Optimal control of resources for species survival

Yannick Privat

Univ. Strasbourg, IRMA

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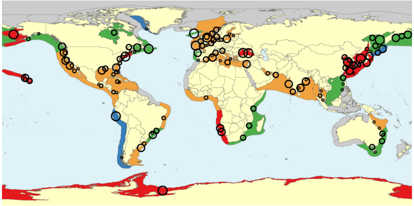
Outline

1. Modeling issues : toward a shape optimization problem
2. Analysis of optimal resources domains
   - Known results about the minimizers of $\lambda(m)$
   - New results on $\lambda(m)$: a Faber-Krahn type inequality ?
   - Maximizing the total population size
3. Biased movement of species
4. Conclusion and open problems



J. Lamboley, A. Laurain, G. Nadin, Y. Privat, *Properties of optimizers of the principal eigenvalue with indefinite weight and Robin conditions*, Calc. Var. Partial Differential Equations 55 (2016), no. 6.

I. Mazari, G. Nadin, Y. Privat, *Optimal location of resources maximizing the total population size in logistic models*, to appear in Journal Math. Pures Appl.

Outline

Biological model : population dynamics

Logistic diffusive equation (Fisher-Kolmogorov 1937, Fleming 1975, Cantrell-Cosner 1989)

Introduce

- $\rightsquigarrow$ $\Omega \subset \mathbb{R}^N$: bounded domain with Lipschitz boundary (habitat)
- $\rightsquigarrow$ μ : diffusion coefficient ($\mu > 0$)
- $\rightsquigarrow$ $u(t, x)$: density of a species at location x and time t
- $\rightsquigarrow$ $m(x)$: control - intrinsic growth rate of species at location x and
  - $\Omega \cap \{m > 0\}$ (resp. $\Omega \cap \{m < 0\}$) is the favorable (resp. unfavorable) part of habitat
  - $\int_\Omega m$ measures the total resources in the spatially heterogeneous environment Ω
  - After renormalization, one is allowed to assume that

$$-1 \leq m(x) \leq \kappa \quad \text{with } \kappa > 0 \quad \text{and } m \text{ changes sign.}$$

Biological model

$$\begin{cases} u_t = \mu \Delta u + u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ u(0, x) \geq 0, \quad u(0, x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Biological model : population dynamics

Choice of boundary conditions

$$\partial_n u = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+ \quad \text{(no-flux boundary condition)}$$

Here, the boundary $\partial\Omega$ acts as a barrier
$\rightsquigarrow$ other kinds of B.C. have been considered in this study

The complete model

$$\begin{cases} u_t = \mu\Delta u + u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

($\rightsquigarrow$ takes into account effects of dispersal and partial heterogeneity)

Analysis of the model : extinction/survival condition

The complete model

$$\begin{cases} u_t = \mu\Delta u + u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

Introduce the eigenvalue problem

$$\boxed{\begin{cases} \Delta\varphi + \lambda m\varphi = 0 & \text{in } \Omega, \\ \partial_n\varphi = 0 & \text{on } \partial\Omega, \end{cases}} \tag{EP}$$

Existence of a positive principal eigenvalue $\lambda(m)$

- if $\int_\Omega m < 0$, then (EP) has a unique principal eigenvalue $\lambda(m)$.
- if $\int_\Omega m \geq 0$, then 0 is the unique nonnegative principal eigenvalue of (EP).

Analysis of the model : extinction/survival condition

The complete model

$$\begin{cases} u_t = \mu\Delta u + u[m(x) - u] & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_n u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0,x) \geq 0, \quad u(0,x) \not\equiv 0 & \text{in } \overline{\Omega}, \end{cases}$$

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Theorem (Cantrell-Cosner 1989, Berestycki-Hamel-Roques 2005)

Let u^* be the unique positive steady solution of the logistic equation above. One has

- $\mu \geq 1/\lambda(m) \implies u(t,x) \underset{t\to\infty}{\longrightarrow} 0$,
- $\mu < 1/\lambda(m) \implies u(t,x) \underset{t\to\infty}{\longrightarrow} u^*(x)$.

Comments on the eigenvalue problem (with a sign changing weight m)

Characterization of $\lambda(m)$

$\lambda(m)$ is the unique principal ($\Leftrightarrow \varphi > 0$) positive eigenvalue of the problem :

$$\begin{cases} \Delta\varphi + \lambda m\varphi = 0 & \text{in } \Omega, \\ \partial_n\varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

Another characterization of $\lambda(m)$

$\lambda(m)$ is also characterized by the min-formula :

$$\lambda(m) = \inf\left\{ \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega m\varphi^2}, \quad \varphi \in H^1(\Omega), \ \int_\Omega m\varphi^2 > 0 \right\}.$$

Optimal arrangements of resources

Conclusion of this part : 2 optimal control problems

$$u_t = \mu \Delta u + \omega u[m(x) - u]$$

Dynamical problem	Static problem
$\Delta \varphi + \lambda m \varphi = 0$	$\mu \Delta u^* + u^*(m - u^*) = 0$
$\rightsquigarrow$ species can be maintained iff $\mu < 1/\lambda(m)$. Hence, the smaller $\lambda(m)$ is, the more likely the species can survive	$\rightsquigarrow$ maximizes the total size of the population
$\inf_{m \in \mathcal{M}_{m_0,\kappa}} \lambda(m)$ $\quad (P_{\text{Dyn}})$	$\sup_{m \in \mathcal{M}_{m_0,\kappa}} \int_\Omega u^*$ $\quad (P_{\text{Stat}})$

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Choice of admissible weights

$$\mathcal{M}_{m_0,\kappa} = \left\{ m \in L^\infty(\Omega, [-1, \kappa]),\ |\{m > 0\}| > 0,\ \int_\Omega m \leq -m_0|\Omega| \right\}$$

Outline

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Bang-bang property of minimizers

Proposition (Lou-Yanagida 2006, Derlet-Gossez-Takac 2010)

Problem (P_{Dyn}) has a solution. Moreover, every minimizer m satisfies

$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

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Shape optimization version of the problem

Consequence : the two following problems

$$\inf\left\{\lambda(m), \quad m \in L^\infty(\Omega,[-1,\kappa]),\ |\{m>0\}|>0,\ \int_\Omega m \leq -m_0|\Omega|\right\} \tag{1}$$

and

$$\inf\left\{\lambda(E) := \lambda(\kappa\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\}, \tag{2}$$

where $c = c(m_0) \in (0,1)$, are equivalent. Moreover, each infimum is in fact a minimum.

State of the art (Highly non-exhaustive)

Proposition (Lou-Yanagida 2006, Derlet-Gossez-Takac 2010)

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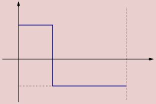
$$\int_\Omega m = -m_0|\Omega| \qquad \text{and} \qquad m = \kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}.$$

- Dirichlet case, with no sign changement on m : symmetrization, regularity in case of symmetry [Krein 1955, Friedland 1977, Cox 1990]
- Periodic case : [Hamel-Roques 2007]
- Neumann 1D case : solved [Lou-Yanagida 2006]
- Robin 1D case : optimization among intervals [Hintermüller-Kao-Laurain 2012]
- Dirichlet 2D case : regularity [Chanillo-Kenig-To 2008]
- Numerics : [Cox, Hamel-Roques, Hintermüller-Kao-Laurain]

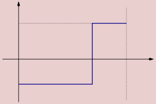
Conjectures in the Neumann case

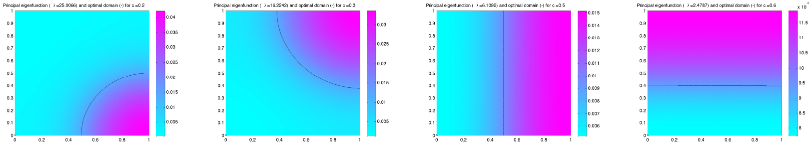
Proposition (Lou & Yanagida 2006)

In 1D (Neumann case), the only solutions of $\inf\left\{\lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}),\ |E| = c|\Omega|\right\}$ are



and





(a) $c = 0.2$ (b) $c = 0.3$ (c) $c = 0.5$ (d) $c = 0.6$

Figure – $\Omega = (0,1)^2$. Optimal domains with $\kappa = 0.5$ and $c \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$

Conjecture (Berestycki - Hamel - Roques)

For c small enough, the free boundaries of minimizers are quarters of circles.

Outline

New results : in dimension $N \geq 2$, is the solution a part of ball ?

$$\inf \left\{ \lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega \setminus E}), \quad |E| = c|\Omega| \right\} \tag{P}$$

Theorem (Lamboley, Laurain, Nadin, YP)

Let assume that $N \geq 2$ and $\partial\Omega$ is connected and C^1. Let E is a critical point for Problem (P). Then, If E or its complement set in Ω is invariant by rotation, then Ω is a ball.

Theorem (Lamboley, Laurain, Nadin, YP)

Let assume that $N \geq 2$ and $\partial\Omega = (0,1)^N$. Let E is a critical point for Problem (P). Then

- E has only one connected component (concentration of minimizers)
- $|\partial E \cap \partial\Omega| > 0$,
- E is not a quarter of ball.

$\rightsquigarrow$The wording "critical" means that E satisfies the 1st order optimality conditions, i.e.

shape derivative of λ at E in direction $V = \langle d\lambda(E), V \rangle \geq 0$,

for all smooth vector fields $V : \mathbb{R}^N \to \mathbb{R}^N$.

It also rewrites : E is a level set of φ, i.e. $E = \{\varphi > \alpha\}$.

Steps of the proof of Theorem 2

Assume $E = B(0, r)$ (or more generally that E in invariant by rotation).

$\rightsquigarrow$ Continuation in E : φ is radial in E : show that $v_{ij} := x_i \partial_{x_j} \varphi - x_j \partial_{x_i} \varphi$ vanishes $(i \neq j)$; to that end use optimality condition.

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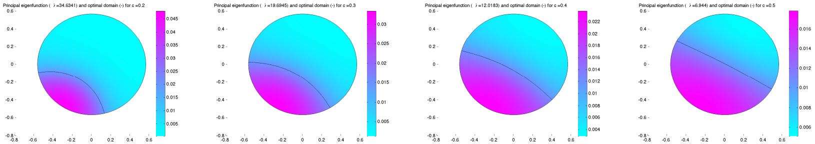
$\rightsquigarrow$ Continuation in Ω : φ is radial in Ω :
Analytic regularity and Cauchy-Kowalevski Theorem.

$\rightsquigarrow$ Ω is a ball.
Geometrical study of the contact angle between the inscribed and circumscribed balls of Ω and $\partial\Omega$.

Neumann case with $\Omega = B(0,1)$

$$\inf\left\{\lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\} \tag{P}$$



(a) $c = 0.2$ (b) $c = 0.3$ (c) $c = 0.4$ (d) $c = 0.5$

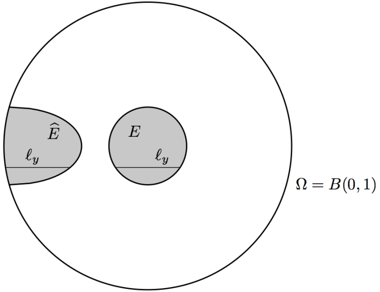
Theorem (Lamboley, Laurain, Nadin, YP)

Let $N \in \{2,3,4\}$ and $\Omega = B(0,1) \subset \mathbb{R}^N$.
Then the centered ball of volume $c|\Omega|$ is not a minimizer for Problem (P).

Ideas of the proof

$\Omega = B(0,1)$, E rotationnally symmetric :

Disymmetrization procedure :



One proves : $\lambda(\widehat{E}) < \left(\dfrac{5N-4}{4N}\right)\lambda(E)$.

Outline

Minimizing the total population size (1)

$$\boxed{\sup_{|E|=c|\Omega|} \int_\Omega u^*} \quad \text{where } u^* \text{ solves the PDE } \begin{cases} \mu\Delta u^* + u^*(\kappa \mathbb{1}_E - u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{cases}$$

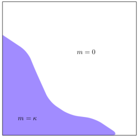
$\rightsquigarrow$ In this model, we always have persistence of species (i.e. $u(t,\cdot) \to u^*$ as $t \to +\infty$)

Theorem (Mazari, Nadin, YP)

Let $\Omega = \prod_{i=1}^N (a_i, b_i)$.

- The problem above has a solution E_μ whenever μ is large enough.
- In 1D, if $\mu \geq \mu^*$: E_μ is an interval meeting one extremity of Ω
- In 1D, if μ is small enough, optimal domains are "fragmented".

$\rightsquigarrow$ Similar conclusions for general domains Ω



Minimizing the total population size (2)

$$\sup_{|E|=c|\Omega|} \int_\Omega u^* \quad \text{where } u^* \text{ solves the PDE } \begin{cases} \mu\Delta u^* + u^*(\kappa \mathbb{1}_E - u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{cases}$$

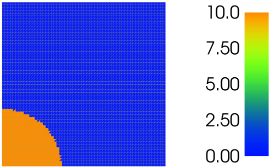
$\rightsquigarrow$ In this model, we always have persistence of species (i.e. $u(t,\cdot) \to u^*$ as $t \to +\infty$)

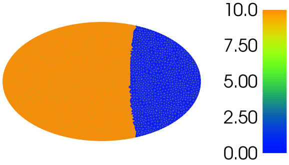
Theorem (Mazari, Nadin, YP)

Let Ω be a convex domain. As $\mu \to +\infty$, E_μ converges in the sense of characteristic functions to a solution of the shape optimization problem

$$\sup_{|E|=c|\Omega|} \int_\Omega |\nabla u^\infty|^2 \quad \text{where } u^\infty \text{ solves the PDE } \begin{cases} \Delta u^\infty + c(\kappa \mathbb{1}_E - c) = 0 & \text{in } \Omega \\ \int_\Omega u^\infty = 0, \quad \partial_n u^\infty = 0 & \text{on } \partial\Omega \end{cases}$$

Simulations by courtesy of Michel Duprez





Sketch of proof : existence of optimal shapes as $\mu \to +\infty$

Let u^* be the solution of $\begin{cases} \mu\Delta u^* + u^*(\kappa \mathbb{1}_E - u^*) = 0 & \text{in } \Omega \\ \partial_n u^* = 0 & \text{on } \partial\Omega \end{cases}$

- Expansion in powers of μ : expands as

$$u^* = c + \frac{\hat{u}}{\mu} + \frac{\mathcal{R}_\mu}{\mu^2},$$

with $\hat{u} = \widehat{\eta} + \beta$, where $\widehat{\eta}$ is the unique solution of

$$\begin{cases} \Delta\widehat{\eta} + c(\kappa \mathbb{1}_E - c) = 0 & \text{in } \Omega \\ \partial_n \widehat{\eta} = 0, & \text{on } \partial\Omega \end{cases}, \text{ with } \int_\Omega \widehat{\eta} = 0$$

- $F_\mu(\mathbb{1}_E) = \int_\Omega u^*$ enjoys a convexity property whenever μ is large enough. One shows that

$$d^2 F_\mu(\mathbb{1}_E)(h,h) = \frac{1}{\mu}\int_\Omega |\nabla \dot{\widehat{\eta}}|^2 + \mathrm{O}\left(\frac{1}{\mu^2}\right) \quad \text{where } \Delta\dot{\widehat{\eta}} + \mathbb{1}_E h = 0.$$

$\rightsquigarrow$ Estimate of the remainder term by using series expansions and Sobolev type estimates

Outline

Similar problem when adding a drift term

$\rightsquigarrow$ We enrich the model by adding an advection term along the gradient of the habitat quality (according to Belgacem and Cosner)

$$\begin{cases} \partial_t u = \mathrm{div}(\nabla u - \alpha u \nabla m) + \lambda u(m-u) & \text{in } \Omega \times (0,\infty), \\ e^{\alpha m}(\partial_n u - \alpha u \partial_n m) + \beta u = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

This models the tendency of the population to move up along the gradient of m.

New shape optimization problem

$$\inf_{m \in \mathcal{M}_{m_0,\kappa}} \lambda_\alpha(m),$$

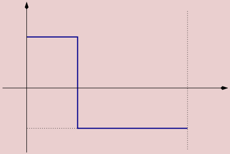
with $\lambda_\alpha(m) = \inf_{\varphi \in \mathcal{S}_0} \dfrac{\int_\Omega e^{\alpha m} |\nabla \varphi|^2}{\int_\Omega m e^{\alpha m} \varphi^2}$ and $\mathcal{S}_0 = \{\varphi \in H^1(\Omega), \int_\Omega m e^{\alpha m} \varphi^2 > 0\}$

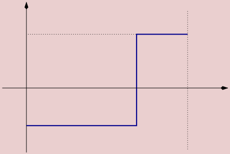
Similar problem when adding a drift term

Theorem (1D model, Caubet, Deheuvels, YP (2017))

Assume that $\Omega = (0,1)$. There exists $\beta^* > 0$ such that

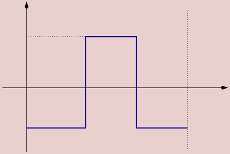
- **if** $\beta < \beta^*$,





are the only solutions.

- **if** $\beta > \beta^*$



is the only solution.

F. Caubet, T. Deheuvels, Y. Privat, *Optimal location of resources for biased movement of species : the 1D case*, SIAM J. Applied Math 77 (2017), no. 6, 1876–1903.

Similar problem when adding a drift term

Theorem (Mazari, Nadin, YP (2019))

Assume that $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is bounded and connected.

- If the problem
$$\inf_{m \in \mathcal{M}_{m_0,\kappa}} \lambda_\alpha(m)$$
has a solution m^*, then necessarily, m^* is bang-bang (i.e. $\exists E^* \subset \Omega$ s.t. $m^* = \kappa \mathbb{1}_{E^*}$)
- In that case, if moreover ∂E^* is a C^2 hypersurface, then Ω is necessarily a ball.
- If Ω is a ball, if α is small enough and if $n = 2, 3$, the centered ball is the unique minimizer of $E \mapsto \lambda_\alpha(\mathbb{1}_E)$ among radial domains E with prescribed volume $c|\Omega|$.

Open problem : case where Ω is a ball.

Existence and characterization of optimal radial domains in any dimension ?

I. Mazari, G. Nadin, Y. Privat, *Shape optimization of a two-phase weighted Dirichlet eigenvalue*, Preprint (2019).

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Conclusion and open questions

On the problem $\inf\left\{\lambda(E) := \lambda(\kappa \mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}), \quad |E| = c|\Omega|\right\}$ (P)

Consider the more general boundary condition

$$\partial_n u + \beta u = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+ \quad \text{(partially inhospitable boundary region)}$$

- If Ω is a ball, is E a concentric ball ?

  $\rightsquigarrow$ Solved if $N = 1$: yes if β is large enough, no else.
  $\rightsquigarrow$ Yes if $\beta = \infty$, No if $\beta = 0$ and $N \in \{2, 3, 4\}$

- Can $\partial E \cap \Omega$ be a piece of sphere ?

  $\rightsquigarrow$ No if $\beta = 0$ and Ω is a square/cube

- Find sufficient conditions so that $\partial E \cap \partial\Omega \neq \emptyset$,

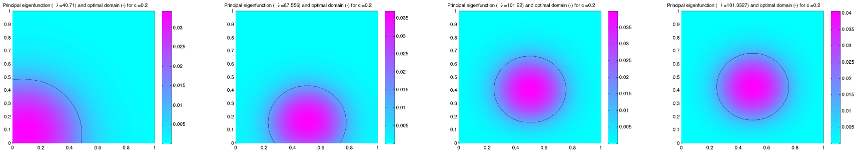
  $\rightsquigarrow$ Expected to be true if $\beta = 0$

Conclusion and open questions

- Can a Faber-Krahn type inequality be expected in the Dirichlet case ($\beta \to +\infty$) ?

$$\text{On the total population size problem } \sup \left\{ \int_\Omega u^*, \quad |E| = c|\Omega| \right\} \qquad \text{(P)}$$

- Existence of *bang-bang* controls for small diffusivities μ ?
- If the answer is yes, the minimizers are fragmented. Can we provide an estimate of the number of connected components wrt μ ?



(a) $\beta = 1$ (b) $\beta = 5$ (c) $\beta = 50$ (d) $\beta = 1000$

Figure – Optimal domains w.r.t. β in the case $\alpha = 0$ (no drift term)

Thank you for your attention