# Generalized derivatives for the solution operator of obstacle problems

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New trends in PDE constrained optimization
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In parts joint work with Lukas Hertlein and Michael Ulbrich, TU München.

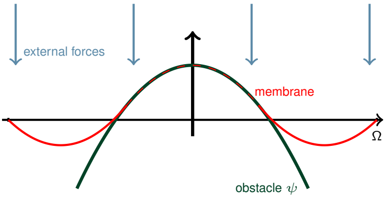
# The Obstacle Problem

We consider the following variational inequality:

$$\text{Find } y \in K_\psi : \quad \langle Ly - f(u), z - y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0 \quad \forall z \in K_\psi$$

- $K_\psi := \{z \in H_0^1(\Omega) : z \geq \psi \text{ q.e.}\}$
- $\psi$ quasi upper-semicontinuous with $K_\psi \neq \emptyset$
- $L \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ coercive and strictly T-monotone, i.e.,

$$\langle L(v - z), (v - z)_+ \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} > 0$$
$$\forall v, z \in H_0^1(\Omega), (v - z)_+ \neq 0$$



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- $U$ partially ordered, separable Banach space

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  - $f\colon \mathbb{R}^n \to H^{-1}(\Omega)$ “finite-dimensional”, $U = \mathbb{R}^n$



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# Notation: Solution Operator and Subsets of $\Omega$

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- solution operator $S\colon U \to H_0^1(\Omega)$ of the obstacle problem
- Lipschitz continuous and monotone



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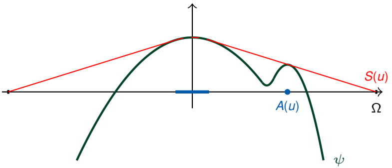

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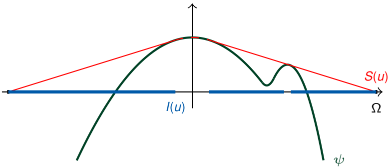

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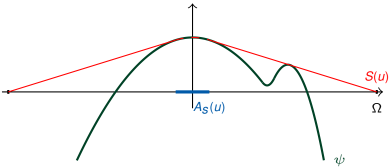

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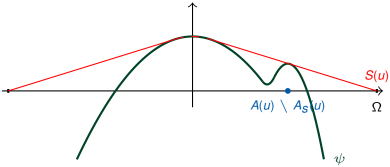
- weakly active set $A(u) \setminus A_s(u)$



# Outline

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Derivation of a Generalized Derivative

Related Problems, Applications and Extensions

Remarks and Conclusion



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# Motivation

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Find an element of a suitable set of generalized derivatives for the solution operator of the infinite dimensional obstacle problem.

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Find an element of a suitable set of generalized derivatives for the solution operator of the infinite dimensional obstacle problem.

- can be used for infinite dimensional bundle methods (current cooperation with Hertlein, M. Ulbrich within SPP 1962) for optimal control of the obstacle problem
- gives theoretical insights into the problem structure of the obstacle problem
- results may be adapted and transferred to related problems

$\rightsquigarrow$ second part of the talk

# Generalized Differential

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- $X$ separable, $Y$ separable, reflexive
- $T\colon X \to Y$ locally Lipschitz
- $D_T \coloneqq \{x \in X : T \text{ Gâteaux differentiable in } x\}$

## Bouligand generalized differential

$$\partial_B T(u) \coloneqq \Big\{ \Sigma \in \mathcal{L}(X, Y) : T'(u_n) \to \Sigma \text{ in the strong operator topology for some sequence } (u_n)_{n\in\mathbb{N}} \subseteq D_T \text{ with } \lim_{n\to\infty} u_n = u \Big\}$$

- strong operator topology: $T'(u_n)v \to \Sigma v$ in $Y$ for all $v \in X$



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# Strategy for Finding an Element $\xi$ in $\partial_B S(u)$

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- characterize Gâteaux derivative $S'(u)$ in points $u$ of differentiability (characterization via (in)active sets)

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  - the property of the sequence guarantees that active sets are decreasing
  - we can show that the derivatives $\big(S'(u_n)\big)_{n\in\mathbb{N}}$ converge to an element $\xi \in \mathcal{L}(U, H_0^1(\Omega))$ with respect to the strong operator topology
  - we can characterize the limit $\xi$ exploiting the monotonicity properties
  - $S$ is Gâteaux differentiable in each $u_n \rightsquigarrow$ Rademacher type argument

$\rightsquigarrow$ then $\xi$ is an element of $\partial_B S(u)$

# Differentiability Properties

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- $S$ is directionally differentiable with directional derivative $S'(u;h)$ in direction $h$ given by the unique solution $\xi$ to

$$\text{Find } \xi \in \mathcal{K}_{K_\psi}(f(u)) : \quad \langle L\xi - f'(u;h), z - \xi \rangle \geq 0 \quad \forall\, z \in \mathcal{K}_{K_\psi}(f(u))$$

(Mignot, 1976)

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- structure of the critical cone $\mathcal{K}_{K_\psi}(f(u)) = \mathcal{T}_{K_\psi}(S(u)) \cap (LS(u) - f(u))^\perp$:

$$\mathcal{K}_{K_\psi}(u) = \left\{ z \in H_0^1(\Omega) : z \geq 0 \text{ q.e. on } A(u), z = 0 \text{ q.e. on } A_s(u) \right\}$$


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- possible source of nondifferentiability: $A(u) \neq A_s(u)$ (violation of strict complementarity condition)

# Points of Gâteaux Differentiability



Assume $S$ is Gâteaux differentiable in $u$, then $S'(u; h)$ also uniquely solves the VI with $\mathcal{K}_{K_\psi}(f(u))$ replaced by

- the largest linear subset of $\mathcal{K}_{K_\psi}(f(u))$:

$$H_0^1(I(u)) = \left\{ z \in H_0^1(\Omega) : z = 0 \text{ q.e. on } A(u) \right\}$$

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- the linear hull of $\mathcal{K}_{K_\psi}(f(u))$:

$$H_0^1(\Omega \setminus A_s(u)) = \left\{z \in H_0^1(\Omega) : z = 0 \text{ q.e. on } A_s(u)\right\}$$

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- all linear subsets $H_0^1(D(u))$ for quasi-open sets $I(u) \subseteq D(u) \subseteq \Omega \setminus A_s(u)$

# Points of Gâteaux Differentiability

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## VE for the Gâteaux Derivative

In points of differentiability, $S'(u;h)$ is the solution $\xi$ of the VE/ Dirichlet problem on a quasi-open domain

$$\text{Find } \xi \in H_0^1(I(u)) : \quad \langle L\xi - f'(u;h), z \rangle = 0 \quad \forall\, z \in H_0^1(I(u)).$$

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$\rightsquigarrow$ need to investigate convergence properties of solutions to VIs/ VEs for a sequence of sets $\left(H_0^1(I(u_n))\right)_{n \in \mathbb{N}}$ to obtain an element of $\partial_B S(u)$



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# Tool: Convergence of Solutions of VIs/ VEs

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## Theorem (Rodrigues, 1987)

*Let $C_n, C$ be nonempty, closed, convex subsets of $X$ with $C_n \to C$ in the sense of Mosco and let $h_n \to h$ in $X^*$. Then the unique solutions $(\xi_n)_{n\in\mathbb{N}}$ of*

$$\text{Find } \xi_n \in C_n : \quad \langle L\xi_n - h_n, z - \xi_n\rangle_{X^*,X} \geq 0 \quad \forall\, z \in C_n$$

*converge to the solution $\xi$ of the limit problem*

$$\text{Find } \xi \in C : \quad \langle L\xi - h, z - \xi\rangle_{X^*,X} \geq 0 \quad \forall\, z \in C.$$

# Influence of Monotonicity

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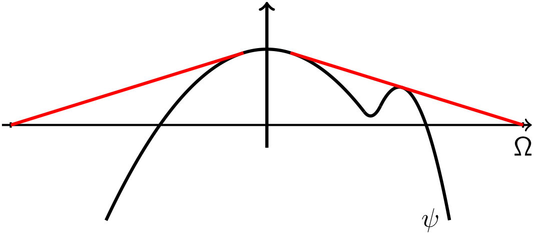


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- $S(0)$ in red
- strict complementarity condition is not satisfied in $u = 0$

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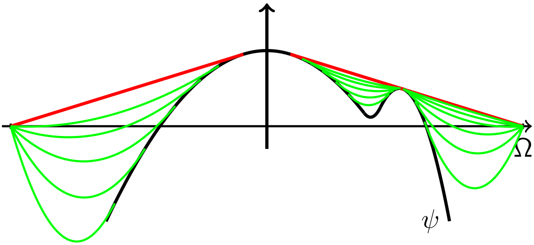


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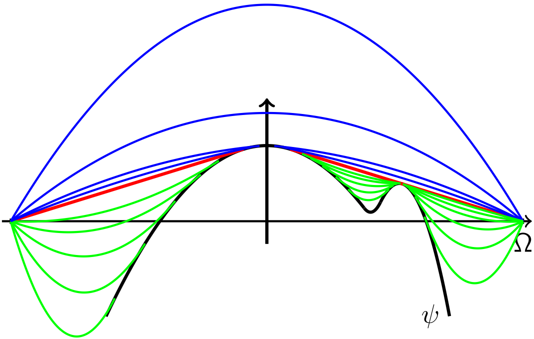


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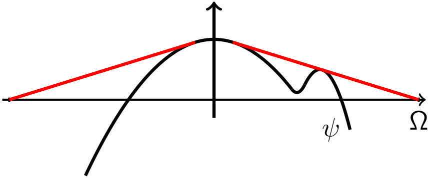
# Influence of Monotonicity: The Sets A(u)



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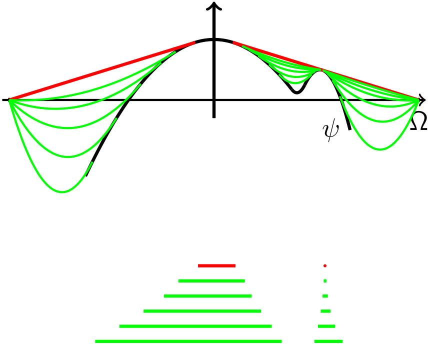


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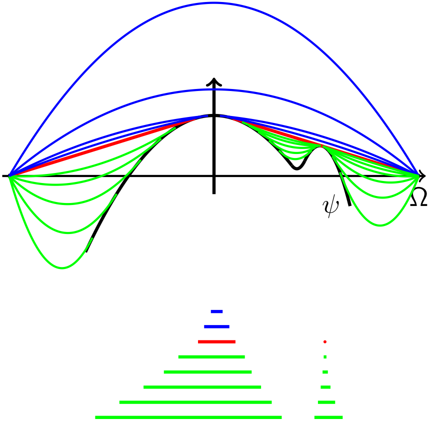


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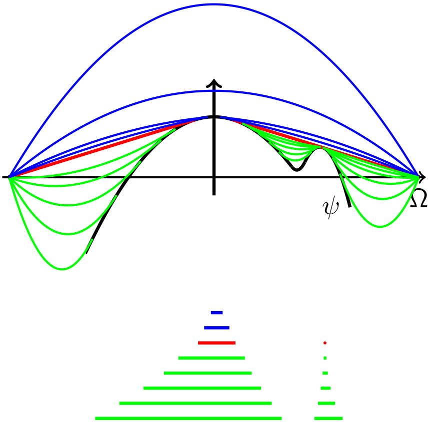


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$\rightsquigarrow$ Mosco convergence of the sets $\left(H_0^1(I(u_n))\right)_{n\in\mathbb{N}}$ towards $H_0^1(I(0))$ might not hold for a decreasing sequence $(u_n)_{n\in\mathbb{N}}$ converging to 0

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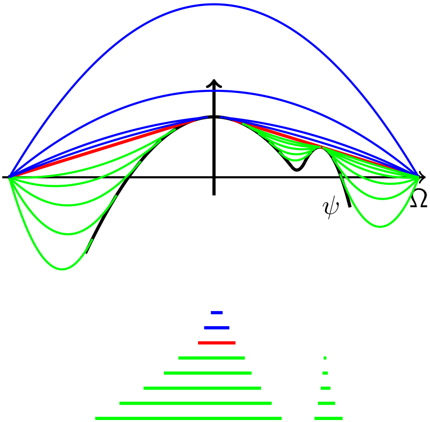


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- opposite behavior occurs for the sets $H_0^1(\Omega \setminus A_s(u_n))$

# Mosco Convergence of the Sets $H_0^1(I(u_n))$

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**Theorem (R., Ulbrich, SICON 2019)**

*Let $(u_n)_{n\in\mathbb{N}}$ be an increasing sequence such that $u_n \to u$. Then $H_0^1(I(u_n)) \to H_0^1(I(u))$ in the sense of Mosco.*



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Proof strategy:

- consequence of the monotonicity structures of the sets $H_0^1(I(u_n))$ and the Lipschitz continuity of $S$
- use tools and approximation results for elements in $H_0^1(I(u))$ (quasi-converings etc.) (Kilpeläinen, Maly, 1992)



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# A Generalization of Rademacher's Theorem

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**Theorem (Mignot, 1976; Aronszajn, 1976)**

*Let $X$ be a separable Banach space and let $Y$ be a Hilbert space. Let $S$ be a Lipschitz function from $X$ into $Y$. Then the set of points in $X$ where $S$ is Gâteaux differentiable is a dense set.*

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## Corollary

*Let $u \in U$ be arbitrary. Then there is an increasing sequence $(u_n)_{n\in\mathbb{N}}$ such that $S$ is differentiable in each $u_n$ and $u_n \to u$.*

# Overview



- for $u \in U$ we find an approximating increasing sequence of points $(u_n)_{n\in\mathbb{N}}$ where $S$ is differentiable
- the monotone sequence of sets $\left(H_0^1(I(u_n))\right)_{n\in\mathbb{N}}$ converges to $H_0^1(I(u))$
- for each $h \in U$ the directional derivatives $(S'(u_n;h))_{n\in\mathbb{N}}$ for $(u_n)_{n\in\mathbb{N}}$ increasing converge to the solution $\xi = \xi(u;h)$ of the variational equation

$$\text{Find } \xi \in H_0^1(I(u)) : \quad \langle L\xi - f'(u;h), z\rangle = 0 \quad \forall\, z \in H_0^1(I(u))$$

as a consequence of the Mosco convergence

$\rightsquigarrow$ $\xi(u;\cdot) \in \mathcal{L}(U, H_0^1(\Omega))$ is an element of the generalized differential $\partial_B S(u)$

# Adjoint Representation of the Subgradient



- $J\colon H_0^1(\Omega) \times U \to \mathbb{R}$ continuously differentiable
- consider reduced objective function $\hat{J}(u) := J(S(u), u)$ for optimal control of the obstacle problem
- $\partial_C \hat{J}(u)$ Clarke subdifferential of $\hat{J}$ in $u \in U$

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## Adjoint Representation

Let $q$ be the unique solution of the VE

$$q \in H_0^1(I(u)) : \quad \langle L^* q, v \rangle = \langle J_y(S(u), u), v \rangle \quad \forall\, v \in H_0^1(I(u)),$$

then we have

$$f'(u)^* q + J_u(S(u), u) \in \partial_C \hat{J}(u).$$



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# Optimal Control of a Stochastic VI: Problem Setting and Assumptions

**Joint work with L. Hertlein and M. Ulbrich.**

## Parameter Dependent Obstacle Problem

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- $(\Xi, \mathcal{A}, P)$ measure space, $\Xi$ separable Banach space

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## Parameter Dependent Obstacle Problem

$$\text{Find } y_\xi \in K_\psi : \quad \langle L_\xi y_\xi - f(\xi, u), z - y_\xi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \geq 0 \quad \forall z \in K_\psi$$

- $(\Xi, \mathcal{A}, P)$ measure space, $\Xi$ separable Banach space
- keep the assumptions on $U$ as before
- $f(\xi, \cdot)$ monotone and continuously differentiable

# Optimal Control of a Stochastic VI: Problem Setting and Assumptions

**Joint work with L. Hertlein and M. Ulbrich.**

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- $\Xi \ni \xi \mapsto L_\xi y \in H^{-1}(\Omega)$ equi-Lipschitz continuous for all $y \in H_0^1(\Omega)$
- $(J_\xi)_{\xi \in \Xi}$ continuously differentiable functions which are equi-Lipschitz continuous on bounded sets, e.g., $J_\xi(y, u) = \frac{1}{2}\|y - y_\xi^d\|_{L^2}^2 + \frac{\alpha}{2}\|u\|_U^2$ for bounded $(y_\xi^d)_{\xi \in \Xi} \subseteq H^1(\Omega)$
- Remark: we impose these assumptions only for $P$-a.a. $\xi \in \Xi$



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## Optimal Control of a Stochastic VI: Problem Setting and Possible Approaches

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- Task: construct Clarke subgradient for $\hat{J}(u) \coloneqq \int_\Xi J_\xi(S_\xi(u)), u)\, dP(\xi)$



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- Problem: regularity of the maps $\hat{J}_\xi(\cdot)$ in $u$ in the sense of Clarke (Clarke, 1990) is needed ↯
- Remedy: find common points of differentiability for almost all $\xi \in \Xi$ to exchange limits and the integral

# Optimal Control of a Stochastic VI: Differentiability

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- Lipschitz continuity of $\Xi \times U \ni (\xi, u) \to S_\xi(u) \in H_0^1(\Omega)$ for $P$-a.a. $\xi \in \Xi$
- exists Lipschitz continuous $T$ on $\Xi \times U$ with $T(\xi, u) = S_\xi(u)$ for $P$-a.a. $\xi \in \Xi$

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## Assumption

There is a nondegenerate Gaussian measure $\mathbb{P}$ on $\Xi$ such that $P \ll \mathbb{P}$.

- Example: $\Xi$ finite-dimensional and $P$ has any nonnegative function as density w.r.t. the Lebesgue measure $\lambda^d$



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# Optimal Control of a Stochastic VI: Differentiability

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## Lemma

*The map $T\colon \Xi \times V \to H_0^1(\Omega)$ is Gâteaux differentiable except on a $P \otimes \mathbb{V}$-null set in $\Xi \times V$. Here, $\mathbb{V}$ is an arbitrary nondegenerate Gaussian measure on $V$ ($\hookrightarrow U$).*

- holds since set of nondifferentiability is Gauss null, i.e., each nondegenerate Gaussian measure vanishes on it (Benyamini, Lindenstrauß, 2000)

# Optimal Control of a Stochastic VI: Result

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## Corollary

There is an increasing sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ converging to $u$ where $T(\xi, \cdot) = S_\xi(\cdot)$ is differentiable for $P$-almost all $\xi \in \Xi$.

## Theorem (Hertlein, R., Ulbrich, Ulbrich, 2019)

*Let $u \in U$ be arbitrary. A Clarke subgradient for $\hat{J} = \int_\Xi J_\xi(S_\xi(\cdot), \cdot)$ is given by $\int_\Xi \Sigma_\xi(u)\, dP(\xi)$, where $\Sigma_\xi(u)$ is the Clarke subgradient of $\hat{J}_\xi = J_\xi(S_\xi(\cdot), \cdot)$ we have constructed in the first part of the talk.*

Proof strategy:

# Optimal Control of a Stochastic VI: Result

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- $(u_n)_{n\in\mathbb{N}}$ increasing sequence of differentiability points of $S_\xi(\cdot)$
- exchange limit and integral to obtain differentiability of $\hat{J}$ in each $u_n$

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- $(u_n)_{n\in\mathbb{N}}$ increasing sequence of differentiability points of $S_\xi(\cdot)$
- exchange limit and integral to obtain differentiability of $\hat{J}$ in each $u_n$
- exchange limit and integral to obtain that $\lim_{n\to\infty} \hat{J}_u(u_n) = \int_\Xi \lim_{n\to\infty} (\hat{J}_\xi)_u(u_n)\, dP(\xi)$



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## Shape Optimization: Problem Setting and Assumptions

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### Obstacle Problem Dependent on Domains

$$\text{Find } y_\tau \in K_{\psi,\Omega_\tau} : \quad \langle -\Delta y_\tau, z - y_\tau \rangle_{H^{-1}(\Omega_\tau), H_0^1(\Omega_\tau)} \geq 0 \quad \forall z \in K_{\psi,\Omega_\tau}$$

- $U = C^1(\mathbb{R}^d)^d$ with norm $\|u\|_U = \|u\|_{L^\infty(\mathbb{R}^d)^d} + \|\nabla u\|_{L^\infty(\mathbb{R}^d)^{d,d}}$
- $\Omega_\tau = \tau(\Omega)$, $\tau = \mathrm{id}_{\mathbb{R}^d} + u$
- $u \in U_\rho := \{u \in U : \|u\|_U < \rho\}$, $\rho > 0$ small enough
- $\Omega_\tau$ bounded domains with $C^1$-boundary



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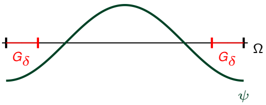
- $u \in U_\rho := \{u \in U : \|u\|_U < \rho\}$, $\rho > 0$ small enough
- $\Omega_\tau$ bounded domains with $C^1$-boundary
- $\psi \in H^1(\mathbb{R}^d)$, $\operatorname{ess\,sup}_{\omega\in\Omega} \psi(\omega) > 0$
- assume there exists $\delta > \rho$ with $\psi|_{G_\delta} < 0$ on $G_\delta = \{\omega \in \Omega : \mathrm{dist}(\omega, \partial\Omega) \leq \delta\}$
- $K_{\psi,\Omega_\tau} = \{z \in H_0^1(\Omega_\tau) : z \geq \psi \text{ q.e. on } \Omega_\tau\}$,






NONLINEAR OPTIMIZATION

- $y^\tau = y_\tau \circ \tau \in H_0^1(\Omega)$, $\psi^\tau = \psi \circ \tau$

## Reformulation of the Problem to a Problem on $\Omega$

$$\text{Find } y^\tau \in K_{\psi^\tau} : \quad a_\tau(y^\tau, z - y^\tau) \geq 0 \quad \forall z \in K_{\psi^\tau}$$

Here, $a_\tau$ is a bilinear form resulting from applying transformation to the above problem and $K_{\psi^\tau} = \{z \in H_0^1(\Omega) : z \geq \psi^\tau\}$.

- $S\colon U_{\rho,0} \to H_0^1(\Omega)$ solution operator of the transformed problem on $U_{\rho,0} = \{u \in U_\rho : \mathrm{supp}(u) \subset G_\delta\}$

# Shape Optimization:
# A Reformulation and Possible Approaches

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- Task: find generalized derivative for $S\colon U_{\rho,0} \to H_0^1(\Omega)$ at $u = 0$
- Strategy: use sequence of transformations that transforms active sets monotonically

# Shape Optimization: Key Properties and Result

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## Theorem (R., Ulbrich, 2019)

*The solution operator of the variational equation*

$$\text{Find } \xi \in H^1_0(I(0)) : \quad a_{\text{id}}(\xi, z) + \langle a'_{\text{id}}(S(0), z), h \rangle = 0 \quad \forall z \in H^1_0(I(0))$$

*is a Bouligand generalized derivative of* $S\colon U_{\rho,0} \to H^1_0(\Omega)$ *at* $u = 0$.

Proof strategy:

# Shape Optimization: Key Properties and Result

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*is a Bouligand generalized derivative of* $S \colon U_{\rho,0} \to H_0^1(\Omega)$ *at* $u = 0$.

Proof strategy:

- Lipschitz continuity of $S$ on $U_{\rho,0}$
- $S$ is directionally differentiable on $U_{\rho,0}$ and directional derivative is solution to variational inequality on critical cone

$$\{z \in H_0^1(\Omega) : z \geq 0 \text{ q.e. on } A(u), a_{\mathrm{id}+u}(S(u), z) = 0\}$$

- monotonicity of the active sets: $u_1, u_2 \in U_{\rho,0}, \Omega_{\mathrm{id}+u_1} \subset \Omega_{\mathrm{id}+u_2} \rightsquigarrow A(u_1) \supset A(u_2)$
- existence of differentiability points with above property

# Outline

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# Remarks

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- it is possible to characterize the whole generalized differential for the solution operator of the basic obstacle problem with distributed forces (no operator $f$)

$$\text{Find } y \in K_\psi : \quad \langle Ly - u, z - y \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \geq 0 \quad \forall z \in K_\psi$$

(R., Wachsmuth, 2019)

- also possible when using different combinations of topologies in the definition of the generalized differential

- current work on error estimates for discretized inexact subgradients



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# Conclusion

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- derivation of a generalized derivative for the obstacle problem
- extension to the optimal control of a stochastic obstacle problem
- extension to a shape optimization problem for the obstacle problem

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