On the use of the damped Newton method to solve direct and controllability problems for parabolic PDEs

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ongoing works with Jérome Lemoine (Clermont-Ferrand) and Irene Gayte (Sevilla)

The talk discusses the approximation of solution of a controllability problem for (nonlinear) PDEs through least-squares method.

For instance, for the Navier-Stokes system: Given $\Omega \in \mathbb{R}^d$, $T > 0$, find a sequence $\{y_k, p_k, v_k\}_{k>0}$ converging (strongly) toward to a solution (y, p, v) of

$$\begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = 0, \quad \nabla\cdot y = 0 & \Omega\times(0,T), \\ y = v, & \partial\Omega\times(0,T), \\ y(0) = y_0, & \Omega\times\{0\} \end{cases} \tag{1}$$

satisfying $y(T) = u_d$, a trajectory (control of flows).

• Largely open question in the context of nonlinear PDEs

• Not straightforward issue, mainly because the fixed point operator (used to prove controllability result) is not a contraction !

Outline

Part 1– Direct Problem for Steady NS - find a sequence $(y_k, p_k)_{k>0}$ converging strongly to a pair (y, p) solution of

$$\begin{cases} \alpha y - \nu \Delta y + (y \cdot \nabla) y + \nabla p = f + \alpha g, \quad \nabla \cdot y = 0 & \Omega, \\ y = 0, & \partial\Omega. \end{cases} \tag{2}$$

(*useful to solve Implicit time schemes for Unsteady NS ....*)

Part 2– Direct problem for Unsteady NS - find a sequence $(y_k, p_k)_{k>0}$ converging strongly to a pair (y, p) solution of

$$\begin{cases} y_t - \nu \Delta y + (y \cdot \nabla) y + \nabla p = f, \nabla \cdot y = 0 & \Omega \times (0, T), \\ y = 0, & \partial\Omega \times (0, T), \\ y(0) = y_0, & \Omega \times \{0\} \end{cases} \tag{3}$$

Part 3– Controllability problem for a sub-linear (controllable) heat equation: find a sequence $(y_k, v_k)_{k>0}$ converging strongly to a pair (y, v) solution of

$$\begin{cases} y_t - \nu \Delta y + g(y) = v 1_\omega, & \Omega \times (0, T), \\ y = 0, & \partial\Omega \times (0, T), \\ y(0) = y_0, & \Omega \times \{0\} \end{cases} \tag{4}$$

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Part 1 – Direct Problem for Steady NS -
Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded connected open set with boundary $\partial\Omega$ Lipschitz. $\boldsymbol{\mathcal{V}} = \{v \in \mathcal{D}(\Omega)^d, \nabla \cdot v = 0\}$, $\boldsymbol{H}$ the closure of $\boldsymbol{\mathcal{V}}$ in $L^2(\Omega)^d$ and $\boldsymbol{V}$ the closure of $\boldsymbol{\mathcal{V}}$ in $H^1(\Omega)^d$.

Find a sequence $(y_k, p_k)_{k>0}$ converging strongly to a pair (y, p) solution of

$$\begin{cases} \alpha y - \nu \Delta y + (y \cdot \nabla) y + \nabla p = f + \alpha g, \quad \nabla \cdot y = 0 & \Omega, \\ y = 0, & \partial\Omega. \end{cases} \tag{5}$$

$f \in H^{-1}(\Omega)^d$, $g \in L^2(\Omega)^d$ and $\underline{\alpha \in \mathbb{R}_+^\star}$.

Part 1- Weak formulation

Let $f \in H^{-1}(\Omega)^d$, $g \in L^2(\Omega)^d$ and $\alpha \in \mathbb{R}_+^\star$. The weak formulation of (5) reads as follows: find $y \in \boldsymbol{V}$ solution of

$$\alpha \int_\Omega y\cdot w + \nu \int_\Omega \nabla y\cdot\nabla w + \int_\Omega y\cdot\nabla y\cdot w = < f, w >_{H^{-1}(\Omega)^d\times H_0^1(\Omega)^d} + \alpha \int_\Omega g\cdot w, \quad \forall w \in \boldsymbol{V}. \tag{6}$$

Proposition

Assume $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz. There exists a least one solution y of (6) satisfying

$$\alpha\|y\|_2^2 + \nu\|\nabla y\|_2^2 \le \frac{c(\Omega)}{\nu}\|f\|^2_{H^{-1}(\Omega)^d} + \alpha\|g\|_2^2 \tag{7}$$

for some constant $c(\Omega) > 0$. If moreover, Ω is C^2 and $f \in L^2(\Omega)^d$, then $y \in H^2(\Omega)^d \cap \boldsymbol{V}$.

Remark- If

$$Q(g, f, \alpha, \nu) := \begin{cases} \dfrac{1}{\nu^2}\left(\|g\|_2^2 + \dfrac{1}{\alpha\nu}\|f\|^2_{H^{-1}(\Omega)^d}\right), & \text{if} \quad d = 2, \\ \dfrac{\alpha^{1/2}}{\nu^{5/2}}\left(\|g\|_2^2 + \dfrac{1}{\alpha\nu}\|f\|^2_{H^{-1}(\Omega)^d}\right), & \text{if} \quad d = 3. \end{cases}$$

is small enough, then the solution of (6) is unique.

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is small enough, then the solution of (6) is unique.

- We introduce the least-squares problem with $E : \boldsymbol{V} \to \mathbb{R}^+$ as follows

$$inf_{y\in \boldsymbol{V}} E(y) := \frac{1}{2}\int_\Omega (\alpha |v|^2 + |\nabla v|^2) \tag{8}$$

where the corrector $v \in \boldsymbol{V}$ is the unique solution of

$$\alpha \int_\Omega v\cdot w + \int_\Omega \nabla v \cdot \nabla w = -\alpha \int_\Omega y\cdot w - \nu \int_\Omega \nabla y\cdot\nabla w - \int_\Omega y\cdot\nabla y\cdot w + < f, w >_{H^{-1}(\Omega)^d\times H_0^1(\Omega)^d} + \alpha \int_\Omega g\cdot w, \quad \forall w \in \boldsymbol{V}. \tag{9}$$

- $inf_{y\in \boldsymbol{V}} E(y) = 0$ reached by a solution of (6). In this sense, the functional E is a so-called error functional which measures, through the corrector variable v, the deviation of the pair y from being a solution of (6).

Remark-

$$E(y) \approx \frac{1}{2}\|\alpha y + \nu B_1(y) + B(y,y) - f + \alpha g\|^2_{V'},$$

$$(B_1(y), w) := (\nabla y, \nabla w)_2, \quad (B(y,z), w) := \int_\Omega y\nabla z\cdot w, \quad y, z, w \in \boldsymbol{V}$$

considered in [1] with experiments but without mathematical justification !

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Proposition

Let $\mathbb{B}_c = \{y \in \boldsymbol{V} : \frac{1}{\nu\alpha}\|\nabla y\|_2^{2(d-1)} < c\}$, $d \in \{2,3\}$, $c > 0$ *There exists a positive constant* C *such that*

$$\sqrt{E(y)} \leq \frac{\nu^{-1}}{\sqrt{2}}\|E'(y)\|_{\boldsymbol{V}'}, \quad \forall y \in \mathbb{B}_C \tag{10}$$

PROOF- • For any $y \in \mathbb{B}_c$, there exists a unique element $Y_1 \in \boldsymbol{V}$ solution of

$$\alpha \int_\Omega Y_1 \cdot w + \nu \int_\Omega \nabla Y_1 \cdot \nabla w + \int_\Omega (y \cdot \nabla Y_1 + Y_1 \cdot \nabla y) \cdot w = -\alpha \int_\Omega v \cdot w - \int_\Omega \nabla v \cdot \nabla w, \forall w \in \boldsymbol{V}$$

where $v \in \boldsymbol{V}$ is the corrector associated to y.

• Y_1 enjoys the following properties: There exists $c > 0$ such that

$$E'(y) \cdot Y_1 = 2E(y), \qquad \text{and} \qquad \|Y_1\|_{\boldsymbol{V}} \leq \sqrt{2}\nu^{-1}\sqrt{E(y)}, \qquad \forall y \in \mathbb{B}_c$$

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$$\begin{cases} y_0 \in \boldsymbol{V}, \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k > 0, \\ \lambda_k = \mathrm{argmin}_{\lambda \in \mathbb{R}^+} E(y_k - \lambda Y_{1,k}) \end{cases} \tag{11}$$

where $Y_{1,k}$ solves the formulation, for all $w \in \boldsymbol{V}$

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leading to $E'(y_k)\cdot Y_{1,k} = 2E(y_k)$.

Theorem

Assume that $y_0 \in \boldsymbol{V}$ satisfies $E(y_0) \le \mathcal{O}(\nu^2(\alpha\nu)^{1/(d-1)})$. Then, $y_k \to y$ strongly in $\boldsymbol{V}$ as $k \to \infty$ where y is a solution of the α-NS equation.
The convergence is quadratic after a finite number of iterate.

Sketch of the proof ($d = 2$): We develop $E(y_k - \lambda Y_{1,k})$ - polynomial of order 4 w.r.t. λ and find that

$$\sqrt{E(y_k - \lambda Y_{1,k})} \le \underbrace{\left(|1-\lambda| + \lambda^2 c_\nu \sqrt{E(y_k)}\right)}_{:=p(\lambda)} \sqrt{E(y_k)}, \quad c_\nu = c(\Omega)\frac{2}{\nu} max(1, \frac{2}{\nu}) = \mathcal{O}(\nu^{-2})$$

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• If $c_\nu\sqrt{E(y_k)} \ge 1$, p reaches a unique minimum for $\lambda_k = 1/(2c_\nu\sqrt{E(y_k)}) \in (0,1/2)$ for which $p(\lambda_k) = 1 - \frac{\lambda_k}{2} \in (0,1)$ leading to

$$c_\nu\sqrt{E(y_{k+1})} \le p(\lambda_k) c_\nu \sqrt{E(y_k)} = \underbrace{\left(1 - \frac{1}{4c_\nu\sqrt{E(y_k)}}\right)}_{\in(0,1)} c_\nu\sqrt{E(y_k)}.$$

and then to

$$c_\nu\sqrt{E(y_{k+p})} \le \left(1 - \frac{1}{4c_\nu\sqrt{E(y_k)}}\right)^p c_\nu\sqrt{E(y_k)} \to 0 \quad \text{as} \quad p\to\infty.$$

• If $c_\nu\sqrt{E(y_k)} < 1$ for some $k \ge m$. Then,

$$\sqrt{E(y_{k+1})} \le p(\lambda_k)\sqrt{E(y_k)} \le p(1)\sqrt{E(y_k)} = c_\nu E(y_k)$$

so that

$$c_\nu\sqrt{E(y_{k+1})} \le (c_\nu\sqrt{E(y_k)})^2, \quad \forall k \ge m$$

The sequence $\{c_\nu\sqrt{E(y_m)}\}_{(m\ge k)}$ decreases to zero with a quadratic rate. In particular, if $c_\nu\sqrt{E(y_0)} \le 1$ and if we fixe $\lambda_k = 1$ for all $k \ge 0$,

Convergence of $E(y_k)$

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• We write that $y_{k+1} = y_0 - \sum_{m=0}^{k} \lambda_m Y_{1,m}$; using that $\lambda_m \in (0,1)$ and $\|Y_{1,m}\|_{\boldsymbol{V}} \leq \nu^{-1}\sqrt{E(y_m)}$, we get

$$\sum_{m=1}^{k} |\lambda_m| \|Y_{1,m}\|_{\boldsymbol{V}} \leq \nu^{-1} \sum_{m=1}^{k} \sqrt{E(y_m)} \leq \nu^{-1} \sum_{m=1}^{k} p(\lambda_{m-1})\sqrt{E(y_{m-1})}$$
$$\leq \nu^{-1} \sum_{m=1}^{k} p(\lambda_0)\sqrt{E(y_{m-1})} \leq \nu^{-1} \sum_{m=1}^{k} p(\lambda_0)^m \sqrt{E(y_0)}$$
$$\leq \frac{\nu^{-1}}{1-p(\lambda_0)}\sqrt{E(y_0)}$$

This implies the strong convergence of y_k toward $y := y_0 - \sum_{m\geq 0} \lambda_m Y_{1,m}$.

• Using that $E(y_k) \to 0$ as $k \to \infty$, the limit in the corrector eq. for v_k,

$$\alpha \int_\Omega v_k \cdot w + \int_\Omega \nabla v_k \cdot \nabla w = -\alpha \int_\Omega y_k \cdot w - \nu \int_\Omega \nabla y_k \cdot \nabla w - \int_\Omega y_k \cdot \nabla y_k \cdot w$$
$$+ < f, w >_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} + \alpha \int_\Omega g \cdot w, \quad \forall w \in \boldsymbol{V}. \tag{12}$$

implies that y solves the α-NS steady equation.

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• The quadratic convergence of the sequence $\{y_k\}_{k>0}$ after a finite number of iterations is due to the inequality

$$
\begin{aligned}
\|y - y_k\|_V &= \Big\| \sum_{m\geq k+1} \lambda_m Y_{1,m} \Big\|_V \\
&\leq \sum_{m\geq k+1} \|Y_{1,m}\|_V \leq \nu^{-1} \sum_{m\geq k+1} \sqrt{E(y_m)} \\
&\leq \nu^{-1} \sum_{m\geq k+1} p(\lambda_{m-1}) \sqrt{E(y_{m-1})} \\
&\leq \nu^{-1} \sum_{m\geq k+1} p(\lambda_k) \sqrt{E(y_{m-1})} \\
&\leq \nu^{-1} \sum_{m\geq k+1} p(\lambda_k)^{m-k} \sqrt{E(y_k)} \\
&\leq \nu^{-1} \frac{p(\lambda_k)}{1-p(\lambda_k)} \sqrt{E(y_k)} \leq \nu^{-1} \frac{p(\lambda_0)}{1-p(\lambda_0)} \sqrt{E(y_k)}, \quad \forall k > 0
\end{aligned}
$$

Rk- The limit $y = y_0 - \sum_{m\geq 0} \lambda_m Y_{1,m}$ is uniquely determined by the initial guess y_0.

The choice $\lambda_k = 1$ converges under the condition that $\sqrt{E(y_0)} \leq \mathcal{O}(\nu^2)$ corresponds to the usual Newton method to solve the variational formulation : find $y \in \boldsymbol{V}$ solution of $F(y,z) = 0, \forall z \in \boldsymbol{V}$,

$$F(y,z) := \int_\Omega \alpha y \cdot z + \nu \nabla y \cdot \nabla z + y \cdot \nabla y \cdot z - < f, z >_{\boldsymbol{V}',\boldsymbol{V}} - \alpha \int_\Omega g \cdot z$$

i.e.

$$\begin{cases} y_0 \in \boldsymbol{V}, \\ \partial_y F(y_k, z) \cdot (y_{k+1} - y_k) = -F(y_k, z), \qquad \forall z \in \boldsymbol{V}, \quad \forall k \geq 0, \end{cases}$$

Remark-

$$E(y) = \frac{1}{2}\left(\sup_{z \in \boldsymbol{V}, z \neq 0} \frac{F(y,z)}{\|z\|_V}\right)^2, \forall y \in \boldsymbol{V}.$$

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Given a discretization $\{t_n\}_{n=0\dots N}$ of $[0,T]$, the backward Euler scheme reads :

$$\begin{cases} \int_\Omega \frac{y^{n+1}-y^n}{\delta t}\cdot w + \nu\int_\Omega \nabla y^{n+1}\cdot\nabla w + \int_\Omega y^{n+1}\cdot\nabla y^{n+1}\cdot w = \langle f^n, w\rangle_{\boldsymbol{V}'\times\boldsymbol{V}},\ \forall n\geq 0,\ \forall w\in \\ y^0(\cdot,0)=u_0, \quad \text{in} \quad \Omega \end{cases} \tag{13}$$

with $f^n := \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} f(\cdot,s)ds$. The piecewise linear interpolation (in time) of $\{y^n\}_{n\in[0,N]}$ weakly converges in $L^2(0,T,\boldsymbol{V})$ toward a solution of Unsteady NS as $\delta t\to 0^+$.
The previous study applied to determine y^{n+1} from y^n, solution of (13) taking $\alpha=\frac{1}{\delta t}$ and $g=y^n$:

Corollary

Assume that $y_0^{n+1}\in\boldsymbol{V}$ satisfies $E(y_0^{n+1})\leq\mathcal{O}(\nu^2(\nu\delta t^{-1})^{1/(d-1)})$. Then, $y_k^{n+1}\to y^{n+1}$ strongly in $\boldsymbol{V}$ as $k\to\infty$ where y^{n+1} solves (13).

Proposition

Assume that $\Omega\in C^2$, that $(f^n)_n$ is a sequence in $L^2(\Omega)^d$ satisfies $\alpha^{-1}\sum_{k=0}^{+\infty}\|f^k\|_2<+\infty$, that $\nabla y^0\in L^2(\Omega)^d$. Then, the sequence $(y^n)_n$ satisfies

$$\|y^{n+1}-y^n\|_2=\mathcal{O}(\delta t^{1/2}\nu^{-3/4}),\quad \forall n\geq 0$$

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Part $2-1$ Direct Problem for unsteady NS -
The weak formulation reads as follows : $f \in L^2(0,T,\boldsymbol{V}')$ and $u_0 \in \boldsymbol{H}$, find a weak solution $y \in L^2(0,T;\boldsymbol{V})$, $\partial_t y \in L^2(0,T;\boldsymbol{V}')$ of the system

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega y\cdot w + \nu\int_\Omega \nabla y\cdot\nabla w + \int_\Omega y\cdot\nabla y\cdot w = \langle f, w\rangle_{\boldsymbol{V}'\times\boldsymbol{V}}, \quad \forall w\in\boldsymbol{V} \\ y(\cdot,0) = u_0, \quad \text{in} \quad \Omega. \end{cases} \tag{14}$$

Let $\mathcal{A} = \{y \in L^2(0,T;\boldsymbol{V}) \cap H^1(0,T;\boldsymbol{V}'),\ y(0) = u_0\}$.

Proposition

There exists a unique $\bar{y} \in \mathcal{A}$ solution in $\mathcal{D}'(0,T)$ of (14). This solution satisfies the following estimates :

$$\|\bar{y}\|^2_{L^\infty(0,T;\boldsymbol{H})} + \nu\|\bar{y}\|^2_{L^2(0,T;\boldsymbol{V})} \leq \|u_0\|^2_{\boldsymbol{H}} + \frac{1}{\nu}\|f\|^2_{L^2(0,T;\boldsymbol{V}')},$$

$$\|\partial_t\bar{y}\|_{L^2(0,T;\boldsymbol{V}')} \leq \sqrt{\nu}\|u_0\|_{\boldsymbol{H}} + 2\|f\|_{L^2(0,T;\boldsymbol{V}')} + \frac{c}{\nu^{\frac{3}{2}}}(\nu\|u_0\|^2_{\boldsymbol{H}} + \|f\|^2_{L^2(0,T;\boldsymbol{V}')}).$$

We introduce the LS functional $E : H^1(0,T,\boldsymbol{V}') \cap L^2(0,T,\boldsymbol{V}) \to \mathbb{R}^+$ by putting

$$E(y) = \frac{1}{2}\int_0^T \|v\|^2_{\boldsymbol{V}} + \frac{1}{2}\int_0^T \|\partial_t v\|^2_{\boldsymbol{V}'}$$

where the corrector $v \in \mathcal{A}_0 = \{y \in L^2(0,T;\boldsymbol{V}) \cap H^1(0,T;\boldsymbol{V}'),\ y(0)=0\}$ is the unique solution in $\mathcal{D}'(0,T)$ of

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Remark- For all $y \in L^2(0,T,\boldsymbol{V}) \cap H^1(0,T;\boldsymbol{V}')$,

$$E(y) \approx \|y_t + \nu B_1(y) + B(y,y) - f\|^2_{L^2(0,T;\boldsymbol{V}')}$$

where $\forall u \in L^\infty(0,T;\boldsymbol{H})$, $v\in L^2(0,T;\boldsymbol{V})$,

$$\langle B(u(t)),v(t)),w\rangle = \int_\Omega u(t)\cdot\nabla v(t)\cdot w \qquad \forall w\in\boldsymbol{V},\ \text{a.e in } t\in[0,T]$$

and $\forall u \in L^2(0,T;\boldsymbol{V})$,

$$\langle B_1(u(t)),w\rangle = \int_\Omega \nabla u(t)\cdot\nabla w \qquad \forall w\in\boldsymbol{V},\ \text{a.e in } t\in[0,T]$$

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Proposition

Let $\bar{y} \in \mathcal{A}$ be the solution of (14), $M \in \mathbb{R}$ such that $\|\partial_t \bar{y}\|_{L^2(0,T,\boldsymbol{V}')} \leq M$ and $\sqrt{\nu}\|\nabla \bar{y}\|_{L^2(Q_T)^4} \leq M$ and let $y \in \mathcal{A}$.
If $\|\partial_t y\|_{L^2(0,T,\boldsymbol{V}')} \leq M$ and $\sqrt{\nu}\|\nabla y\|_{L^2(Q_T)^4} \leq M$, then there exists a constant $c(M)$ such that

$$\|y - \bar{y}\|_{L^\infty(0,T;\boldsymbol{H})} + \sqrt{\nu}\|y - \bar{y}\|_{L^2(0,T;\boldsymbol{V})} + \|\partial_t y - \partial_t \bar{y}\|_{L^2(0,T,\boldsymbol{V}')} \leq c(M)\sqrt{E(y)}.$$

Let $m \geq 1$.

$$\begin{cases} y_0 \in \mathcal{A}, \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k \geq 0, \\ E(y_k - \lambda_k Y_{1,k}) = \min\limits_{\lambda \in [0,m]} E(y_k - \lambda Y_{1,k}) \end{cases} \tag{16}$$

with $Y_{1,k} \in \mathcal{A}_0$ the solution of the formulation

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega Y_{1,k} \cdot w + \nu \int_\Omega \nabla Y_{1,k} \cdot \nabla w + \int_\Omega y_k \cdot \nabla Y_{1,k} \cdot w \\ \qquad + \displaystyle\int_\Omega Y_{1,k} \cdot \nabla y_k \cdot w = -\dfrac{d}{dt}\int_\Omega v_k \cdot w - \int_\Omega \nabla v_k \cdot \nabla w, \quad \forall w \in \boldsymbol{V} \\ Y_{1,k}(0) = 0, \end{cases}$$

where $v_k \in \mathcal{A}_0$ is the corrector (associated to y_k) solution of (15) leading to $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$.

Theorem

Let $\{y_k\}_{k\in\mathbb{N}}$ the sequence of $\mathcal{A}$ defined by (29). Then $y_k \to \bar{y}$ in $H^1(0,T;\boldsymbol{V}') \cap L^2(0,T;\boldsymbol{V})$ where $\bar{y} \in \mathcal{A}$ is the unique solution of (14). Moreover, there exists a $k_0 \in \mathbb{N}$ such that the sequence $\{\|y_k - \bar{y}\|_{\mathcal{A}}\}_{(k\geq k_0)}$ decays quadratically.

The key lemma is

Lemma

Let $\{y_k\}_{k\in\mathbb{N}}$ the sequence of $\mathcal{A}$ defined by (29). Then

$$\sqrt{E(y_{k+1})} \leq \sqrt{E(y_k)}\Big(|1-\lambda| + \lambda^2 C_1\sqrt{E(y_k)}\Big), \qquad \forall\lambda \in [0,m]. \tag{17}$$

where $C_1 = \frac{c}{\nu\sqrt{\nu}}\exp\Big(\frac{c}{\nu^2}\|u_0\|^2_{\boldsymbol{H}} + \frac{c}{\nu^3}\|f\|^2_{L^2(0,T;\boldsymbol{V}')} + \frac{c}{\nu^3}E(y_0)\Big)$ does not depend on y_k.

PROOF -

$$E(y_k - \lambda Y_{1,k}) \leq E(y_k)\Big(|1-\lambda| + \lambda^2\frac{c}{\nu\sqrt{\nu}}\sqrt{E(y_k)}\exp\big(\frac{c}{\nu}\int_0^T \|y_k\|^2_{\boldsymbol{V}}\big)\Big)^2.$$

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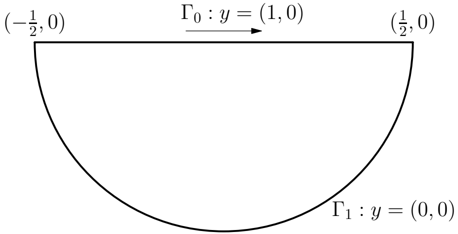
Case considered by Glowinski [2006] [2] for which a Hopf bifurcation phenomenon occurs : for $Re = \nu^{-1} \geq 6650$, the unsteady solution does not converge toward the steady solution.



Semi-disk geometry: $\Omega = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 \leq 1/4, x_2 \leq 0\}$

For $\alpha = 0$ (Pure steady NS) Initialized with the solution of the corresponding Stokes problem,

- Newton algorithm ($\lambda_k = 1$) converges up to $Re \approx 500$.
- Damped Newton algorithm converges up to $Re \approx 910$.

Continuation technic w.r.t. ν is used for $Re > 910$.

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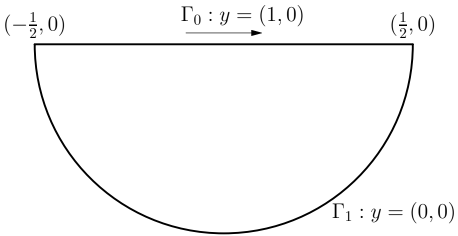

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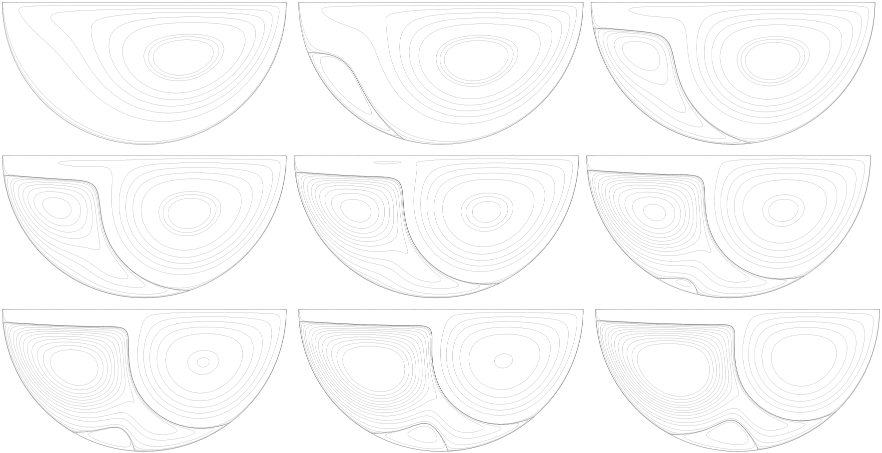
Experiment : The driven semi-disk



Streamlines of the steady state solution for
$Re = 500, 1000, 2000, 3000, 4000, 5000, 6000, 7000$ and $Re = 8000$.

Experiment: Damped Newton Method vs. Newton method; $T = 10$

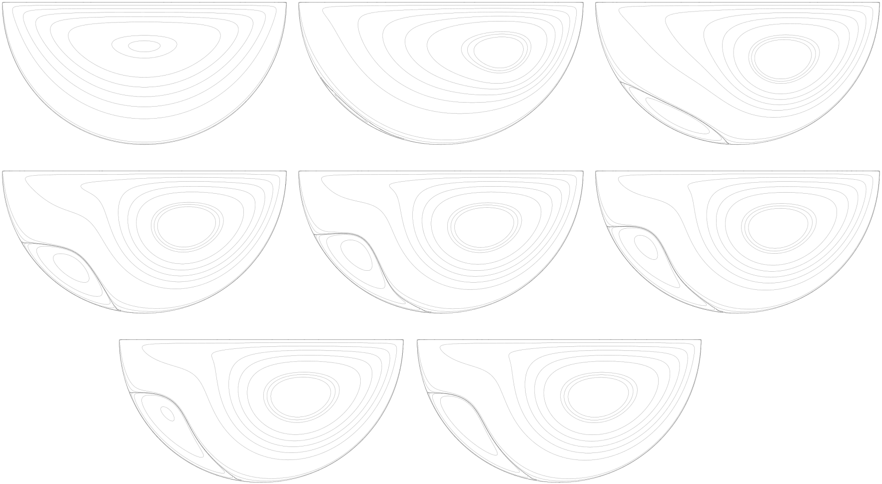
Initialization y_0 (independent of ν) with the Stokes solutions associated to $\nu = 1$.

♯iterate k	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(\mathbf{V})}}{\Vert y_{k-1}\Vert_{L^2(\mathbf{V})}}$	$\sqrt{2E(y_k)}$	λ_k	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(\mathbf{V})}}{\Vert y_{k-1}\Vert_{L^2(\mathbf{V})}}$ $(\lambda_k = 1)$	$\sqrt{2E(y_k)}$ $(\lambda_k = 1)$
0	–	2.690×10^{-2}	0.8112	–	2.690×10^{-2}
1	4.540×10^{-1}	1.077×10^{-2}	0.7758	5.597×10^{-1}	1.254×10^{-2}
2	1.836×10^{-1}	3.653×10^{-3}	0.8749	2.236×10^{-1}	5.174×10^{-3}
3	7.503×10^{-2}	7.794×10^{-4}	0.9919	7.830×10^{-2}	6.133×10^{-4}
4	1.437×10^{-2}	2.564×10^{-5}	1.0006	9.403×10^{-3}	1.253×10^{-5}
5	4.296×10^{-4}	3.180×10^{-8}	1.	1.681×10^{-4}	4.424×10^{-9}
6	5.630×10^{-7}	6.384×10^{-11}	–	–	–

$Re = \nu^{-1} = 500$

♯iterate k	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(\mathbf{V})}}{\Vert y_{k-1}\Vert_{L^2(\mathbf{V})}}$	$\sqrt{2E(y_k)}$	λ_k	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(\mathbf{V})}}{\Vert y_{k-1}\Vert_{L^2(\mathbf{V})}}$ $(\lambda_k = 1)$	$\sqrt{2E(y_k)}$ $(\lambda_k = 1)$
0	–	2.690×10^{-2}	0.6344	–	2.690×10^{-2}
1	5.138×10^{-1}	1.493×10^{-2}	0.5803	8.101×10^{-1}	2.234×10^{-2}
2	2.534×10^{-1}	7.608×10^{-3}	0.3496	4.451×10^{-1}	2.918×10^{-2}
3	1.345×10^{-1}	5.477×10^{-3}	0.4025	5.717×10^{-1}	5.684×10^{-2}
4	1.105×10^{-1}	3.814×10^{-3}	0.5614	3.683×10^{-1}	2.625×10^{-2}
5	8.951×10^{-2}	2.295×10^{-3}	0.8680	2.864×10^{-1}	1.828×10^{-2}
6	6.394×10^{-2}	8.679×10^{-4}	1.0366	1.423×10^{-1}	4.307×10^{-3}
7	1.788×10^{-2}	4.153×10^{-5}	0.9994	6.059×10^{-2}	9.600×10^{-4}
8	7.982×10^{-4}	9.931×10^{-8}	0.9999	1.484×10^{-2}	5.669×10^{-5}
9	2.256×10^{-6}	4.000×10^{-11}	–	9.741×10^{-4}	3.020×10^{-7}
10	–	–	–	4.267×10^{-6}	3.846×10^{-11}

$Re = \nu^{-1} = 1000$



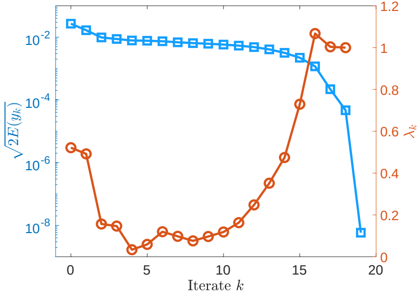
Streamlines of the unsteady state solution for $Re = 1000$ at time $t = i$, $i = 0, \cdots, 7$s.

♯iterate k	$\frac{\|y_k - y_{k-1}\|_{L^2(\mathbf{V})}}{\|y_{k-1}\|_{L^2(\mathbf{V})}}$	$\sqrt{2E(y_k)}$	λ_k	$\frac{\|y_k - y_{k-1}\|_{L^2(\mathbf{V})}}{\|y_{k-1}\|_{L^2(\mathbf{V})}}$ $(\lambda_k = 1)$	$\sqrt{2E(y_k)}$ $(\lambda_k = 1)$
0	–	2.691×10^{-2}	0.6145	–	2.691×10^{-2}
1	5.241×10^{-1}	1.530×10^{-2}	0.5666	8.528×10^{-1}	2.385×10^{-2}
2	2.644×10^{-1}	8.025×10^{-3}	0.3233	4.893×10^{-1}	3.555×10^{-2}
3	1.380×10^{-1}	5.982×10^{-3}	0.3302	7.171×10^{-1}	8.706×10^{-2}
4	1.115×10^{-1}	4.543×10^{-3}	0.4204	4.849×10^{-1}	3.531×10^{-2}
5	9.429×10^{-2}	3.221×10^{-3}	0.5875	1.125×10^{0}	3.905×10^{-1}
6	7.664×10^{-2}	1.944×10^{-3}	0.9720	–	1.337×10^{4}
7	5.688×10^{-2}	5.937×10^{-4}	1.022	–	8.091×10^{27}
8	1.009×10^{-2}	1.081×10^{-5}	0.9998	–	–
9	2.830×10^{-4}	1.332×10^{-8}	1.	–	–
10	2.893×10^{-7}	4.611×10^{-11}	–	–	–

Table: $Re = 1100$: Damped Newton method vs. Newton method.

♯iterate k	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(\boldsymbol{V})}}{\Vert y_{k-1}\Vert_{L^2(\boldsymbol{V})}}$	$\sqrt{2E(y_k)}$	λ_k
0	–	2.691×10^{-2}	0.5215
1	6.003×10^{-1}	1.666×10^{-2}	0.4919
2	3.292×10^{-1}	9.800×10^{-3}	0.1566
3	1.375×10^{-1}	8.753×10^{-3}	0.1467
4	1.346×10^{-1}	7.851×10^{-3}	0.0337
5	5.851×10^{-2}	7.688×10^{-3}	0.0591
6	7.006×10^{-2}	7.417×10^{-3}	0.1196
7	9.691×10^{-2}	6.864×10^{-3}	0.0977
8	8.093×10^{-2}	6.465×10^{-3}	0.0759
9	6.400×10^{-2}	6.182×10^{-3}	0.0968
10	6.723×10^{-2}	5.805×10^{-3}	0.1184
11	6.919×10^{-2}	5.371×10^{-3}	0.1630
12	7.414×10^{-2}	4.825×10^{-3}	0.2479
13	8.228×10^{-2}	4.083×10^{-3}	0.3517
14	8.146×10^{-2}	3.164×10^{-3}	0.4746
15	7.349×10^{-2}	2.207×10^{-3}	0.7294
16	6.683×10^{-2}	1.174×10^{-3}	1.0674
17	3.846×10^{-2}	2.191×10^{-4}	1.0039
18	5.850×10^{-3}	4.674×10^{-5}	0.9998
19	1.573×10^{-4}	5.843×10^{-9}	–

$Re = 2000$



$Re = 3000$: 39 iterations ; $Re = 4000$: 75 iterations.

Part 2 $-$ 2 Direct Problem for unsteady NS -
Let $\Omega \subset \mathbb{R}^3$ be a bounded connected open set whose boundary $\partial\Omega$ is $\mathcal{C}^2$
For $f \in L^2(Q_T)^3$ and $u_0 \in \boldsymbol{V}$, there exists $T^* = T^*(\Omega, \nu, u_0, f) > 0$ and a unique solution $\overline{y} \in L^\infty(0, T^*; \boldsymbol{V}) \cap L^2(0, T^*; H^2(\Omega)^3)$, $\partial_t \overline{y} \in L^2(0, T^*; \boldsymbol{H})$ of the equation

$$\begin{cases} \frac{d}{dt}\int_\Omega y \cdot w + \nu \int_\Omega \nabla y \cdot \nabla w + \int_\Omega y \cdot \nabla y \cdot w = \int_\Omega f \cdot w, \quad \forall w \in \boldsymbol{V} \\ y(\cdot, 0) = u_0, \quad \text{in} \quad \Omega. \end{cases} \tag{18}$$

For any $t > 0$, let

$$\mathcal{A}(t) = \{y \in L^2(0, t; H^2(\Omega)^3 \cap \boldsymbol{V}) \cap H^1(0, t; \boldsymbol{H}),\ y(0) = u_0\}$$

and

$$\mathcal{A}_0(t) = \{y \in L^2(0, t; H^2(\Omega)^3 \cap \boldsymbol{V}) \cap H^1(0, t; \boldsymbol{H}),\ y(0) = 0\}.$$

Endowed with the scalar product $\langle y, z\rangle_{\mathcal{A}_0(t)} = \int_0^t \langle P(\Delta y), P(\Delta z)\rangle_{\boldsymbol{H}} + \langle \partial_t y, \partial_t z\rangle_{\boldsymbol{H}}$ and the norm $\|y\|_{\mathcal{A}_0(t)} =< y, y >_{\mathcal{A}_0(t)}$ is a Hilbert space.
P is the orthogonal projector in $L^2(\Omega)^3$ onto $\boldsymbol{H}$

We introduce our least-squares functional $E : \mathcal{A}(T^*) \to \mathbb{R}^+$ by putting

$$E(y) = \frac{1}{2}\int_0^{T^*} \|P(\Delta v)\|^2_{\boldsymbol{H}} + \frac{1}{2}\int_0^{T^*} \|\partial_t v\|^2_{\boldsymbol{H}} = \frac{1}{2}\|v\|^2_{\mathcal{A}_0(T^*)} \tag{19}$$

Proposition

Let $\bar{y} \in \mathcal{A}(T^*)$ be the solution of (18), $M \in \mathbb{R}$ such that $\|\partial_t \bar{y}\|_{L^2(Q_{T^*})^3} \leq M$ and $\sqrt{\nu}\|P(\Delta\bar{y})\|_{L^2(Q_{T^*})^3} \leq M$ and let $y \in \mathcal{A}(T^*)$. If $\|\partial_t y\|_{L^2(Q_{T^*})^3} \leq M$ and $\sqrt{\nu}\|P(\Delta y)\|_{L^2(Q_{T^*})^3} \leq M$, then there exists a constant $c(M)$ such that

$$\|y-\bar{y}\|_{L^\infty(0,T^*;\boldsymbol{V})}+\sqrt{\nu}\|P(\Delta y)-P(\Delta\bar{y})\|_{L^2(Q_{T^*})^3)}+\|\partial_t y-\partial_t\bar{y}\|_{L^2(Q_{T^*})^3)} \leq c(M)\sqrt{E(y)}.$$

Therefore, we can define, for any $m \geq 1$, a minimizing sequence y_k as follows:

$$\begin{cases} y_0 \in \mathcal{A}(T^*), \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k \geq 0, \\ E(y_k - \lambda_k Y_{1,k}) = \min_{\lambda \in [0,m]} E(y_k - \lambda Y_{1,k}) \end{cases} \tag{20}$$

where $Y_{1,k}$ in $\mathcal{A}_0(T^*)$ solves the formulation

$$\begin{cases} \frac{d}{dt}\int_\Omega Y_{1,k} \cdot w + \nu \int_\Omega \nabla Y_{1,k} \cdot \nabla w + \int_\Omega y_k \cdot \nabla Y_{1,k} \cdot w \\ \qquad + \int_\Omega Y_{1,k} \cdot \nabla y_k \cdot w = -\frac{d}{dt}\int_\Omega v_k \cdot w - \int_\Omega \nabla v_k \cdot \nabla w, \quad \forall w \in \boldsymbol{V} \\ Y_{1,k}(0) = 0, \end{cases}$$

and v_k in $\mathcal{A}_0(T^*)$ is the corrector (associated to y_k) leading to $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$.

Proposition

Let $\{y_k\}_{k\in\mathbb{N}}$ the sequence of $\mathcal{A}(T^*)$ defined by (20). Then $y_k \to \bar{y}$ in $H^1(0, T^*; \boldsymbol{H}) \cap L^2(0, T^*; H^2(\Omega)^3 \cap \boldsymbol{V})$ where $\bar{y} \in \mathcal{A}(T^*)$ is the unique solution of (14).

based on the estimate

$$\sqrt{E(y_{k+1})} \leq \sqrt{E(y_k)}\Big(|1-\lambda| + \lambda^2 C_1 \sqrt{E(y_k)}\Big), \quad \forall \lambda \in \mathbb{R}_+$$

where

$$\begin{cases} C_1 = \dfrac{c}{\nu^{5/4}} \exp\Big(c\Big(\dfrac{C_2}{\nu^2} + (\dfrac{C_2}{\nu^2})^2\Big)\Big), \\ C_2 = \|u_0\|^2_{\boldsymbol{V}} + \dfrac{8}{\nu}\|f\|^2_{L^2(Q_{T^*})^3} + \dfrac{16}{\nu} E(y_0) \end{cases} \tag{21}$$

does not depend on y_k, $k \in \mathbb{N}^*$.

Part 3– Controllability problem for a sub-linear (controllable) heat equation: find a sequence $(y_k, v_k)_{k>0}$ converging strongly to a pair (y, v) solution of

$$\begin{cases} y_t - \nu\Delta y + g(y) = f 1_\omega \quad \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases}, \tag{22}$$

such that $y(\cdot, T) = 0$.

• $u_0 \in L^2(\Omega)$, $f \in L^\infty(q_T)$ is a *control* function.

• $g : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz-continuous and satisfies

$$|g'(s)| \leq C(1 + |s|^m) \quad \text{a.e., with } 1 \leq m \leq 1 + 4/d. \tag{23}$$

so that (22) possesses exactly one local in time solution.

If g is “not too super-linear" at infinity, then the control can compensate the blow-up phenomena occurring in $\Omega\backslash\overline{\omega}$.

Theorem (Fernandez-Cara,Zuazua (2000), Barbu (2000))

Let $T > 0$ be given. Assume that $g(0) = 0$ and that $g : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz-continuous and satisfies (23) and

$$\frac{g(s)}{|s|\log^{3/2}(1+|s|)} \to 0 \quad as \quad |s| \to \infty. \tag{24}$$

Then (22) is null-controllable at time T.

The proof is based on a fixed point method. Precisely, it is shown that the operator $\Lambda : L^2(Q_T) \to L^2(Q_T)$, where $y_z := \Lambda z$ is a null controlled solution of the linear boundary value problem

$$\begin{cases} y_{z,t} - \nu\Delta y_z + y_z\,\tilde{g}(z) = f_z 1_\omega, & \text{in} \quad Q_T \\ y_z = 0 \text{ on } \Sigma_T, \quad y_z(\cdot,0) = u_0 & \text{in} \quad \Omega \end{cases}, \qquad \tilde{g}(s) := \begin{cases} g(s)/s & s \neq 0, \\ g'(0) & s = 0 \end{cases},$$

maps the closed ball $B(0,M) \subset L^2(Q_T)$ into itself, for some $M > 0$. The Kakutani's theorem provides the existence of at least one fixed point for Λ, which is also a controlled solution for (22).

We define the convex space

$$\mathcal{A} = \Big\{(y,f) : \rho\, y \in L^2(Q_T), \rho_1 \nabla y \in L^2(Q_T),\ \rho_0 f \in L^2(q_T),$$
$$\rho_0(y_t - \Delta y) \in L^2(0,T;H^{-1}(\Omega)), y(\cdot,0) = 0 \,\text{in}\, \Omega, y = y_0 \,\text{on}\, \Sigma_T\Big\}.$$

where ρ, ρ_1 and ρ_0 defines Carleman type weights, continuous, $\geq \rho_* > 0$ in Q_T and blowing up as $t \to T^-$. $\rho_i \approx exp(\beta(x)/(T-t))$ then the least-squares problem, with $E : \mathcal{A} \to \mathbb{R}$ as

$$\inf_{(y,f)\in\mathcal{A}} E(y,f) = \frac{1}{2}\Big\|\rho_0\Big(y_t - \nu\Delta y + g(y) - f\,1_\omega\Big)\Big\|^2_{L^2(0,T;H^{-1}(\Omega))} \tag{25}$$

Actually, for any $(\overline{y},0) \in \mathcal{A}$, we consider the extremal problem $\inf_{(y,f)\in\mathcal{A}_0} E(\overline{y}+y,f)$ where $\mathcal{A}_0$ is the Hilbert space

$$\mathcal{A}_0 = \Big\{(y,f) : \rho\, y \in L^2(Q_T), \rho_1 \nabla y \in L^2(Q_T),\ \rho_0 f \in L^2(q_T),$$
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$$\mathcal{A}_0 = \Big\{(y,f) : \rho\, y \in L^2(Q_T), \rho_1\, \nabla y \in L^2(Q_T),\ \rho_0 f \in L^2(q_T),$$
$$\rho_0(y_t - \Delta y) \in L^2(0,T;H^{-1}(\Omega)), y(\cdot,0) = 0 \,\text{in}\, \Omega, y = 0 \,\text{on}\, \Sigma_T\Big\}.$$

For any $(y,f)\in\mathcal{A}$, we now look for a pair $(Y^1,F^1)\in\mathcal{A}_0$ solution of

$$\begin{cases} Y^1_t-\Delta Y^1+g'(y)\cdot Y^1=F^1 1_\omega+\big(y_t-\Delta y+g(y)-f\,1_\omega\big), & \text{in} \quad Q_T \\ Y^1=0 \text{ on } \Sigma_T, \quad Y^1(\cdot,0)=0 \text{ in } \Omega. \end{cases}, \tag{26}$$

$(Y^1,F^1)\in\mathcal{A}_0$ so that F^1 is a null control for Y^1.

Proposition

Assume that g is differentiable. Then, $E((\overline{y},\overline{f})+\cdot)$ is differentiable over $\mathcal{A}_0$. Let $(y,f)\in\mathcal{A}$ and let $(Y^1,F^1)\in\mathcal{A}_0$ be a solution of (26). Then the derivative of E at the point $(y,f)\in\mathcal{A}$ along the direction (Y^1,F^1) satisfies

$$E'(y,f)\cdot(Y^1,F^1)=2E(y,f).$$

Part 3: a least-square approach

For any $(y, f) \in \mathcal{A}$, we now look for a pair $(Y^1, F^1) \in \mathcal{A}_0$ solution of

$$\begin{cases} Y^1_t - \Delta Y^1 + g'(y)\cdot Y^1 = F^1 1_\omega + \big(y_t - \Delta y + g(y) - f\,1_\omega\big), & \text{in} \quad Q_T \\ Y^1 = 0 \text{ on } \Sigma_T, \quad Y^1(\cdot,0) = 0 \text{ in } \Omega. \end{cases}, \tag{27}$$

Proposition

Assume that $g \in W^{1,\infty}(\mathbb{R})$. For any $(y, f) \in \mathcal{A}$, we define the unique pair (Y^1, F^1) solution of (27), which minimizes the functional $J : L^2(\rho_0, q_T) \times L^2(\rho, Q_T) \to \mathbb{R}^+$ defined by

$$J(u, z) := \|\rho_0\, u\|^2_{L^2(q_T)} + \|\rho\, z\|^2_{L^2(Q_T)}.$$

$(Y^1, F^1) \in \mathcal{A}_0$ satisfies

$$\|\rho(T-t)\nabla Y^1\|_{L^2(q_T)} + \|\rho_0 F^1\|_{L^2(q_T)} + \|\rho\, Y^1\|_{L^2(Q_T)} \le C\sqrt{E(y,f)} \tag{28}$$

for some $C = C(T, \Omega, \|g'(y)\|_{L^\infty(Q_T)}) > 0$ of the form

$$C = e^{c(\Omega)\left(1+T^{-1}+T+(T^{1/2}+T)\|g'(y)\|_{L^\infty(Q_T)}+\|g'(y)\|^{2/3}_{L^\infty(Q_T)}\right)}.$$

For any $(y,f)\in\mathcal{A}$, we now look for a pair $(Y^1,F^1)\in\mathcal{A}_0$ solution of

$$\begin{cases} Y^1_t-\Delta Y^1+g'(y)\cdot Y^1=F^1 1_\omega+\big(y_t-\Delta y+g(y)-f\,1_\omega\big), & \text{in} \quad Q_T \\ Y^1=0 \text{ on } \Sigma_T, \quad Y^1(\cdot,0)=0 \text{ in } \Omega. \end{cases}, \qquad (27)$$

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$$\begin{cases} Y^1_t - \Delta Y^1 + g'(y) \cdot Y^1 = F^1 1_\omega + \big(y_t - \Delta y + g(y) - f\, 1_\omega\big), & \text{in} \quad Q_T \\ Y^1 = 0 \text{ on } \Sigma_T, \quad Y^1(\cdot, 0) = 0 \text{ in } \Omega. \end{cases}, \qquad (27)$$

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$$C=e^{c(\Omega)\left(1+T^{-1}+T+(T^{1/2}+T)\|g'(y)\|_{L^\infty(Q_T)}+\|g'(y)\|^{2/3}_{L^\infty(Q_T)}\right)}.$$

Therefore, we can define a minimizing sequence $\{y_k, f_k\}_{k>0}$ as follows:

$$\begin{cases} (y_0, f_0) \in \mathcal{A}, \\ (y_{k+1}, f_{k+1}) = (y_k, f_k) - \lambda_k (Y_k^1, F_k^1), \quad k > 0, \\ \lambda_k = argmin_{\lambda \in \mathbb{R}^+} E\big((y_k, f_k) - \lambda (Y_k^1, F_k^1)\big) \end{cases} \tag{29}$$

where $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is such that F_k^1 is a null control for Y_k^1, solution of

$$\begin{cases} Y^1_{k,t} - \Delta Y_k^1 + g'(y_k) \cdot Y_k^1 = F_k^1 1_\omega - (y_{k,t} - \Delta y_k + g(y_k) - f_k 1_\omega), \quad \text{in} \quad Q_T \\ Y_k^1 = 0 \text{ on } \Sigma_T, \quad Y_k^1(\cdot, 0) = 0 \text{ in } \Omega, \end{cases},$$

and minimizes the functional J.

Theorem

Assume that $g \in W^{2,\infty}(\mathbb{R})$. Then, for any $(y_0, f_0) \in \mathcal{A}$, the sequence $\{y_k, f_k\}_{k>0}$ strongly converges to $\{y, f\} \in \mathcal{A}$ as $k \to \infty$.

Theorem

Assume that $g \in W^{2,\infty}_{loc}(\mathbb{R})$ and that $e^{\|g'(y_0)\|_{L^\infty}} \sqrt{E(y_0, f_0)} < e^{1/2}$. Then, the sequence $\{y_k, f_k\}_{k>0}$ strongly converges to $\{y, f\} \in \mathcal{A}$ as $k \to \infty$.

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Theorem

Assume that $g \in W^{2,\infty}_{loc}(\mathbb{R})$ and that $e^{\|g'(y_0)\|_{L^\infty}} \sqrt{E(y_0, f_0)} < e^{1/2}$. Then, the sequence $\{y_k, f_k\}_{k>0}$ strongly converges to $\{y, f\} \in \mathcal{A}$ as $k \to \infty$.

Take $g(s) = -5\, s\, log^{1.4}(1 + |s|)$; $g' \notin L^\infty(\mathbb{R})$ but $g'' \in L^\infty(\mathbb{R})$!

$$\begin{cases} y_t - 0.1 y_{xx} - 5y\, log^{1.4}(1+|y|) = f\, 1_{(0.2,0.6)}, & (x,t) \in (0,1) \times (0,1/2), \\ y(\cdot,0) = 40\, sin(\pi x), & x \in (0,1), \\ y(0,t) = y(1,t) = 0, & t \in (0,1/2) \end{cases} \quad (30)$$

The uncontrolled solution blows up at $t_c \approx 0.339$. [3]

At each iterates k, the pair (Y_k^1, F_k^1), minimizer of J is computed through a mixed space-time variational formulation, well-suited for mesh adaptivity.

Conformal approximation in time and space leads to strong convergent approximation $(Y_k^1, F_k^1)_h$ of (Y_k^1, F_k^1), [4]

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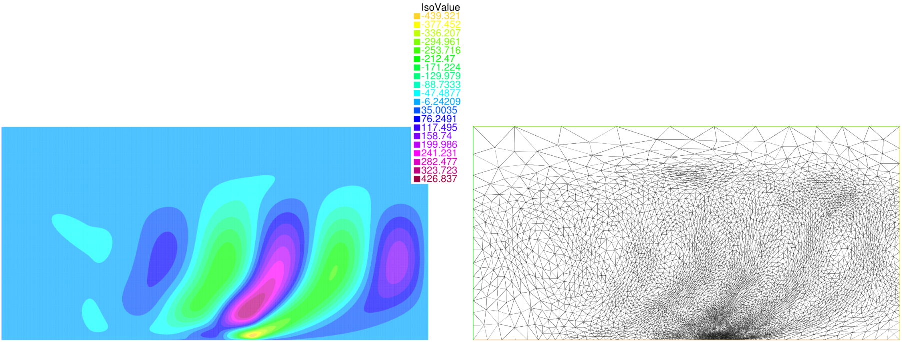
[3] E. Fernandez-Cara, A. Munch, Numerical null controllability of semi-linear 1D heat equations : fixed point, least squares and Newton methods, Mathematical Control and Related Fields (2012).

[4] E. Fernandez-Cara, A. Munch, Strong convergent approximations of null controls for the heat equation, SEMA, 2013

Table

$\sharp$iterate k	$\frac{\Vert y_k - y_{k-1}\Vert_{L^2(Q_T)}}{\Vert y_{k-1}\Vert_{L^2(Q_T)}}$	$\sqrt{2E(y_k, f_k)}$	λ_k	$\Vert Y_k^1, F_k^1\Vert_{\mathcal{A}_0}$
0	–	46.17	0.3192	1252.5
1	3.767	38.96	0.4512	854.6
2	1.442	27.61	0.2120	449.60
3	7.034×10^{-1}	16.904	0.3100	178.01
4	2.292×10^{-1}	7.229	0.5040	67.56
5	7.987×10^{-2}	3.107	0.6120	26.00
6	3.162×10^{-2}	1.240	0.3801	10.18
7	5.427×10^{-3}	4.547×10^{-1}	0.5321	4.080
8	2.458×10^{-3}	1.489×10^{-1}	0.5823	1.684
9	1.177×10^{-3}	4.515×10^{-2}	0.6203	0.720
10	5.939×10^{-4}	1.380×10^{-2}	0.7831	0.3214
11	3.134×10^{-4}	4.629×10^{-3}	0.6932	0.1512
12	1.727×10^{-4}	1.861×10^{-3}	0.6512	0.07616
13	9.950×10^{-5}	9.659×10^{-4}	0.7921	0.04182
14	6.018×10^{-5}	4.840×10^{-4}	0.8945	0.02553
15	3.845×10^{-5}	3.933×10^{-4}	0.9230	0.01741
16	2.607×10^{-5}	3.268×10^{-4}	0.9412	0.01306
17	1.876×10^{-5}	2.725×10^{-4}	0.9582	0.01047
18	1.426×10^{-5}	2.262×10^{-4}	0.9356	0.00877
19	1.134×10^{-5}	1.862×10^{-4}	0.9844	0.0075
20	9.339×10^{-6}	9.515×10^{-5}	–	–



Iso-values of the controlled solution in $(0,1)\times(0,0.5)$ and space-time adapted mesh.

• Analysis of weak LS method/ damped Newton method for NS leading to globally convergent approximation

• Theoretical justification of the H^{-1}-LS introduced by Glowinski in 79.

• Can be efficient to solve exact controllability problems.

• Possibly useful at the numerical analysis since (coercivity type) inequality like

$$\|y_{k,h} - \overline{y}\|_V \leq C\sqrt{E(y_{k,h})}, \quad \forall y_{k,h} \in V_h \subset \boldsymbol{V}$$

remains true.

• The analysis can be extended to other "reasonable" nonlinearity (visco-elastic NS, nonlinear hyperbolic PDEs, ...).

• Damped Newton method is possibly useful to solve (nonlinear) inverse problems.

Details and experiments are available here:

Analysis of $\boldsymbol{V'}$-Least-squares pb. (interior and exterior case) based on the gradient (Conjugate gradient / Barzilai Borwein)

- J. Lemoine, A.Münch, P. Pedregal, Analysis of continuous H^{-1}-least-squares methods for the steady Navier-Stokes system Applied. Math. Optimization 2020

Analysis of $\boldsymbol{V'}$ and $L^2(\boldsymbol{V'})$-Least-squares pb. based on the Newton-direction

- J. Lemoine, A.Münch, Resolution of the Implicit Euler scheme for the Navier-Stokes equation through a least-squares method. hal-01996429
- J. Lemoine, A. Münch, A fully space-time least-squares method for the unsteady Navier-Stokes system arxiv.org/abs/1909.05034
- J. Lemoine, I. Marin-Gayte, A. Münch, Stong convergent approximation of null controls for sublinear heat equation using a least-squares approach. arxiv.org/abs/1910.0018.

THANK YOU FOR YOUR ATTENTION