Shape Optimization for Geometrically Inverse Problems

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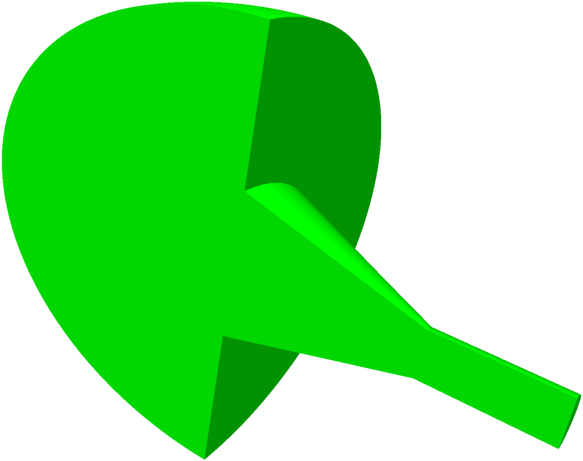
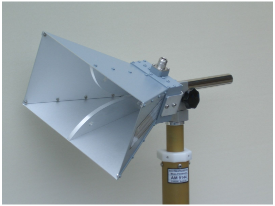
14 October 2019

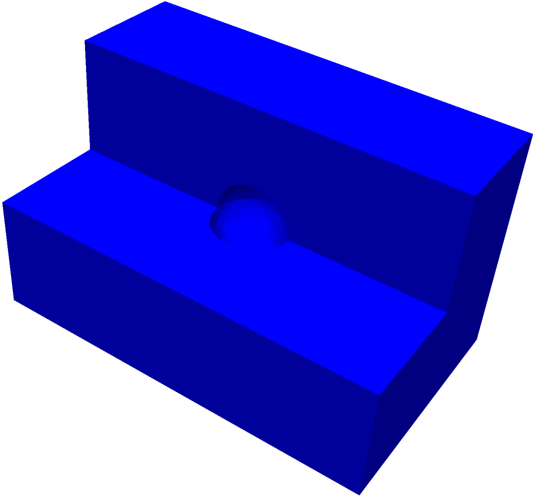
Acoustic Horn Design (joint with M. Berggren, E. Wadbro)



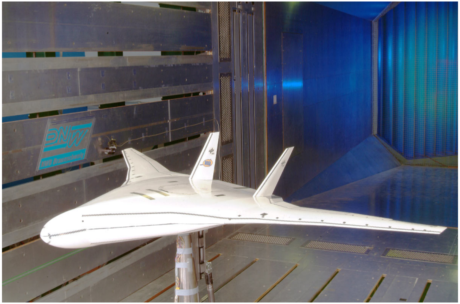


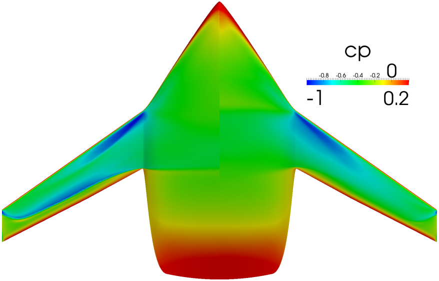
- General problem formulation allows treatment of general problems
- Design of acoustic (linear wave) horn antenna, $3.5 \cdot 10^9$ unknowns!

Maxwell Scattering Problem (joint with M. Schütte, O. Ebel, A. Walther)



VELA Aircraft (joint with V. Schulz and DLR Braunschweig)





Design study for blended wing-body configurations

- Transonic Inviscid Incompressible CFD
- $> 460,000$ surface node positions to be optimized
- Planform constant

Geometric Inverse Problem

$$\min_{(\varphi,\Gamma_{\text{inc}})} J(\varphi,\Omega) := \frac{1}{2}\int_0^T \int_{\Gamma_{\text{i/o}}} \|B(n)(\varphi - \varphi_{\text{meas}})\|_2^2 \, \mathrm{d}\,t \, \mathrm{d}\,s + \delta \int_{\Gamma_{\text{inc}}} 1 \, \mathrm{d}\,s$$

subject to

$$\dot{\varphi} + \operatorname{div} F(\varphi) = 0 \quad \text{in} \quad \Omega$$
$$\text{BCs} = g \quad \text{on} \quad \Gamma$$

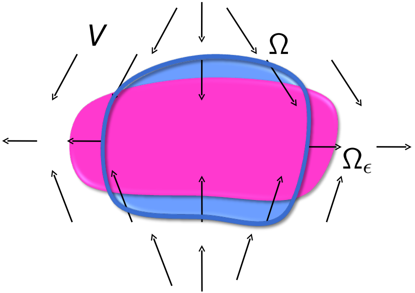
Acoustics:

$$\frac{\partial u}{\partial t} + \nabla p = 0 \text{ in } \Omega,$$
$$\frac{\partial p}{\partial t} + c^2 \operatorname{div} u = 0 \text{ in } \Omega,$$
$$\frac{1}{2}(p - c\langle u, n\rangle) = g \text{ on } \Gamma_{\text{i/o}}$$

Electromagnetism:

$$\mu \frac{\partial H}{\partial t} = -\nabla \times E \text{ in } \Omega,$$
$$\varepsilon \frac{\partial E}{\partial t} = \nabla \times H - \sigma E \text{ in } \Omega,$$
$$\text{BCs} = g \text{ on } \Gamma_{\text{i/o}}$$

Introduction to Shape Optimization



- Shape is modeled by set Ω
- $\Omega_\epsilon := \{x + \epsilon V(x) : x \in \Omega\} \subset \mathcal{D}$
- $J : \mathcal{P}(\mathcal{D}) \to \mathbb{R}$: target function
- (Directional) derivative of J with respect to Ω?

- Directional Derivative

$$dJ(\Omega)[V] := \lim_{\epsilon \to 0^+} \frac{J(\Omega_\epsilon) - J(\Omega)}{\epsilon}$$

The Shape Derivative

- Objective function:

$$J_1(\epsilon, \Omega) := \int_{\Omega(\epsilon)} f(\epsilon, x_\epsilon)\, \mathrm{d}\, x_\epsilon \;\text{ or }\; J_2(\epsilon, \Omega) := \int_{\Gamma(\epsilon)} g(\epsilon, s_\epsilon)\, \mathrm{d}\, s_\epsilon$$

- Take Limit:

$$dJ_1(\Omega)[V] = \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\Big|_{\epsilon=0} \int_{\Omega(\epsilon)} f(\epsilon, x_\epsilon)\, \mathrm{d}\, x_\epsilon \;\text{ or }\; dJ_2(\Omega)[V] = \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\Big|_{\epsilon=0} \int_{\Gamma(\epsilon)} g(\epsilon, s_\epsilon)\, \mathrm{d}\, s_\epsilon$$

- Change of Variables = Change in Domain

$$dJ_1(\Omega)[V] = \int_{\Omega} \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\Big|_{\epsilon=0} \Big[f(\epsilon, T_\epsilon(x)) \cdot |\det DT_\epsilon(x)| \Big]\, \mathrm{d}\, x$$

$$dJ_2(\Omega)[V] = \int_{\Gamma} \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\Big|_{\epsilon=0} \Big[g(\epsilon, T_\epsilon(s)) \cdot |\det DT_\epsilon(s)| \|(DT_\epsilon(s))^{-T} n(s)\|_2 \Big]\, \mathrm{d}\, s$$

The Shape Derivative (Weak vs Strong)

- Material Derivative: $df[V] := \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\Big|_{\epsilon=0} f(\epsilon, T_\epsilon(x)) = \langle \nabla f, V \rangle + f'[V]$
- Local / Shape Derivative: $f'(x)[V] := \frac{\partial}{\partial \epsilon} f(0, x)$

$$dJ_1(\Omega)[V] = \int_\Omega f(0,x)\mathrm{div}\,(V) + \frac{\mathrm{d}}{\mathrm{d}\,\epsilon}\Big|_{\epsilon=0} f(\epsilon, T_\epsilon(x))\,\mathrm{d}\,x$$

$$= \int_\Omega f\mathrm{div}\; V + df[V]\;\mathrm{d}\,x \quad \text{(Weak/Volume/Distributed Formulation)}$$

$$= \int_\Omega \mathrm{div}\,(fV) + f'[V]\;\mathrm{d}\,x = \int_\Gamma \langle V, n\rangle f\,\mathrm{d}\,s + \int_\Omega f'[V]\;\mathrm{d}\,x \quad \text{(Surface Formulation)}$$

$$dJ_2(\Omega)[V] = \int_\Gamma g\mathrm{div}_\Gamma\; V + dg[V]\;\mathrm{d}\,s \quad \text{(Weak/Volume/Distributed Formulation)}$$

$$\overset{V\text{ normal}}{=} \int_\Gamma \mathrm{div}_\Gamma\,(gV) + \langle V, n\rangle \frac{\partial g}{\partial n} + g'[V]\,\mathrm{d}\,s$$

$$= \int_\Gamma \langle V, n\rangle \left[\frac{\partial g}{\partial n} + \kappa g\right] + g'[V]\,\mathrm{d}\,s$$

Some Remarks...

- if f or g is "glued to mesh" (i.e. FEM function), then $df[V] = f'[V]$.
- if f or g is "fixed w.r.t. V", then $f'[V] = 0$.
- Material derivative typically same regularity as state (Berggren, 2010)
- Correction terms make surface formulation exact (Berggren, 2010)

Dido's Problem

Find shape of maximum volume for given surface:

$$\max_{\Omega} J(\Omega) := \int_{\Omega} 1 \, \mathrm{d}\, x$$

s.t.

$$\int_{\Gamma} 1 \, \mathrm{d}\, s = A_0$$

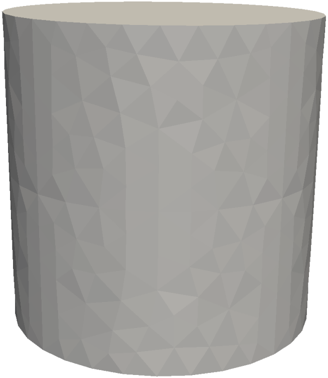
Lagrangian:

$$F(\Omega, \lambda) = \int_{\Omega} -1 \, \mathrm{d}\, x + \lambda \left(\int_{\Gamma} 1 \, \mathrm{d}\, s - A_0 \right)$$

$$dF(\Omega, \lambda)[V] = \int_{\Gamma} \langle V, n \rangle \left[-1 + \lambda\kappa \right] \mathrm{d}\, s \stackrel{!}{=} 0 \quad \forall V$$

Because $\lambda \in \mathbb{R}$: $\kappa = \frac{1}{\lambda} \in \mathbb{R}$. Thus, curvature constant!
Optimality fulfilled by sphere!!

Dido's Problem (Gradient Descent + Newton)



Regularization, Approximate Newton, H^1-Descent

Shape-descent in H^1 / Sobolev Gradient Method / Approximate Newton can all be motivated by surface area penalization:

$$R(\Gamma) = \int_\Gamma 1 \; \mathrm{d}s$$

Then:

$$dR(\Gamma)[V] = \int_\Gamma \langle V, n\rangle \kappa \; \mathrm{d}s \qquad \text{Curvature Flow, Minimal Surface}$$

$$d^2R(\Gamma)[V, W] = \int_\Gamma \langle \nabla_\Gamma \langle V, n\rangle, \nabla_\Gamma \langle W, n\rangle\rangle + \langle V, n\rangle\langle W, n\rangle\kappa^2 \; \mathrm{d}s$$

Shape Linearization of General Conservation Law

Hyperbolic PDE:

$$\dot{\varphi} + \operatorname{div} F(\varphi) = 0$$

Find $\varphi'[V]$ such that

$$\begin{aligned}
0 = &\int_0^T \int_\Gamma \langle V, n\rangle \left[\langle \lambda, \dot{\varphi}\rangle - \langle F(\varphi), \nabla\lambda\rangle\right] \, \mathrm{d}\,s \, \mathrm{d}\,t \\
&+ \int_0^T \int_\Gamma \langle V, n\rangle [\langle \nabla(\lambda \cdot F_\mathrm{b}^*(\varphi, n)), n\rangle \\
&\quad + \kappa\,(\lambda \cdot F_\mathrm{b}^*(\varphi, n) - D_n(\lambda \cdot F_\mathrm{b}^*(\varphi, n)) \cdot n) + \operatorname{div}_\Gamma\,(D_n^T(\lambda \cdot F_\mathrm{b}^*(\varphi, n)))] \, \mathrm{d}\,s \, \mathrm{d}\,t \\
&+ \int_0^T \int_\Omega \langle \lambda, \dot{\varphi}'[V]\rangle - \langle DF(\varphi)\varphi'[V], \nabla\lambda\rangle \, \mathrm{d}\,x \, \mathrm{d}\,t + \int_0^T \int_\Gamma \langle \lambda, D_\varphi F_\mathrm{b}^*(\varphi, n)\varphi'[V]\rangle \, \mathrm{d}\,s \, \mathrm{d}\,t
\end{aligned}$$

(Existence Results: Cagnol/Eller/Marmorat/Zolésio):

Adjoint Equation

Adjoint equation can be read from the shape-linearized equation: Find λ such that

$$0 = \int_0^T \int_\Omega \langle -\dot{\lambda}, \varphi'[V] \rangle - \langle \varphi'[V], D^T F(\varphi) \nabla \lambda \rangle \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_\Gamma \langle \varphi'[V], D_\varphi^T F_\mathrm{b}^*(\varphi, n) \cdot \lambda \rangle \ \mathrm{d}s \ \mathrm{d}t$$

$$+ \int_0^T \int_{\Gamma_{\mathrm{i/o}}} \langle B^T(n) B(n) \cdot (\varphi - \varphi_{\mathrm{meas}}), \varphi'[V] \rangle \ \mathrm{d}s \ \mathrm{d}t$$

$\Rightarrow$ Flux for adjoint can be read: $D_\varphi^T F^*(\varphi, n) \cdot \lambda$

Shape Derivative for Tomography Problems

Maxwell (Existence Results: Cagnol/Eller/Marmorat/Zolésio):

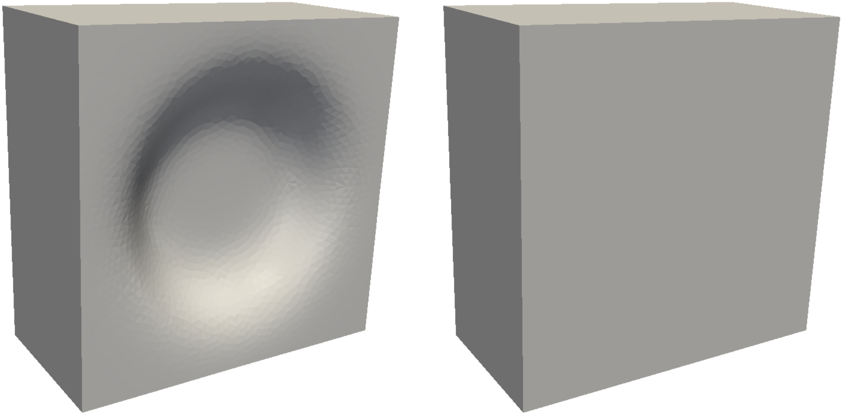
$$
\begin{aligned}
& dJ(H, E, \Omega)[V] \\
&= \int_0^T \int_{\Gamma_{\text{inc}}} \langle V, n \rangle \left[\langle \lambda_H, \dot{H} \rangle + \frac{1}{\mu} \langle E, \operatorname{curl} \lambda_H \rangle + \langle \lambda_E, \dot{E} \rangle - \frac{1}{\epsilon} \langle H, \operatorname{curl} \lambda_E \rangle + \frac{\sigma}{\epsilon} \langle \lambda_E, E \rangle \right] \, \mathrm{d}s \, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Gamma_{\text{inc}}} \langle V, n \rangle \operatorname{div} \left(Zc(H \times \lambda_E) \right) \, \mathrm{d}s \, \mathrm{d}t
\end{aligned}
$$

Horn/Linear Wave:

$$
\begin{aligned}
dJ(u, p, \Omega)[V] &= \int_0^T \int_{\Gamma_{\text{horn}}} \langle V, n \rangle \left[\langle \lambda_u, \dot{u} \rangle - p \operatorname{div} \lambda_u + \lambda_p \, \dot{p} - c^2 \langle u, \nabla \lambda_p \rangle \right] \, \mathrm{d}s \, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Gamma_{\text{horn}}} \langle V, n \rangle \operatorname{div} \left(c^2 \lambda_p \cdot u \right) \, \mathrm{d}s \, \mathrm{d}t
\end{aligned}
$$

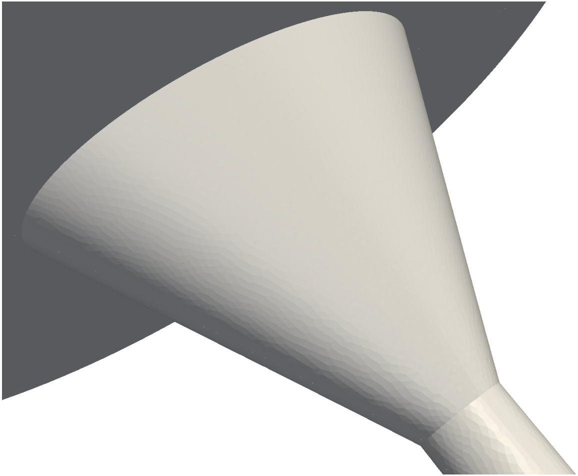
“Backwards in time” adjoint equations for (λ_H, λ_E) and (λ_u, λ_p)

Obstacle Without Antenna



- 4.1 – 12.3 Ghz SINC-puls
- 2.4 – 7.3 cm waves, 3.65 cm obstacle

Optimal Emitter for Acoustics

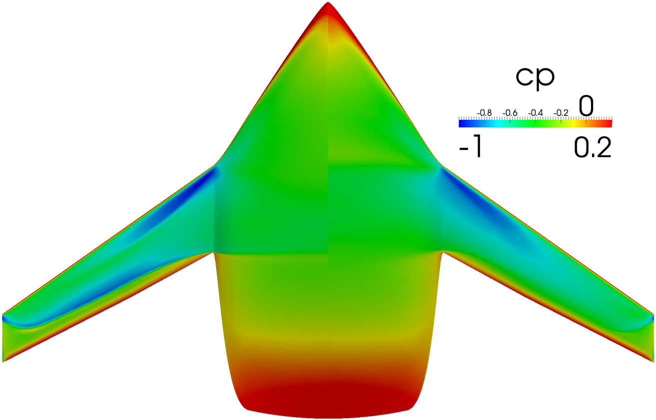


Boundary Data Compression:

$3.5 \cdot 10^9$ unknowns: 26 TB to 3.26 GB, 3 Months on 48 CPUs

(S., Wadbro, Berggren, 2016)

3D Euler Flow: VELA



Shape	C_D	%	C_L	%
460, 517	$3.342 \cdot 10^{-3}$	-30.06%	$1.775 \cdot 10^{-1}$	-0.67%

DLR Flow Solver TAU for primal and adjoint equation

Shape Hessians

$$d^2 J_1[V,W] = \int_\Omega f \operatorname{div} V \operatorname{div} W - f \operatorname{tr}(DVDW) \\ + df[V]\operatorname{div} W + df[W]\operatorname{div} V + d^2 f[V,W] \, \mathrm{d}x$$

- Material and Spatial Derivatives do not commute:

$$d(Df)[V] = Ddf[V] - DfDV$$

- Excessively long expressions with normal, curvature or PDEs
- Typically rank deficient

One-Shot Volume Hessian CFD (joint with S. Funke, J. Dokken, SIMULA)

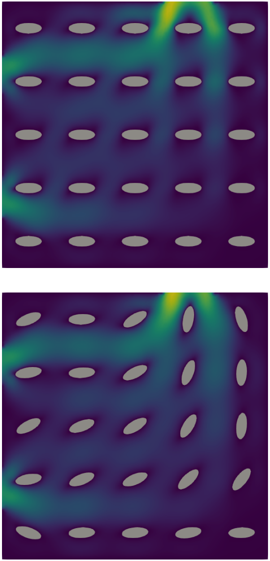
Incompressible CFD
cooling and tidal turbine placement

$$\min_{(u,p,\Omega)} E_{NS}(u,p,\Omega) := \frac{1}{2}\int_\Omega \mu \sum_{j,k=1}^{3} \left(\frac{\partial u_k}{\partial x_j}\right)^2 dA$$

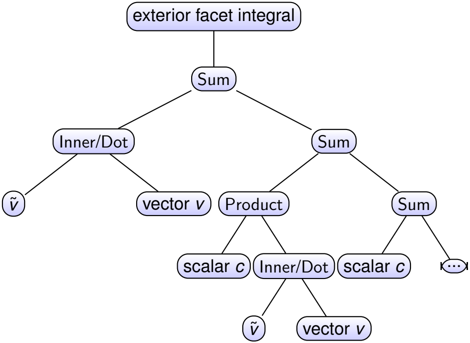
subject to

$$\begin{aligned} -\mu\Delta u + \rho u \nabla u + \nabla p = 0 \quad & \text{in} \quad \Omega \\ \operatorname{div} u = 0 \quad & \\ u = u_+ \quad & \text{on} \quad \Gamma_+ \\ u = 0 \quad & \text{on} \quad \Gamma_0 \\ pn - \mu\frac{\partial u}{\partial n} = 0 \quad & \text{on} \quad \Gamma_- \end{aligned}$$

FEM-Multimesh Implementation



Automatic Shape Derivatives in FEniCS/UFL



- Change expression into "maximally expanded form"
- Sort all sums closest to integral
- Apply rules to each sub-branch
- Pattern Recognition Problems
- Agument UFL derivatives in dolfin-adjoint/pyadjoint with release 2018.1

Freely Available: `www.bitbucket.org/Epoxid/femorph`

Weak Navier–Stokes Shape Derivative

Optimality:

$$0 = \int_\Omega \|\nabla u\|^2 \mathrm{div}\, V + 2\langle (DV)^T \nabla u, \nabla u\rangle - \mu\langle (DV)^T \nabla u, \nabla \lambda^u\rangle$$
$$- \mu\langle \nabla u, (DV)^T \nabla \lambda^u\rangle - \rho\langle \lambda^u, DuDVu\rangle + p\mathrm{tr}(D\lambda^u DV) - \lambda^p \mathrm{tr}(DuDV)\ \mathrm{d}\,x$$

Use adjoint to eliminate material derivatives: Find (λ^u, λ^p) such that:

$$0 = \int_\Omega 2\langle \nabla(du), \nabla u\rangle + \mu\langle \nabla(du), \nabla \lambda^u\rangle + \rho[\langle \lambda^u, D(du)u\rangle + \langle \lambda^u, Du \cdot du\rangle]$$
$$- dp \cdot \mathrm{div}\, \lambda^u + \lambda^p \mathrm{div}\, du\ \mathrm{d}\,x, \quad \text{for all } (du, dp) \quad \text{and Dirichlet BCs}$$

Not shown: Volume and Centroid, Hessian (MUCH too long)
See also:

- (Yang, Stadler, Moser, Ghattas (2011))
- (Brandenburg, Lindemann, Ulbrich, Ulbrich (2012))

SQP Strategy

- Build the full KKT-System (state + shape + adjoint)
- No approximations

Volume Hessian has large Kernel:

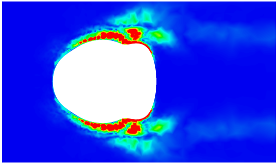
- SQP: Find W, such that

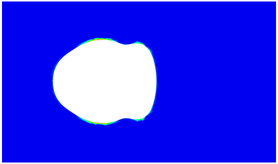
$$KKT(V, W, ...) + \langle V, W \rangle_\Omega + 0.1 \langle \nabla V, \nabla W \rangle_\Omega = dL(V, ...)$$

- Testfunction: V, Trialfunction W
- KKT and dL generated with automatic symbolic calculation

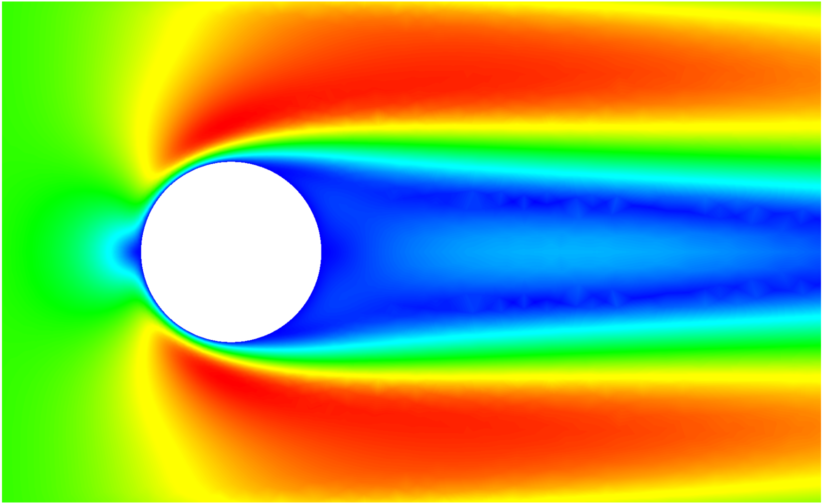
Mesh Defo:

- Boundary trace of W as Dirichlet BC in Laplace mesh deformation
- Inexact PDE $\Rightarrow$ Spurious volume movement (One Shot)





SQP / Newton Results



3rd Order Shape Derivatives

Validation of the Hessian: (joint with J. Dokken, SIMULA)

Numerical tests have shown that for certain types of integrals, the Taylor expansion can be truncated after the Hessian with no error!!

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Numerical tests have shown that for certain types of integrals, the Taylor expansion can be truncated after the Hessian with no error!!

Suppose $df[V] = 0$. Then, in 1D:

$$\mathrm{d}^2 J(\Omega)[V, W] = \int_\Omega \mathrm{div}\, V \mathrm{div}\, W - \mathrm{tr}(DV\, DW)\,\mathrm{d}\,x$$
$$= \int_\Omega \frac{\partial V}{\partial x}\frac{\partial W}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial W}{\partial x}\,\mathrm{d}\,x = 0 \quad \forall V, W$$

What about 2D?

The 3rd Shape Derivative in 2D

3rd order shape derivative:

$$
\begin{aligned}
& d^3 J(\Omega)[V, W, X] \\
=& \int_\Omega f \mathrm{div}\, V \mathrm{div}\, W \mathrm{div}\, X - f\,\mathrm{tr}(DV\, DX)\mathrm{div}\, W - f\,\mathrm{tr}(DW\, DX)\mathrm{div}\, V \\
& + f\,\mathrm{tr}(DV\, DX\, DW + DV\, DW\, DX) \\
& + df[V]\mathrm{div}\, W \mathrm{div}\, X + df[W]\mathrm{div}\, V \mathrm{div}\, X + df[X]\mathrm{div}\, V \mathrm{div}\, W \\
& - df[X]\mathrm{tr}(DV\, DW) - df[V]\mathrm{tr}(DW\, DX) - df[W]\mathrm{tr}(DV\, DX) \\
& + d^2 f[V, X]\mathrm{div}\,(W) + d^2 f[W, X]\mathrm{div}\,(V) + d^2 f[V, W]\mathrm{div}\,(X) + d^3 f[V, W, X] \;\mathrm{d}\,x
\end{aligned}
$$

Suppose $df[V] = 0$. Then what?

$$
\begin{aligned}
\operatorname{tr}(DA\,DB)\operatorname{div} C &= \left(\frac{\partial a_1}{\partial x_1}\frac{\partial b_1}{\partial x_1} + \frac{\partial a_1}{\partial x_2}\frac{\partial b_2}{\partial x_1} + \frac{\partial a_2}{\partial x_1}\frac{\partial b_1}{\partial x_2} + \frac{\partial a_2}{\partial x_2}\frac{\partial b_2}{\partial x_2}\right)\left(\frac{\partial c_1}{\partial x_1} + \frac{\partial c_2}{\partial x_2}\right) \\
&= \frac{\partial a_1}{\partial x_1}\frac{\partial b_1}{\partial x_1}\frac{\partial c_1}{\partial x_1} + \frac{\partial a_1}{\partial x_1}\frac{\partial b_1}{\partial x_1}\frac{\partial c_2}{\partial x_2} + \frac{\partial a_1}{\partial x_2}\frac{\partial b_2}{\partial x_1}\frac{\partial c_1}{\partial x_1} + \frac{\partial a_1}{\partial x_2}\frac{\partial b_2}{\partial x_1}\frac{\partial c_2}{\partial x_2} \\
&+ \frac{\partial a_2}{\partial x_1}\frac{\partial b_1}{\partial x_2}\frac{\partial c_1}{\partial x_1} + \frac{\partial a_2}{\partial x_1}\frac{\partial b_1}{\partial x_2}\frac{\partial c_2}{\partial x_2} + \frac{\partial a_2}{\partial x_2}\frac{\partial b_2}{\partial x_2}\frac{\partial c_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2}\frac{\partial b_2}{\partial x_2}\frac{\partial c_2}{\partial x_2}
\end{aligned}
$$

Sum up all similar terms... get zero!

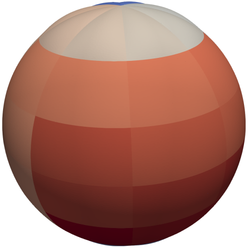
Proposition:
If the material derivatives vanish, then the $n+1$ directional shape derivative in n-dimensions is always zero.

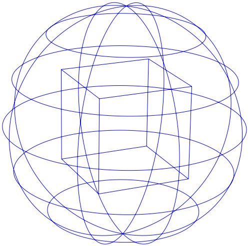
The Regularization Term (joint with R. Herzog, J.Vidal-Nuñez, R. Bergmann, M. Herrmann)

Example:
Geoelectrical Impedance Tomography:

$$\text{Minimize} \quad \frac{1}{2}\sum_{i=1}^{r}\int_{\Gamma_2} |u_i - z_i|^2 \; ds + \beta\, R(\Gamma_1)$$

$$\text{s.t.} \quad \begin{cases} -\Delta u_i = 0 & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial n} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial u_i}{\partial n} + \alpha\, u_i = f_i & \text{on } \Gamma_2 \end{cases}$$





- z_i given measurement data
- Previously:
  R surface area:
  Laplace Smoothing/Curvature Flow
- Idea: Regularization to favor kinks
- Idea: Total Variation of the Normal!

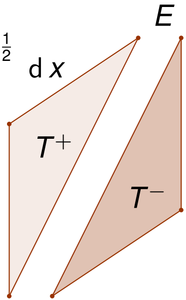
Total Variation for Surfaces and Manifold $\mathcal{S}^2$

Classical Total Variation:

$$|u|_{TV(\Omega)} := \int_\Omega \|\nabla u\|_2 \,\mathrm{d}\,x = \int_\Omega \left(\|(Du)\,e_1\|^2 + \|(Du)\,e_2\|^2\right)^{\frac{1}{2}} \,\mathrm{d}\,x$$

$$|u|_{DTV(\Omega)} := \sum_T \int_T \|\nabla u\|_2 \,\mathrm{d}\,x + \sum_E \int_E |[\![u]\!]| \,\mathrm{d}\,s$$

$$\overset{r=0}{=} \sum_E |E|\,\left|u^+ - u^-\right|$$



New difficulty here: Γ is a manifold and n maps to $\mathcal{S}^2$

$$|n|_{TV(\Gamma)} := \int_\Gamma \left(\|(D_\Gamma n)\,\xi_1\|_{\mathfrak{g}}^2 + \|(D_\Gamma n)\,\xi_2\|_{\mathfrak{g}}^2\right)^{1/2} \,\mathrm{d}\,s$$

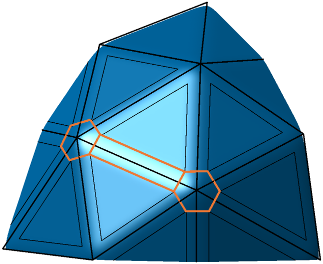
$$|n|_{DTV(\Gamma)} := \sum_E |E|\, d(n_E^+, n_E^-) = \sum_E |E| \left|\log_{n_E^+} n_E^-\right|_2$$

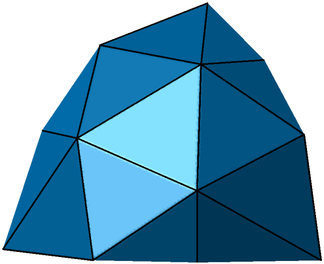
Properties of $|n|_{TV(\Gamma)}$ and $|n|_{DTV(\Gamma)}$

Let $\{\Gamma_\varepsilon\}$ denote a family of smooth approximations of Γ_h obtained by mollification, with normal vector fields n_ε. Then

$$|n_\varepsilon|_{TV(\Gamma_\varepsilon)} \to |n|_{DTV(\Gamma_h)} \quad \text{as } \varepsilon \searrow 0.$$

Proof: Utilize different convergence orders for edges and vertex caps





Bergmann, Herrmann, Herzog, Schmidt, Vidal-Núñez (SPP 1962 preprint, in review)

Properties of $|n|_{TV(\Gamma)}$ and $|n|_{DTV(\Gamma)}$

Properties of $|n|_{TV(\Gamma)}$:

Spheres are stationary points among all surfaces Γ of constant area.

Proof: By construction: Derive shape derivative and use that integrand is spatially constant on a sphere

Properties of $|n|_{TV(\Gamma)}$ and $|n|_{DTV(\Gamma)}$

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Proof: By construction: Derive shape derivative and use that integrand is spatially constant on a sphere

Properties of $|n|_{DTV(\Gamma)}$:

The icosahedron and the cube with crossed diagonals are stationary points within the class of triangulated surfaces Γ_h of constant area and identical connectivity.

Proof: By construction

Bergmann, Herrmann, Herzog, Schmidt, Vidal-Núñez (in review)

Sketch of Proof for Continuous Case

Same strategy as with Dido's Problem:

$$g(\varepsilon, s_\varepsilon) = \left(k_{1,\varepsilon}^2(s_\varepsilon) + k_{2,\varepsilon}^2(s_\varepsilon)\right)^{\frac{1}{2}} = \left(|(D_\Gamma n_\varepsilon)\,\xi_{1,\varepsilon}|_{\mathfrak{g}}^2 + |(D_\Gamma n_\varepsilon)\,\xi_{2,\varepsilon}|_{\mathfrak{g}}^2\right)^{\frac{1}{2}}$$

Shape/Material derivative:

$$g'[V] \overset{\text{tangent argument}}{=} \mathrm{d}\, g[V] = \frac{1}{g(s)} \sum_{i=1}^{2} \mathfrak{g}((D_\Gamma n)\,\xi_i, \mathrm{d}[(D_\Gamma n)\,\xi_i][V])$$

Tangents with Gram-Schmidt:

$$\mathrm{d}\,\xi_1[V] = (DV)\,\xi_1 - (\xi_1^\top (DV)\,\xi_1)\,\xi_1$$
$$\mathrm{d}\,\xi_2[V] = (DV)\,\xi_2 - (\xi_2^\top (DV)\,\xi_2)\,\xi_2 - (\xi_1^\top (DV + DV^\top)\,\xi_2)\,\xi_1.$$

On sphere: $g(s) = (\kappa_1^2(s) + \kappa_2^2(s))^{\frac{1}{2}} = \frac{\sqrt{2}}{r}$

Shape Derivative Lagrangian with tangential Stokes:

$$\mathrm{d}\,\mathcal{L}(0,\lambda)[V] = \left[\frac{2}{r}\left(\frac{1}{\sqrt{2}r} + \lambda\right)\right] \int_\Gamma \langle V, n\rangle \;\mathrm{d}\,s$$

ADMM Optimization for $|n|_{DTV(\Gamma)}$

Not (shape-) differentiable:

$$\min_{\Gamma} \frac{1}{2} |u(\Gamma) - z|_2^2 + \beta |n|_{DTV(\Gamma)}$$

Idea: ADMM: Solve independently for Γ, d and b

$$\min_{\Gamma, d} \frac{1}{2} |u(\Gamma) - z|_2^2 + \beta \sum_E |E| |d_E|_2 + \frac{\lambda}{2} \sum_E |E| \left| d_E - \log_{n_E^+} n_E^- - b_E \right|_2^2$$

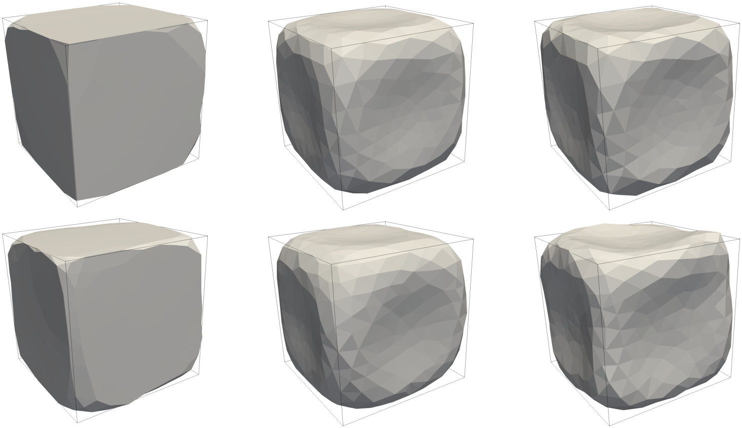
- Γ-problem: smooth shape problem, adjoint calculus
- Parallel transport: $b_E \in \mathcal{T}_{n_E^+} \mathcal{S}^2$
- d-problem:

$$d_E = \text{shrink}(b_E + \log_{n_E^+} n_E^-, \frac{\beta}{\lambda})$$

- b-update:

$$b_E := b_E + \log_{n_E^+} n_E^- - d_E$$

Geoelectric Reconstruction of a Cube

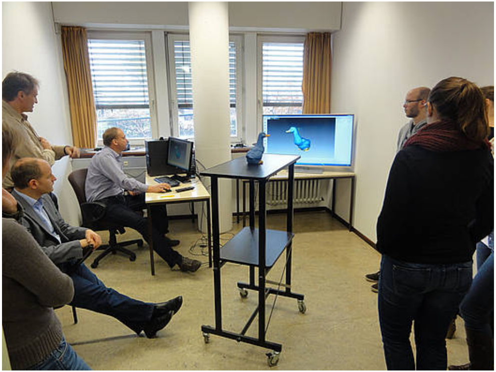


noise free

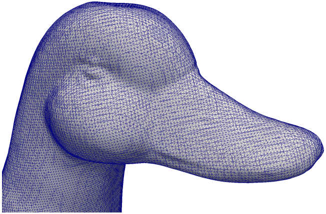
input noise

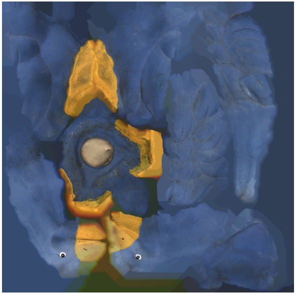
$\|n\|_{DTV(\Gamma)}$	perimeter	perimeter
$\beta = 10^{-6}$	$\beta = 5 \cdot 10^{-5}$	$\beta = 2 \cdot 10^{-5}$

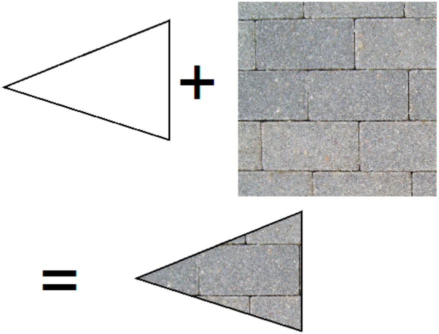
Fully integrated DG-Suite



3D Scan ⇒ FEM/DG/Optimization (FEniCS) ⇒ 3D Print







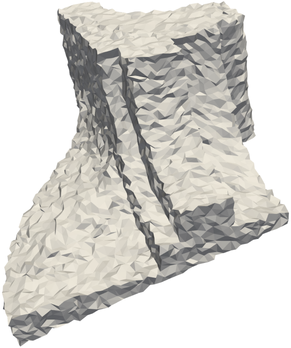
(Source: Wikipedia)

Computer Graphics:
Texture Mapping: Multiple Pixels per Triangle on Surface

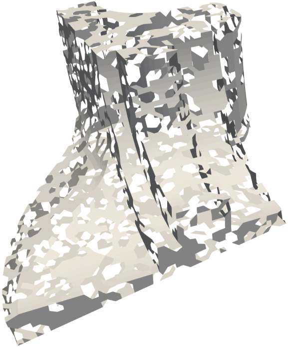
Here:

- Convert Geometry + Texture (Bitmap) to DG-FEM on Surface
- Information in Higher Order or Refinement!

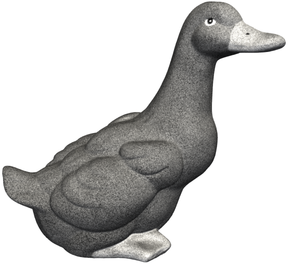
Common 3D Scan Problems



Noisy Geometry:
Tracking Term



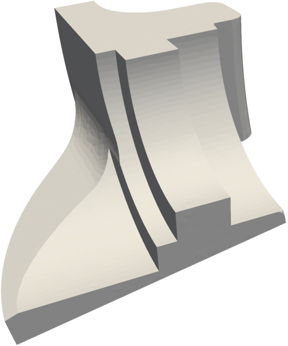
Missing Geometry:
Subdomain



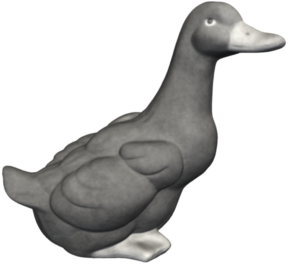
Noisy/Missing Textures:
No shape, different
manifold

Common 3D Scan Problems: Results

Noisy Geometry



Missing Geometry



Noisy/Missing Textures

Conclusions and Outlook

- Inverse Problems and Surfaces with Kinks!
- Optimization for Variable Geometries and HPC
- 1st, 2nd and 3rd order Shape Derivatives
- $|n|_{TV(\Gamma)}$ and $|n|_{DTV(\Gamma)}$
  non-smooth reconstructions