Optimization of Time Delays in a Parabolic Delay Equation

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New trends in PDE constrained optimization

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Joint work with

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Martin Gugat	(Erlangen, Germany)
Mariano Mateos	(Gijón, Spain)

Outline

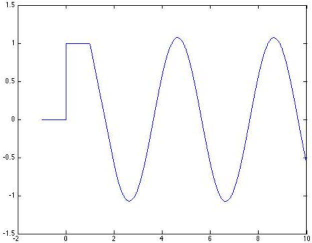
A linear ODE with time delay

$$
\begin{aligned}
y'(t) &= \kappa\, y(t-1), \quad && t > 0 \\
y(t) &= y_0(t), && -1 \le t \le 0.
\end{aligned}
$$

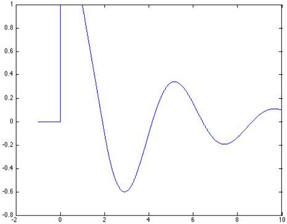
T. Erneux,

Applied delay differential equations,

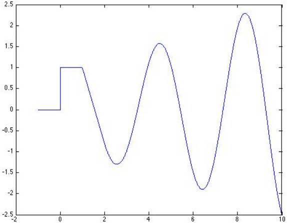
Springer, 2009



$\kappa = -\pi/2$



$\kappa = -1.8$, $y_0(0) = 1$, $y_0(t) = 0$, $t < 0$



$\kappa = -1.1$

Nonlinear ODE with delay

We consider nonlinear equations with cubic nonlinearity, e.g.

$$\begin{aligned} y'(t) + y^3(t) &= \kappa\, y(t - \tau), \quad t > 0 \\ y(t) &= y_0(t), \qquad -1 \le t \le 0. \end{aligned}$$

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Instead of $R(y) = y^3$, consider more general reaction terms like

$$R(y) = (y - y_1)(y - y_2)(y - y_3)$$

and

$$y'(t) + R(y(t)) = \kappa\, y(t-\tau).$$

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Pyragas feedback control:

$$y'(t) + R(y(t)) = \kappa\,(y(t-\tau) - y(t)).$$

PDE case

So far, we had $y : [0, T] \to \mathbb{R}$. Let y also depend on a spatial variable $x \in \Omega \subset \mathbb{R}^n$, $y = y(x, t)$, and consider

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$$\begin{aligned}
(\partial_t y - \Delta_x y + R(y))(x, t) &= \kappa\, y(x, t - \tau) && \text{in } \Omega \times (0, T) \\
y &= y_0, && \text{in } \Omega \times [-\tau, 0] \\
\partial_n y &= 0 && \text{in } \partial\Omega \times (0, T).
\end{aligned}$$

Reaction term:

$$R(y) = \rho\,(y - y_1)(y - y_2)(y - y_3),$$

$\rho > 0,\ y_1 \le y_2 \le y_3.$

Let

$$m_R := \min_y R'(y).$$

Multiple time delays

More general are multiple time delays $\quad 0 \le \tau_1 < \tau_2 \ldots < \tau_m \le T$,

$$
\begin{aligned}
(\partial_t y - \Delta y + R(y))(x,t) &= \sum_{i=1}^m \kappa_i \, y(x, t-\tau_i) && (x,t) \in Q = \Omega \times (0,T) \\
y &= y_0 && \text{in } Q_- = \Omega \times [-T, 0] \\
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$$u(x,t) = \sum_{i=1}^m \kappa_i \, y(x, t - \tau_i).$$

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We will optimize the weights κ_i and the delays τ_i for fixed m.

Set $\quad \tau := (\tau_1, \ldots, \tau_m), \quad \kappa := (\kappa_1, \ldots, \kappa_m), \quad u := (\tau, \kappa)$.

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Nonlocal problem with Borel measure $\mu \in \mathcal{M}[0, T]$

$$\begin{aligned}
\partial_t y(x,t) - \Delta y(x,t) + R(y(x,t)) &= \int_0^T y(x,t-s)\,d\mu(s) && (x,t) \in Q \\
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Assume $y_0 \in C(\overline{Q_-})$.

The nonlocal problem with measures

Theorem (Casas, Mateos, Tr. 2017)

For all $T > 0$ and every $\mu \in \mathcal{M}[0,T]$, the nonlocal problem has a unique solution $y_\mu \in \mathcal{Y} = W(0,T) \cap C(\bar{Q})$. We have

$$\|y_\mu\|_{L^2(0,T;H^1(\Omega))} \le C\big(\|y_0\|_{L^2(Q_-)}\|\mu\|_{\mathcal{M}[0,T]} + \|y_0(\cdot,0)\|_{L^2(\Omega)} + |R(0)|\big)$$

$$\|y_\mu\|_{C(\bar{Q})} \le C\big(\|y_0\|_{C(\bar{Q}_-)}\|\mu\|_{\mathcal{M}[0,T]} + \|y_0(\cdot,0)\|_{C(\bar{\Omega})} + |R(0)|\big),$$

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$\Longrightarrow$

Theorem (Casas, Mateos, Tr. 2018)

To each weight $\kappa \in \mathbb{R}^m$ and delay $\tau \in \mathbb{R}^m$, there exists a unique solution $y_{\tau,\kappa} \in \mathcal{Y}$. The mapping $(\tau,\kappa) \mapsto y_{\tau,\kappa}$ is continuous from $\mathbb{R}^{2m}$ to $\mathcal{Y}$.

Differentiability of the mapping $(\tau, \kappa) \mapsto y_{\tau,\kappa}$

Theorem (Partial derivatives)

The mapping $G : (\tau, \kappa) \mapsto y_{(\tau,\kappa)}$ from $\mathbb{R}^{2m}$ to $\mathcal{Y}$ has all partial derivatives $\partial_{\kappa_i} G$ and $\partial_{\tau_i} G$, $i = 1, \ldots, m$.

The result cannot be directly obtained as a particular case of the problem with measures μ, where we proved the differentiability w.r. to μ.

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Formal computation of derivatives: Consider e.g. $\partial_{\kappa_i} G$; insert

$$y = (G(\tau, \kappa))(x, t)$$

$$\Big(\partial_t G(\tau, \kappa) - \Delta G(\tau, \kappa) + R(G(\tau, \kappa))\Big)(x, t) = \sum_{j=1}^{m} \kappa_j \, G(\tau, \kappa)(x, t - \tau_j)$$

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Partial derivatives

Theorem (Partial derivatives)

The partial derivatives of $G : \mathbb{R}^{2m} \to \mathcal{Y}$, $(\tau,\kappa) \mapsto y_{\tau,\kappa}$ *are given as follows: For every* $(\tau,\kappa) = u$ *and* $1 \le i \le m$, *we have* $\partial_{\tau_i} G(u) = z_i$, *where* z_i *satisfies the equation*

$$
\left\{
\begin{array}{l}
\partial_t z - \Delta z + R'(y_u)\, z = \displaystyle\sum_{j=1}^{m} \kappa_j\, z(x, t-\tau_j) - \kappa_i\, \partial_t y_u(x, t-\tau_i) \ \text{ in } Q \\[2ex]
\partial_n z = 0 \ \text{ on } \Sigma, \ z = 0 \ \text{ in } Q_-,
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and $\partial_{\kappa_i} G(u) = \eta_i$, *where* η_i *satisfies*

$$\left\{ \begin{array}{l} \partial_t \eta - \Delta \eta + R'(y_u)\, \eta = \displaystyle\sum_{j=1}^m \kappa_j\, \eta(x,t-\tau_j) + y_u(x,t-\tau_i) \ \ \text{in } Q \\[2ex] \partial_n \eta = 0 \ \ \text{on } \Sigma, \ \ \eta = 0 \ \ \text{in } Q_-. \end{array} \right.$$

Outline

The optimization problem

Admissible set:

$$\mathrm{U}_{ad} = \{u = (\tau, \kappa) \in \mathbb{R}^m \times \mathbb{R}^m : \ a_i \le \tau_i \le b_i, \ \alpha_i \le \kappa_i \le \beta_i, 1 \le i \le m\},$$

Desired state $y_Q \in L^2(Q)$

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Optimization problem

$$(P) \qquad \min_{u \in \mathrm{U}_{ad}} J(u) = \frac{1}{2} \int_Q (y_u - y_Q)^2 \, dxdt + \frac{\nu}{2} |\kappa|^2,$$

where $\nu \ge 0$ is fixed; y_u solves

$$\left\{ \begin{array}{rcll} \partial_t y - \Delta y + R(y) & = & \displaystyle\sum_{i=1}^m \kappa_i \, y(x, t - \tau_i) & \text{in } Q \\ \partial_n y & = & 0 & \text{on } \Sigma \\ y & = & y_0 & \text{in } Q_- . \end{array} \right.$$

Adjoint state

Theorem

The partial derivatives of J are given by

$$\partial_{\tau_i} J(u) = -\kappa_i \int_Q \varphi_u(x,t)\, \partial_t y_u(x, t-\tau_i)\, dxdt,$$

$$\partial_{\kappa_i} J(u) = \nu\kappa_i + \int_Q \varphi_u(x,t)\, y_u(x, t-\tau_i)\, dxdt,$$

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$1 \le i \le m$, *where the adjoint state* $\varphi_u \in \mathcal{Y}$ *is the unique solution to the advanced adjoint equation*

$$\begin{cases} -\partial_t\varphi - \Delta\varphi + R'(y_u)\varphi = y_u - y_Q + \displaystyle\sum_{i=1}^m \kappa_i\, \varphi(x,t+\tau_i) \ \ \text{in } Q \\ \partial_n\varphi(x,t) = 0 \ \ \text{on } \Sigma, \ \ \varphi(x,t) = 0 \ \ \text{if } t \ge T. \end{cases}$$

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The theorem follows from the last one by the chain rule.

Existence of optimal solutions

Theorem (Existence of an optimal solution)

If $\nu > 0$ or $-\infty < \alpha_i \leq \beta_i < \infty$ for all $i \in \{1, \ldots, m\}$, then Problem (P) has a solution $\bar{u} = (\bar{\tau}, \bar{\kappa})$.

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Proof.

Let $\kappa^k \to \kappa$ and $\tau^k \to \tau$ in $\mathbb{R}^m$ be infimal sequences, then

$$\sum_{i=1}^{m} \kappa_i^k \delta_{\tau_i^k} \overset{*}{\rightharpoonup} \sum_{i=1}^{m} \kappa_i \delta_{\tau_i} \qquad \text{in } \mathcal{M}[0,T]$$

as $k \to \infty$.

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By a weak* $\to$ strong continuity result for the mapping $\mu \mapsto y_\mu$ (Casas/Mateos/Tr. 2017), it follows that

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Therefore, J is continuous in U. Either the objective functional is coercive w.r.t. to κ or U_{ad} is compact in $\mathbb{R}^{2m}$. Therefore, (P) has a global solution. □

Optimality conditions

Theorem (Necessary optimality conditions)

Let $\bar{u} \in \mathrm{U}_{ad}$ be a local solution of (P) and let $\bar{y} := y_{\bar{u}}$ be the associated state. Then there exists a unique associated adjoint state $\bar{\varphi} := \varphi_{\bar{u}} \in \mathcal{Y}$ such that the variational inequalities

$$-\bar{\kappa}_i \int_Q \partial_t \bar{y}(x, t - \bar{\tau}_i)\, \bar{\varphi}(x,t)\, dxdt\, (\tau_i - \bar{\tau}_i) \geq 0 \quad \forall \tau_i \in [a_i, b_i],$$

and

$$\left(\nu \bar{\kappa}_i + \int_Q \bar{y}(x, t - \bar{\tau}_i)\, \bar{\varphi}(x,t)\, dxdt \right) (\kappa_i - \bar{\kappa}_i) \geq 0 \quad \forall \kappa_i \in [\alpha_i, \beta_i] \cap \mathbb{R},$$

are satisfied for $i = 1, \ldots, m$.

Outline

Discretization of the state equation

- Ω: polygonal or polyhedral domain of $\mathbb{R}^d$, $d \leq 3$,
- $\{\mathcal{K}_h\}_{h>0}$: quasi-uniform family of triangulations of $\bar{\Omega}$, size h,
- $0 = t_0 < t_1 < \cdots < t_{N_\delta} = T$: quasi-uniform partition of $[0, T]$,
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Finite dimensional spaces

$$Y_h = \{z_h \in C(\bar{\Omega}) : \; z_{h|K} \in \mathcal{P}^1(K) \; \forall K \in \mathcal{K}_h\},$$

$$\mathcal{Y}_\sigma^0 = \{\phi_\sigma \in L^2(0, T; Y_h) : \phi_{\sigma|I_k} \in \mathcal{P}^0(I_k; Y_h) \; \forall k = 1, \ldots, N_\delta\},$$

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- $\mathcal{P}^1(K)$: set of polynomials of degree 1 in K
- $\mathcal{P}^i(I_k; Y_h)$: set of polynomials of degree i defined in I_k with values in Y_h, $i = 0, 1$

For $\phi_\sigma \in \mathcal{Y}_\sigma^0$, we denote by $\phi_\sigma^k \in Y_h$ the value of ϕ_σ in I_k.

Discretization of the state equation

Discrete state equation in variational form:

The discrete state $y_\sigma(u) \in \mathcal{Y}_\sigma^1$ is the unique solution to (cf. Becker-Meidner-Vexler (2007))

$$y_{\sigma,0}(x,0) = \Pi_h y_0(x,0),$$

$$\int_Q \partial_t y_\sigma \phi_\sigma \, dxdt + \int_Q \nabla_x y_\sigma \cdot \nabla_x \phi_\sigma \, dxdt + \int_Q R(y_\sigma)\phi_\sigma \, dxdt$$
$$= \sum_{i=1}^m \kappa_i \left[\int_\Omega \int_0^{\tau_i} y_0(x, t-\tau_i)\phi_\sigma \, dxdt + \int_\Omega \int_{\tau_i}^T y_\sigma(x, t-\tau_i)\phi_\sigma \, dxdt \right] \forall \phi_\sigma \in \mathcal{Y}_\sigma^0,$$

where $\Pi_h : L^2(\Omega) \to Y_h$ is the projection onto Y_h in the $L^2(\Omega)$-sense.

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Discretized optimization problem: Defined upon the discretized state.

Outline

Example 1 (ODE)

In all examples:

$$R(y) = y\,(y - 0.25)(y - 1), \quad T = 80, \quad \nu = 0.$$

$0 \le \tau_i \le T$, $|\kappa_i| \le 1000$, $i = 1, \ldots, m$.

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$$y'(t) + R(y(t)) = \sum_{i=1}^{m} \kappa_i y(t - \tau_i), \; t \in (0, T], \quad y(t) = y_0(t), \; t \le 0,$$

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$$y'(t) = -\frac{\pi}{2} y(t-1),\ t \in (0,T], \quad y(t) = 1,\ t \in [-1, 0].$$

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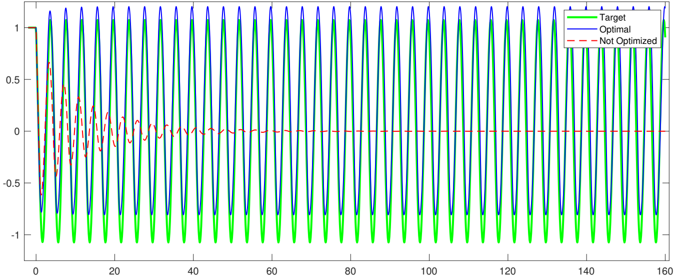
Take $m = 1$. Aim: Find $u = (\tau, \kappa)$, to best approximate y_Q by the solution of the nonlinear delay equation with initial function $y_0(t) = 1$.

Example 1, result

Computed optimal solution for $m = 1$: $\bar{\tau} = 1.2409$, $\bar{\kappa} = -1.7668$, $J(\bar{u}) = 1.8701$, $|\nabla J(\bar{u})| = 3.8 \cdot 10^{-7}$.



Uncontrolled state: red, Target state: green, Optimal state: blue

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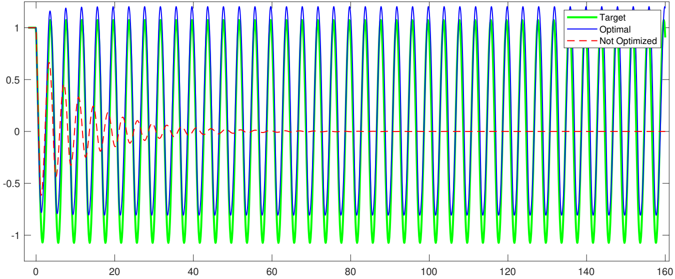

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Optimization code: MATLAB code `fmincon`, derivatives by adjoint calculus

Example 2

Data of Nestler/Schöll/Tr. (2016)

- $\Omega = (-20, 20)$
- $y_0(x,t) = \frac{1}{2}\left[1 - \tanh\left(\frac{x - vt}{2}\right)\right]$ with $v = 0.25\sqrt{2}$
- $y_Q(x,t) = 3\sin\left(t - \cos\left(\frac{\pi}{20}(x+20)\right)\right)$
- 2^7 finite elements in space, 2^7 steps in time
- $m = 6$ delays

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Desired state y_Q

i	$\bar{\tau}_i$	$\bar{\kappa}_i$
1	0.0000	0.9846
2	0.9367	−1.5039
3	6.7481	0.4542
4	28.3843	−2.2799
5	32.2258	3.7013
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$$J(\bar{u}) = 4209.3, \quad \partial_{\tau_1} J(\bar{u}) = 486,$$
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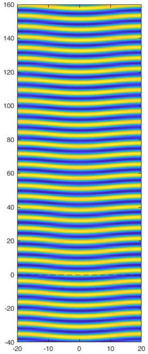

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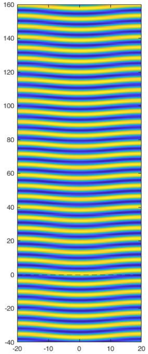
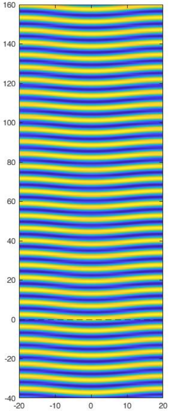
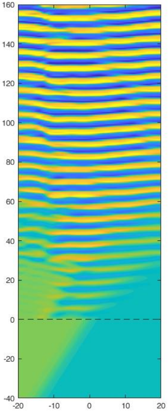


Notice: $Q = (-20, 20) \times (0, 80)$, $|Q| = 3200$, $\|1\|^2_{L^2(Q)} = 3200$.

Example 2





Desired state

Computed optimal state

References

Nestler, P., Schöll, E., T. F.
Optimization of nonlocal time-delayed feedback controllers,
COAP 2016

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K. Pyragas,
Continuous control of chaos by self-controlling feedback.
Phys. Rev. Lett. 1992.

J. Löber, R. Coles, J. Siebert, H. Engel, E. Schöll,
Control of chemical wave propagation in Engineering of Chemical Complexity II.
A. S. Mikhailov, G. Ertl (Eds.), World Scientific, Singapore, 2014.

E. Schöll, H.G. Schuster,
Handbook of Chaos Control.
Wiley-VCH (2008).

Outline

Pyragas feedback control

Take $u(t) = \kappa \, (y(t-\tau) - y(t))$.

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Pyragas feedback: Let $\tau > 0$ and $y_0 : Q_- \to \mathbb{R}$ be given,

$$\begin{aligned} \partial_t y(t) - \Delta y(t) + R(y(t)) &= \kappa\,(y(t-\tau) - y(t)), && t > 0, \\ y(t) &= y_0(t), && t \in [-\tau, 0]. \end{aligned}$$

If y has period τ, then the feedback vanishes. This type of feedback is very popular in Theoretical Physics.

Pyragas feedback with multiple delays:

$$\partial_t y(t) - \Delta y(t) + R(y(t)) = \sum_{i=1}^{k} \kappa_i y(t-\tau_i) - \kappa_0\, y(t)$$

Nonlocal Pyragas feedback

Nonlocal version with "distributed time delay"

$$\begin{aligned} \partial_t y(t) - \Delta y(t) + R(y(t)) &= \kappa \left(\int_0^T g(\tau) y(t-\tau)\, d\tau - y(t) \right), \quad t > 0, \\ y &= y_0, \qquad\qquad \text{in } [-\tau, 0], \end{aligned}$$

with given feedback gain κ and a kernel $g \in L^\infty(0, T)$.

Forward problem: $g \mapsto y$

Depending on the chosen feedback kernel g, different solutions y are generated.

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$$\Omega = (0,200),\ T = 400, \quad y_0(x,t) := \frac{1}{2}\left(1 - \tanh\left(\frac{x-vt}{2\sqrt{2}}\right)\right)$$

"Weak gamma delay kernel" $g(t) = e^{-t}$, $y_1 = 0,\ y_3 = 1$

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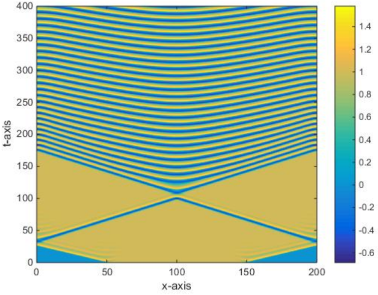
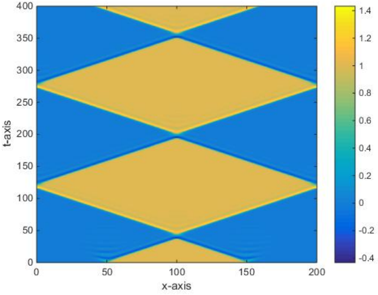


$y_2 = 0.25,\ \kappa = -1.65$



$y_2 = 0.5,\ \kappa = -1.4$

Pyragas feedback with Borel measures

More general idea: Cover standard and nonlocal Pyragas type feedback in a unified way by using a Borel measure.

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y + R(y) & = & \displaystyle\int_0^T y(t-s)\,d\mu(s) & \text{in } Q, \\ \partial_n y & = & 0 & \text{on } \Sigma, \\ y & = & y_0 & \text{in } Q_- \end{array} \right.$$

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Find the measure $\bar{\mu}$ such that the solution of this equation best approximates a desired solution.

Optimal feedback design problem

Design problem:

Optimal feedback design problem

Design problem:

$$\min \frac{1}{2}\|y - y_Q\|^2_{L^2(Q)} + \frac{\nu}{2}\|\mu\|_{\mathcal{M}(0,T)}$$

where y solves

$$\begin{cases} \partial_t y(t) - \Delta y(t) + R(y(t)) = \int_0^T y(t-\tau)\,d\mu(\tau) & \text{in } Q, \\ y = y_0 & \text{in } Q_- \\ \partial_n y = 0 & \text{in } \Sigma. \end{cases}$$

Casas, E., Mateos, M., T. F.
Measure control of a semilinear parabolic equation with a nonlocal time delay, Accepted by SICON

Outline

A stability theorem

Desired state: $y^{desi} \in L^\infty(Q_{2\tau})$, τ-periodic, $y^{desi} = y^{desi}(t,\, x)$

Assume: y^{desi} has Neumann traces $y_x^{desi}(\cdot,\, 0) \in L^p(0,\tau)$ and $y_x^{desi}(\cdot,\, L) \in L^p(0,\tau),\ p > 2.$

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– Extend y^{desi} to a $\tau-$periodic function on $[0, \infty) \times [0, L]$.

– Let $y_0 \in C([-\tau, 0], L^\infty(0, L))$; consider

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(S)

$$
\begin{aligned}
y_t &= y_{xx} - y_{xx}(\cdot - \tau, \cdot) - \rho\,[R(y) - R(y(\cdot - \tau, \cdot)] \\
&\quad + y_t(\cdot - \tau, \cdot) - \kappa[y - y(\cdot - \tau, \cdot)], && \text{in } (0,\infty) \times (0,\, L), \\
y_x(t,\, 0) &= y_x^{desi}(t,\, 0), && t > 0, \\
y_x(t,\, L) &= y_x^{desi}(t,\, L), && t > 0, \\
y(t,\, x) &= y_0(t,\, x) && (t,\, x) \in [-\tau,\, 0] \times (0,\, L).
\end{aligned}
$$

Stability theorem

Theorem (Exponential Stability)

Let $\tau > 0$ and a τ-periodic state $y^{desi} \in H^2(Q_{2\tau})$ be given. Define $\mu = 2\,\kappa - 2\,\rho\,|m_R|$. Assume that $\mu > 0$. Then

$$V(t) = \frac{1}{2}\int_0^L (y(t,x) - y(t-\tau,x))^2\;dx$$

is a strict Lyapunov function for **(S)**, *i.e. for $t \geq \tau$ we have*

$$V(t) \leq \exp(-\mu\,(t-\tau))\;V(\tau).$$

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$$V(t) \leq \exp(-\mu\,(t-\tau))\, V(\tau).$$

Moreover, there are a τ*-periodic* $y^* \in L^2(Q_{2\tau})$ *and* $c_0 > 0$ *such that*

$$\int_0^L (y(t,\,x) - y^*(t,\,x))^2 \, dx \leq c_0 \, \exp\,(-\mu t)$$

for all $t \geq \tau$*.*

Example 1 continued

Let $\quad R(y) = y(y - 0.25)(y - 1).$

For delay $\tau = 1.240683838477202$ and weight $\kappa = -1.766552137106608$, let $y : [-\tau, \infty) \to \mathbb{R}$ be the solution y of the nonlinear delay equation

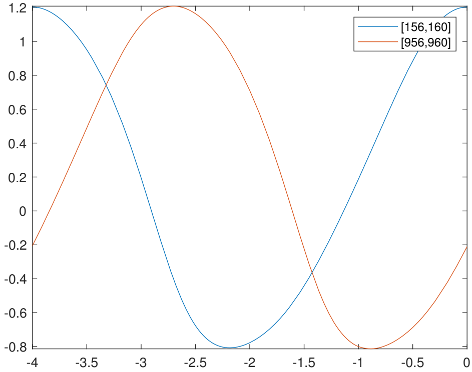
$$
\begin{aligned}
y'(t) + R(y(t)) &= \kappa y(t - \tau) \quad \text{for } t \in (0, \infty), \\
y(t) &= 1 \quad \text{if } t \leq 0
\end{aligned}
$$

that in $[0, 160]$ minimizes the L^2-distance to the solution v of

$$
\begin{aligned}
v'(t) &= -\frac{\pi}{2} v(t - 1) \quad \text{for } t \in [0, \infty), \\
v(t) &= 1 \quad \text{if } t \leq 0.
\end{aligned}
$$

This v has period $\tau = 4$.

The long-term behaviour in $[0, 960]$ shows that y is possibly not periodic



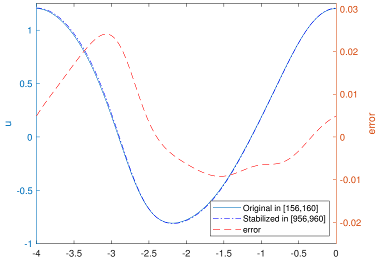
We stabilize the solution by Pyragas feedback with $\tau = 4$.

Stabilization by Pyragas feedback

We solve the equation

$$u'(t) + R(u(t)) = \kappa\, u(t-\tau) + 100\,(u(t-4) - u(t)), \qquad t \in [160, 960],$$
$$u(t) = y(t), \qquad t \in [156, 160)$$

and obtain a very good numerical adjustment to 4-periodicity;



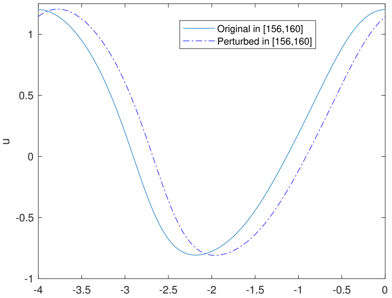
(error graph is magnified 40 times.)

Example 1 under perturbation

The solution is very sensitive w.r. to perturbations of the data: For the tiny modification $\hat{\tau} = 1.24,\ \hat{\kappa} = -1.77,$ the solution of

$$\begin{aligned} \hat{y}'(t) + R(\hat{y}(t)) + &= \hat{\kappa}\hat{y}(t-\hat{\tau}), \quad t \in (0,\infty), \\ \hat{y}(t) &= 1, \qquad\qquad\qquad t \le 0 \end{aligned}$$

considerably differs from the unperturbed solution.

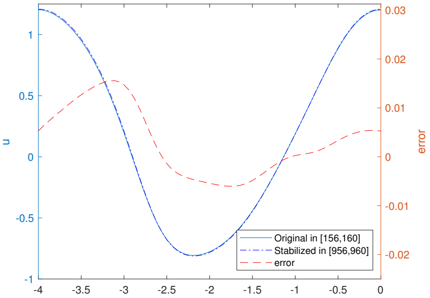


Example 1, perturbation and Pyragas feedback

Again, Pyragas feedback stabilizes the solution: The solution of

$$\begin{aligned} y'(t) + R(y(t)) &= \hat{\kappa} y(t - \hat{\tau}) + \kappa(y(t-4) - y(t)), \quad && t \in (160, 960], \\ y(t) &= 1.01\, v(t), && t \in [156, 160] \end{aligned}$$

not only appears to be periodic, but also is very close to the solution of the original unperturbed problem.



Reference

Gugat, M., Mateos, M., T. F.
Exponential stability for the Schlögl system by Pyragas feedback,
2019, submitted

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Thank you