if } x \text{ is discontinuity point} \end{cases}$$

Algorithm $p_h^{T,r}$ Given $r > 0$ small do

- Compute the discrete state y_h and approximate shock locations x_k^h, $k = 1, \dots, K$, of $y_h(T,\cdot)$ as midpoints of the K regions with $\Delta^+ y_j^{N_T} = -O(\sqrt{h})$.
- Define the weight function $w^r(x) = \begin{cases} 1 & \text{if } |x| \le r, \\ \max\left\{\frac{2r-|x|}{r}, 0\right\} & \text{if } |x| > r. \end{cases}$
- Set $\quad p^T_{x_k^h} = \omega(x_k^h)\,\frac{\psi(y_h(T,x_k^h+h^{1/3}),y_d(x_k^h))-\psi(y_h(T,x_k^h-h^{1/3}),y_d(x_k^h))}{y_h(T,x_k^h+h^{1/3})-y_h(T,x_k^h-h^{1/3})}$.
- Now approximate p^T by

$$p_j^{N_T,r} = \begin{cases} \omega(x_j)\,\psi_y(y_j^{N_T}, y_{d,j}) & \text{if } |x_j - x_k^h| > 2r, \\ w^r(x_j - x_k^h)p^T_{x_k^h} + (1 - w^r(x_j - x_k^h))\omega(x_j)\,\psi_y(y_j^{N_T}, y_{d,j}) & \text{else.} \end{cases}$$

Convergence of the discrete adjoint: Discontinuous end data

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Theorem. Let $u \in L^1(\mathbb{R}) \cap PC^1(\mathbb{R})$. Consider the EO-scheme with $1/2$-CFL condition or modified LF-scheme with $\min(\gamma, 1-\gamma)$-CFL condition.

Then: There exists a piecewise constant function $r(h) > 0$ with $r(h) \to 0$ as $h \to 0$ such that: adjoint scheme with end data $p_j^{N_T, r(h)}$ obtained from Algorithm $p_h^{T,r}$ yields

$$p_h \to p \text{ in } C([0,T]; L^1_{loc}(\mathbb{R})) \text{ and boundedly everywhere on } [0,T] \times \mathbb{R} \text{ as } h \to 0$$

with the unique reversible solution p of the adjoint equation.

Proof: See Schäfer Aguilar, Schmitt, S.U., Moos 2019.

Choice of $r(h)$: For the EO-scheme and a stationary Riemann problem we proved that the choice $r(h) = O(h^\alpha)$ with $\alpha \in [1/3, 1/2)$ is possible in the above theorem.

Remark: One can also use y_h of any convergent scheme satisfying a DOSLC.

Numerical example

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Consider Burgers equation, i.e., $f(y) = y^2/2$.

Initial data: $u(x) = \begin{cases} 2 & \text{for } x \leq 0, \\ -1 & \text{for } x > 0 \end{cases}.$

Objective function: $J(y) = \int_{\mathbb{R}} \omega(x) \frac{y(1,x)^2}{2} \, dx, \quad \omega \in C_c^1(\mathbb{R}), \; \omega \equiv 1 \text{ on } [-2, 2].$

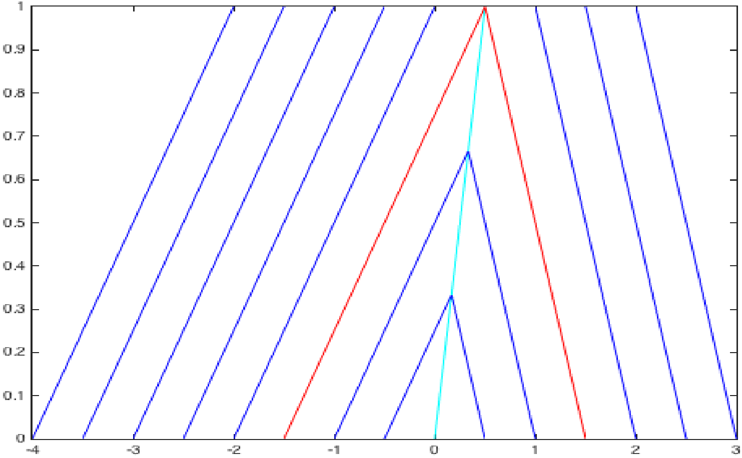
Entropy solution: Has a single shock with speed $s = 1/2$ and is given by

$$y(t,x) = \begin{cases} 2 & \text{for } x \leq t/2, \\ -1 & \text{for } x > t/2. \end{cases}$$

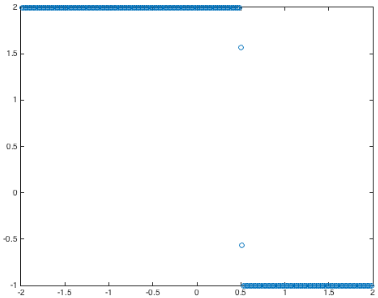
Adjoint state: The reversible solution of the adjoint equation on $[0, T] \times [-2, 2]$ is

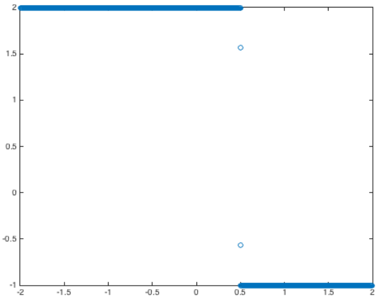
$$p(t,x) = \begin{cases} 2 & \text{for } -2 \leq x < 1/2 - 2(1-t), \\ -1 & \text{for } 1/2 + (1-t) < x \leq 2, \\ \frac{1}{2} & \text{for } 1/2 - 2(1-t) \leq x \leq 1/2 + (1-t). \end{cases}$$

Characteristics of the state



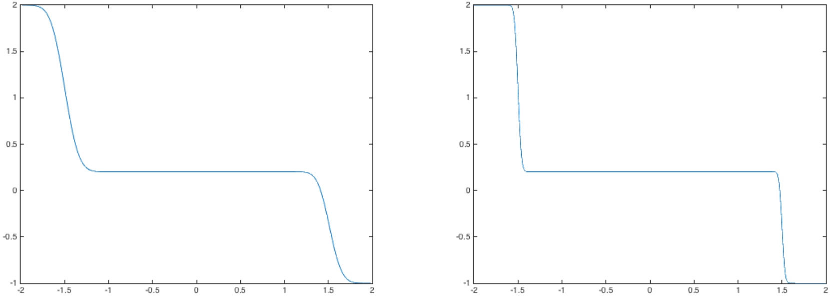
Discrete state
EO-scheme $1/2$-CFL





Discrete state EO-scheme $y_h(1,\cdot)$ for $h = 2^{-6}$ (left), $h = 2^{-10}$ (right).

Discrete adjoint: Original end data
EO-scheme $1/2$-CFL



Discrete adjoint EO-scheme $p_h(0,\cdot)$ for $h = 2^{-6}$ (left), $h = 2^{-10}$ (right) original end data. No convergence to p in shock funnel!

Discrete adjoint: Preprocessed end data

EO-scheme $1/2$-CFL, $r(h) = h^{9/20}$

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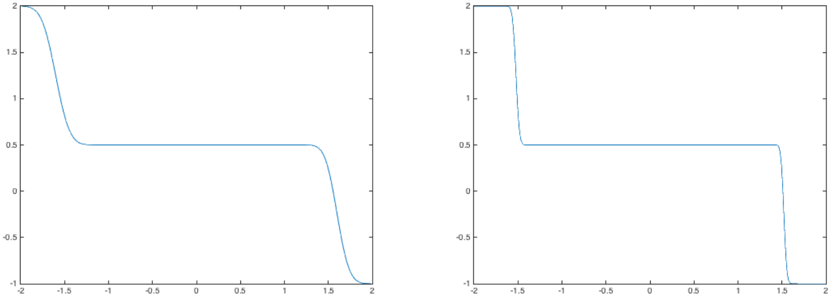

Discrete adjoint EO-scheme $p_h(0, \cdot)$ for $h = 2^{-6}$ (left), $h = 2^{-10}$ (right) preprocessed end data .

Errors for the discrete adjoints

h	$\|(p_h - p)(0)\|_{L^1}$ orig. data	$\|(p_h - p)(0)\|_{L^1}$ proc. data	exp. ord. of conv.
2^{-6}	1.0749	0.3785	
2^{-7}	1.0194	0.2579	0.5536
2^{-8}	0.9815	0.1856	0.4741
2^{-9}	0.9552	0.1273	0.5443
2^{-10}	0.9369	0.0887	0.5215

Current work

Current work for the initial-boundary value control problem:

- Extend above results to boundary control.
- Characterization of reversible solutions of the adjoint equation by monotonicity properties (with P. Schäfer Aguilar).
- Higher order methods for the adjoint?
- Extension to networks to handle Nash equilibrium problems on networks (with M. Ulbrich, M. Moos, J. Wachter)

Current work for systems of conservation laws:

- Analogous differentiability result for generalized Riemann problem and piecewise C^1-solutions. See also revious results for directional variational calculus by (Bressan, Marson 1995, Bressan, Shen 2007).
- Adjoint representation of reduced gradients for objective functions.
- The results for state constraints can then be extended to systems.
- Consider numerical approximations in the case of piecewise C^1-solutions.

Summary

- Sensitivity analysis of boundary control for hyperbolic conservation laws, especially for controls with switching times
- Necessary optimality conditions for problems with state constraints
- Convergence of Moreau-Yosida regularization
- Convergence of numerical approximations of the optimal control problem

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Thank you for your attention!

Optimality conditions for (P_γ)

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$$
\begin{aligned}
&\tfrac{d}{dw} J\left(y(w_\gamma), w_\gamma\right)(w - w_\gamma) \\
&+ \sum_{k=0}^{K} \Bigg[\int_{x_k(w_\gamma)}^{x_{k+1}(w_\gamma)} \tfrac{\partial}{\partial x}\left[y(T,x,w_\gamma) - \bar{y}(x)\right] \tfrac{x_{k+1}(w_\gamma) - x}{x_{k+1}(w_\gamma) - x_k(w_\gamma)} \lambda_k(x, w_\gamma)\, \mathrm{d}x \cdot \tfrac{d}{dw} x_k(w_\gamma)(w - w_\gamma) \\
&\quad + \int_{x_k(w_\gamma)}^{x_{k+1}(w_\gamma)} \tfrac{\partial}{\partial x}\left[y(T,x,w_\gamma) - \bar{y}(x)\right] \tfrac{x - x_k(w_\gamma)}{x_{k+1}(w_\gamma) - x_k(w_\gamma)} \lambda_k(x, w_\gamma)\, \mathrm{d}x \cdot \tfrac{d}{dw} x_{k+1}(w_\gamma)(w - w_\gamma) \\
&\quad + \int_{x_k(w_\gamma)}^{x_{k+1}(w_\gamma)} \tfrac{d}{dw} y(T,x,w_\gamma)(w - w_\gamma) \lambda_k(x, w_\gamma)\, \mathrm{d}x \\
&\quad + \underbrace{\int_{x_k(w_\gamma)}^{x_{k+1}(w_\gamma)} \frac{\left(y(T,x,w_\gamma) - \bar{y}(x)\right)_+^2}{2\gamma\left(x_{k+1}(w_\gamma) - x_k(w_\gamma)\right)}\, \mathrm{d}x \cdot \tfrac{d}{dw}\left(x_{k+1}(w_\gamma) - x_k(w_\gamma)\right)}_{\to\, 0 \text{ for } \gamma \to 0 \text{ can be proven}} \left(w - w_\gamma\right) \Bigg] \geq 0
\end{aligned}
$$

Sketch of the proof

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Proof of the first assertion (similar to the usual procedure)

- Show by using Robinson's CQ that $\lambda_0(\cdot, w_{\gamma_l}), \dots, \lambda_K(\cdot, w_{\gamma_l})$ are uniformly bounded in $L^1([a, b])$.
- Hence, there exists a subsequence w_{γ_l} and nonnegative Borel measures $\mu_0, \dots, \mu_K \in \mathcal{M}([a, b])$

$$\lambda_k(\cdot, w_{\gamma_l}) \xrightarrow{w^*} \mu_k(\cdot) \in \mathcal{M}([a, b]), \quad k = 0, \dots, K. \qquad (*)$$

Moreover, one can show

$$\lambda_k \left(x_k(w_{\gamma_l}) + (x - x_k(\bar{w})) \frac{x_{k+1}(w_{\gamma_l}) - x_k(w_{\gamma_l})}{x_{k+1}(\bar{w}) - x_k(\bar{w})} \right) \xrightarrow{w^*} \mu_k(\cdot) \in \mathcal{M}([x_k(\bar{w}), x_{k+1}(\bar{w})]).$$

Proof of the second assertion:

- $\bar{w}$ local solution for (P) $\Rightarrow$ (F)
- Regularity of the extensions $Y_k(T, \cdot, w_\gamma)$, $k = 0, \dots, K$, and $(*)$ $\Rightarrow$ (C),(S)