Optimal Control of Perfect Plasticity

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joint work with

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Outline

Outline

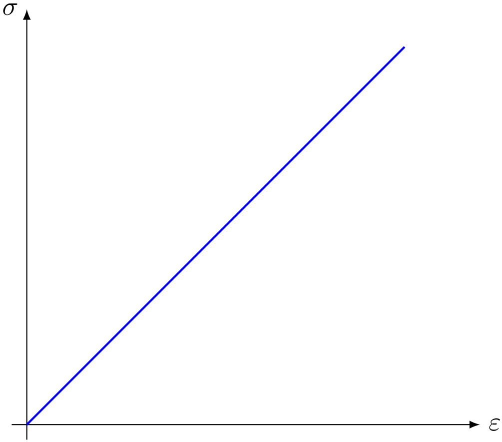
Introduction to Perfect Plasticity

Stress Tracking
- Existence of Optimal Solutions and their Approximation
- Optimality System

Displacement Tracking
- Existence of Optimal Solutions
- Reverse Approximation

Conclusion and Outlook

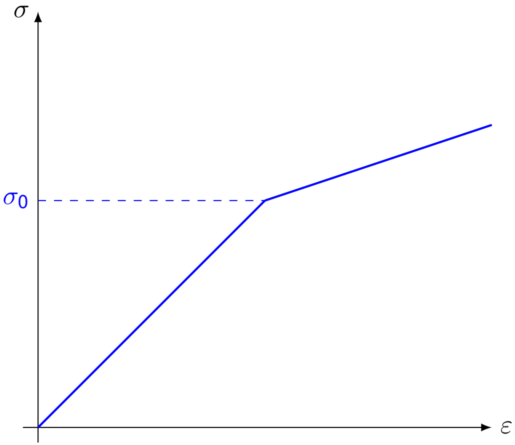
Plasticity in a nutshell



Linear elasticity

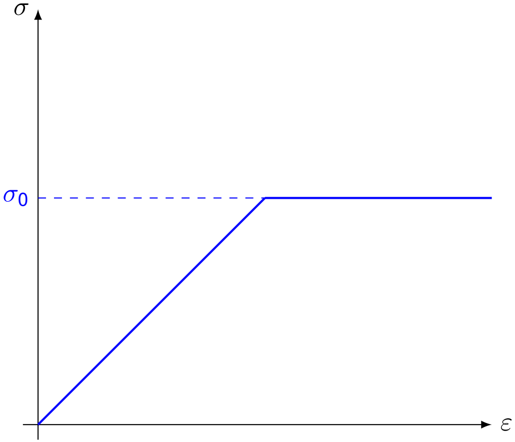
Stress-Strain Relation

Plasticity in a nutshell



Plasticity with hardening

Plasticity in a nutshell



Perfect plasticity

Formal Strong Formulation

Linear elasticity

$$
\begin{aligned}
-\operatorname{div}\sigma &= 0 && \text{in } \Omega \times (0,T),\\
\sigma &= \mathbb{C}\nabla^{s} u && \text{in } \Omega \times (0,T),\\
\\
u &= u_D && \text{on } \Gamma_D \times (0,T),\\
\sigma\nu &= 0 && \text{on } \Gamma_N \times (0,T),\\
u(0) &= u_0, \quad \sigma(0) = \sigma_0 && \text{in } \Omega
\end{aligned}
$$

with

- $u : \Omega \to \mathbb{R}^d$ displacement, $\sigma : \Omega \to \mathbb{R}^{d\times d}_{\text{sym}}$ stress
- $\mathbb{C}$ linear and coercive elasticity tensor
- $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \neq \emptyset$, ν outward normal
- u_D given Dirichlet boundary data, u_0, σ_0 initial data

Formal Strong Formulation

Perfect plasticity

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\begin{aligned}
-\operatorname{div}\sigma &= 0 && \text{in } \Omega \times (0, T),\\
\sigma &= \mathbb{C}(\nabla^s u - z) && \text{in } \Omega \times (0, T),\\
\partial_t z &\in \partial I_{\mathcal{K}}(\sigma) && \text{in } \Omega \times (0, T),\\
u &= u_D && \text{on } \Gamma_D \times (0, T),\\
\sigma\nu &= 0 && \text{on } \Gamma_N \times (0, T),\\
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\end{aligned}
$$

with

- $u : \Omega \to \mathbb{R}^d$ displacement, $\sigma : \Omega \to \mathbb{R}^{d\times d}_{\text{sym}}$ stress, z plastic strain
- $\mathbb{C}$ linear and coercive elasticity tensor
- $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \neq \emptyset$, ν outward normal
- u_D given Dirichlet boundary data, u_0, σ_0 initial data
- $\mathcal{K}$ set of admissible stresses, closed and convex

Challenges in Perfect Plasticity

- Displacement and plastic strain are in general **not unique**
- **Lack of regularity**:
  - Time derivative of the displacement field only in $L^2_w(0, T; BD(\Omega))$
  - Space of bounded deformation, not Bochner measureable
  - Plastic strain is only a regular Borel measure
- Existence only under a **safe load condition**:
  Applied loads must admit an elastic solution not obeying the Dirichlet boundary conditions such that the associated stress is in the interior of $\mathcal{K}$

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BUT, if the safe load condition is fulfilled, then ...

For every Dirichlet displacement u_D there exists a unique stress field

Outline

Notation and Standing Assumptions

Spaces

- Stress space: $\mathcal{H}^p := L^p(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})$, $\mathcal{H} := \mathcal{H}^2$
- Test space for displacements:

$$\mathcal{V}^p := W^{1,p}(\Omega; \mathbb{R}^d), \quad \mathcal{V} := \mathcal{V}^2,$$

$$\mathcal{V}^p_D := \overline{\{\psi|_\Omega : \psi \in C^\infty_0(\mathbb{R}^n),\ \text{supp}(\psi) \cap \Gamma_D = \emptyset\}}^{W^{1,p}(\Omega;\mathbb{R}^n)}, \quad \mathcal{V}_D := \mathcal{V}^2_D$$

Standing assumptions

- $K \subset \mathbb{R}^{d\times d}_{\text{sym}}$ nonempty, closed, and convex
- $\mathbb{C} : \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}$ linear, symmetric, and coercive, $\mathbb{A} := \mathbb{C}^{-1}$
- $u_D \in H^1(0,T;\mathcal{V})$, $\sigma_0 \in \mathcal{H}$ with $-\operatorname{div}\sigma_0 = 0$, $\sigma_0 \in K$ a.e. in Ω
- Γ_D relatively closed subset of $\partial\Omega$ with positive measure, $\Omega \cup \Gamma_N$ regular in the sense of Gröger

Definition (Reduction to the stress only, Johnson'76)

A function $\sigma \in H^1(0, T; \mathcal{H})$ is called **reduced solution** (with respect to u_D), if fa.a. $t \in (0, T)$, it holds

Equilibrium condition:	$\sigma(t) \in \mathcal{E} := \{\tau \in \mathcal{H} : \langle \tau, \nabla^s \varphi \rangle_{\mathcal{H}} = 0 \; \forall \, \varphi \in \mathcal{V}_D\}$	(E)
Yield condition:	$\sigma(t) \in \mathcal{K} := \{\tau \in \mathcal{H} : \tau(x) \in K \text{ f.a.a. } x \in \Omega\}$	(Y)
Flow rule:	$\langle \mathbb{A}\partial_t \sigma(t) - \nabla^s \partial_t u_D(t), \tau - \sigma(t) \rangle_{\mathcal{H}} \geq 0 \quad \forall \tau \in \mathcal{E} \cap \mathcal{K}$	(F)
Initial condition:	$\sigma(0) = \sigma_0$	(0)

Yosida Regularization

$$
\begin{aligned}
-\operatorname{div}\sigma &= 0 && \text{in } \Omega \times (0, T),\\
\sigma &= \mathbb{C}(\nabla^s u - z) && \text{in } \Omega \times (0, T),\\
\partial_t z &\in \partial I_{\mathcal{K}}(\sigma) && \text{in } \Omega \times (0, T),\\
u &= u_D && \text{on } \Gamma_D \times (0, T),\\
\sigma\nu &= 0 && \text{on } \Gamma_N \times (0, T),\\
u(0) = u_0, &\quad \sigma(0) = \sigma_0 && \text{in } \Omega
\end{aligned}
$$

Yosida Regularization

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\begin{aligned}
-\operatorname{div}\sigma &= 0 && \text{in } \Omega \times (0,T),\\
\sigma &= \mathbb{C}(\nabla^s u - z) && \text{in } \Omega \times (0,T),\\
\partial_t z &= \partial I_\lambda(\sigma) && \text{in } \Omega \times (0,T),\\
u &= u_D && \text{on } \Gamma_D \times (0,T),\\
\sigma\nu &= 0 && \text{on } \Gamma_N \times (0,T),\\
u(0) = u_0, \quad \sigma(0) &= \sigma_0 && \text{in } \Omega
\end{aligned}
$$

with

$$
\partial I_\lambda(\tau) = \tfrac{1}{\lambda}(\tau - \pi_K(\tau)) \quad \text{and} \quad \pi_K(\tau) = \operatorname*{arg\,min}_{\varsigma \in K} |\varsigma - \tau|_F^2
$$

Proposition (Existence of a reduced solution)

There exists a unique reduced solution $\sigma \in H^1(0,T;\mathcal{H})$.

Proof:

- Existence for the Yosida regularization by standard contraction arguments
- A priori bounds for σ_λ in $H^1(0,T;\mathcal{H}) \Rightarrow$ existence of a weak limit σ for $\lambda \searrow 0$
- Passage to the limit in (E) & (F), feasibility $\sigma(t) \in \mathcal{K}$ by Yosida regularization
- Uniqueness of σ by coercivity of $\mathbb{A}$ $\square$

Theorem (Continuity properties of reduced solutions)

Assume that $u_D^n \rightharpoonup u_D$ *in* $H^1(0,T;\mathcal{V})$, $u_D^n \to u_D$ *in* $L^2(\mathcal{V})$, $u_{D,n}(T) \to u_D(T)$ *in* $\mathcal{V}$.

Then $\sigma_n \rightharpoonup \sigma$ *in* $H^1(0,T;\mathcal{H})$ *and, if* $\lambda_n \searrow 0$, *then* $\sigma_\lambda^n \rightharpoonup \sigma$ *in* $H^1(0,T;\mathcal{H})$.

Regularization – Extension and Remarks

- Instead of Yosida regularization, one could also use **hardening** to prove existence:

$$\partial_t z \in \partial I_{\mathcal{K}}(\sigma - \varepsilon\, z) \quad \text{with } \varepsilon > 0$$

  (and, of course, both, Yosida and hardening, together)
- If $u_D^n \to u_D$ in $H^1(0, T; \mathcal{V})$, then the convergence is strong, i.e., $\sigma_n \to \sigma$ and $\sigma_\lambda^n \to \sigma$ in $H^1(0, T; \mathcal{H})$

C. Johnson, *Existence theorems for plasticity problems*, Journal de Matématiques Pures et Appliquées, 55 (1976), pp. 431–444.

P.-M. Suquet, *Sur les équations de la plasticité: existence et régularité des solutions*, J. Mécanique, 20 (1981), pp. 3–39.

S. Bartels, A. Mielke, and T. Roubček, *Quasi-static small-strain plasticity in the limit of vanishing hardening and its numerical approximation*, SIAM Journal on Numerical Analysis, 50 (2012), pp. 951–976

Optimal control of the stress

$$\left.\begin{aligned} \min \quad & \tfrac{1}{2}\|\sigma(T)-\sigma_d\|_{\mathcal{H}}^2 + \tfrac{\alpha}{2}\|\partial_t \ell\|_{L^2(0,T;\mathcal{X}_c)} \\ \text{s.t.} \quad & \sigma \text{ is a reduced solution associated with } u_D = G(\ell) + a \\ \text{and} \quad & \ell(0) = \ell(T) = 0 \end{aligned}\right\} \qquad (\mathrm{P}_\sigma)$$

with

- $\alpha > 0$
- **Control space**: $\mathcal{X}_c \hookrightarrow \mathcal{X}$, $\mathcal{X}_c$ Hilbert space, $\mathcal{X}$ Banach space
- $G : \mathcal{X} \to \mathcal{V}$ linear and continuous, $a \in \mathcal{V}$ given offset, Example:
  - $\Lambda \subset \partial\Omega$, relatively closed, $\mathrm{dist}(\Lambda, \Gamma_D) > 0$
  - $\mathcal{X} := H^{-1}_\Lambda(\Omega; \mathbb{R}^d)$, $\mathcal{X}_c := L^2(\Omega; \mathbb{R}^d)$
  - G solution operator of **linear elasticity** with zero Dirichlet boundary conditions on Λ

Theorem

There exists at least one globally optimal control of (P_σ).

Proof: based on the continuity results by standard direct method □

Possible extensions:

- More general objectives (weakly lower semicontinuous functionals)
- Directly use u_D as control (boundary controls in $H^{1/2}$)

Approximation of Optimal Solutions

- Regularized solution operator:

$$S_\lambda : H^1(0,T;\mathcal{X}) \ni \ell \mapsto u_D = G(\ell) + a \mapsto \sigma_\lambda \in H^1(0,T;\mathcal{H})$$

- **Regularized optimal control problem**:

$$\left.\begin{aligned} \min \quad & \tfrac{1}{2}\,\|S_\lambda(\ell)(T) - \sigma_d\|_{\mathcal{H}}^2 + \tfrac{\alpha}{2}\,\|\partial_t \ell\|_{L^2(0,T;\mathcal{X}_c)} \\ \text{s.t.} \quad & \ell(0) = \ell(T) = 0 \end{aligned}\right\} \qquad (\mathrm{P}_\sigma^\lambda)$$

Based on the previous convergence results:

Theorem

Let $\lambda \searrow 0$ and $\{\ell_\lambda\}$ be a sequence of optimal solutions of $(\mathrm{P}_\sigma^\lambda)$. Every weak accumulation point of $\{\ell_\lambda\}$ is a strong accumulation point and a minimizer of (P_σ). There is at least one accumulation point.

(Extension to isolated local minimizers possible by standard arguments)

Smoothing

- Yosida regularization of $\partial I_{\mathcal{K}}$ is still **not** Gâteaux-differentiable

$\Rightarrow$ Further **smoothing** necessary: Let

$$K := \{\tau \in \mathbb{R}^{d\times d}_{\text{sym}} : \tau^D| \leq \gamma\} \quad \text{with} \quad \tau^D := \tau - \frac{1}{d}\operatorname{tr}(\tau) \ \ \text{(deviator)}$$

(von Mises yield condition). Then replace ∂I_λ by

$$A_\delta : \tau \mapsto \frac{1}{\lambda}\max_\delta\Big(1 - \frac{\gamma}{|\tau^D|_F}\Big)\tau^D \quad \text{with} \quad \max_\delta : r \mapsto \begin{cases} \max\{0, r\}, & |r| \geq \delta \\ \frac{1}{4\delta}(r+\delta)^2, & |r| < \delta \end{cases}$$

- Under a suitable coupling of λ and δ, the above convergence results also hold with A_δ instead of ∂I_λ ($\delta \sim o(\lambda^2 \exp(-\lambda^{-1}))$ is sufficient, but probably not optimal)
- Smoothed equation: $S_\delta : H^1(0, T; \mathcal{X}) \ni \ell \mapsto \sigma \in H^1(0, T; \mathcal{H})$ solution operator of:

$$-\operatorname{div}\sigma = 0, \quad \sigma = \mathbb{C}(\nabla^s u - z), \quad \partial_t z = A_\delta(\sigma) \quad \text{in } \Omega \times (0, T),$$
$$u = G(\ell) + a \quad \text{on } \Gamma_D \times (0, T), \quad \sigma\nu = 0 \quad \text{on } \Gamma_N \times (0, T),$$
$$u(0) = u_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega$$

Differentiability of the Regularized Solution Map

Assumption: Let G be continuous from $\mathcal{X}_c$ to $\mathcal{V}^p$ for some $p > 2$

(Fulfilled in case of linear elasticity)

Proposition

Under the above assumption, the smooth solution operator S_δ is Fréchet-differentiable from $H^1(0, T; \mathcal{X}_c)$ to $H^1(0, T; \mathcal{H})$. For $\ell, h \in H^1(0, T; \mathcal{X}_c)$, $\tau := S'_\delta(\ell)h$ solves

$$
\begin{aligned}
-\operatorname{div}\tau = 0, \quad \tau = \mathbb{C}(\nabla^s v - \eta), \quad \partial_t \eta = A'_\delta(\sigma)\tau \quad &\text{in } \Omega \times (0, T), \\
v = G\,h \quad \text{on } \Gamma_D \times (0, T), \quad \tau\nu = 0 \quad &\text{on } \Gamma_N \times (0, T), \\
v(0) = 0, \quad \tau(0) = 0 \quad &\text{in } \Omega
\end{aligned}
$$

Proof: Direct consequence of differentiability of A_δ from $\mathcal{H}^p$ to $\mathcal{H}$ for $p > 2$ (norm gap required) in combination with $W^{1,p}$-regularity results for linear elasticity □

R. Herzog, C. Meyer, G. Wachsmuth, *Integrability of displacement and stresses in linear and nonlinear elasticity with mixed boundary conditions*, Journal of Mathematical Analysis and Applications, 382 (2011), pp. 802–813.

Theorem (Necessary optimality conditions for the smoothed problems)

Let $\bar{\ell}$ be locally optimal for the smoothed optimal control problem with associated state $(\bar{\sigma}, \bar{u}, \bar{z}) \in H^1(\mathcal{H} \times \mathcal{V} \times \mathcal{H})$. Then there exists an **adjoint state** *$(w, \varphi) \in H^1(0, T; \mathcal{V}_D) \times H^1(0, T; \mathcal{H})$ and $w_T \in \mathcal{V}_D$ such that*

$$
\begin{gathered}
-\operatorname{div} \mathbb{C}\nabla^s w = -\operatorname{div} \mathbb{C} A_\delta'(\bar{\sigma})\varphi \quad \text{in } \Omega \times (0, T), \\
w = 0 \quad \text{on } \Gamma_D \times (0, T), \quad (\mathbb{C}\nabla^s w)\nu = 0 \quad \text{on } \Gamma_N \times (0, T),
\end{gathered}
$$

$$
\begin{gathered}
\partial_t \varphi = \mathbb{C} A_\delta'(\bar{\sigma})\varphi - \mathbb{C}\nabla^s w \quad \text{in } \Omega \times (0, T), \\
\varphi(T) = \mathbb{C}(\bar{\sigma}(T) - \sigma_d - \nabla^s w_T) \quad \text{in } \Omega,
\end{gathered}
$$

$$
\begin{gathered}
-\operatorname{div} \mathbb{C}\nabla^s w_T = -\operatorname{div} \mathbb{C}(\bar{\sigma}(T) - \sigma_d) \quad \text{in } \Omega, \\
w_T = 0 \quad \text{on } \Gamma_D, \quad (\mathbb{C}\nabla^s w_T)\nu = 0 \quad \text{on } \Gamma_D
\end{gathered}
$$

$$
\begin{gathered}
\alpha \partial_{tt}^2 \bar{\ell} + G^* \big(div(\mathbb{C}\nabla^s w - A_\delta'(\bar{\sigma})\varphi) \big) = 0 \quad \text{in } \Omega \times (0, T), \\
\bar{\ell}(0) = \bar{\ell}(T) = 0.
\end{gathered}
$$

Comments on KKT-Conditions

- Analogous results for optimal control of plasticity systems with *hardening*:

  G. Wachsmuth, *Optimal control of quasistatic plasticity*, Ph.D. thesis, Dr. Hut, 2011.

  H. Meinlschmidt, C. Meyer, S. Walther, *Optimal Control of an Abstract Evolution Variational Inequality with Application to Homogenized Plasticity*, SPP 1962, Preprint 123, 2019.

- **Limit analysis** for vanishing regularization/smoothing:
  - Adjoint variables lack boundedness in "nice" spaces (even in case *with hardening*)
  - Only **weak stationarity** conditions are obtained in the limit without any sign condition on the dual variables

- Similar smoothing of shape optimization problems in (static) perfect plasticity:

  A. Maury, G. Allaire, F. Jouve, *Elasto-plastic shape optimization using the level set method*, SICON, 56:556–581, 2018.

- Gradient-based optimization algorithms ▸ Preliminary numerical results

Outline

Definition (inspired by Suquet'81)

A tuple $(u, \sigma) \in H^1(0, T; \mathcal{V}) \times H^1(0, T; \mathcal{H})$ is called **H^1-solution** of the perfect plasticity equation (w.r.t. u_D), if f.a.a. $t \in (0, T)$

Equilibrium condition and yield condition: $\sigma(t) \in \mathcal{E} \cap \mathcal{K}$

Flow rule:
$$\langle \mathbb{A}\partial_t\sigma(t) - \nabla^s \partial_t u_D(t), \tau - \sigma(t)\rangle_{\mathcal{H}} + (\partial_t u(t) - \partial_t u_D(t), \operatorname{div}\tau)_{L^2(\Omega;\mathbb{R}^d)} \geq 0 \quad \forall \tau \in \mathcal{N} \cap \mathcal{K}$$

Initial condition: $u(0) = u_0, \quad \sigma(0) = \sigma_0$

with $\mathcal{N} := \{\tau \in \mathcal{H} : \operatorname{div}\tau \in L^d(\Omega; \mathbb{R}^d),\ (\tau, \nabla^s v)_{\mathcal{H}} + (\operatorname{div}\tau, v)_{L^2} = 0 \ \forall\, v \in \mathcal{V}_D\}$.

(If u satisfies in addition $u = u_D$ a.e. on $\Gamma_D \times (0, T)$, then (u, σ) is a strong solution)

Definition (inspired by Suquet'81)

A tuple $(u, \sigma) \in H^1(0, T; \mathcal{V}) \times H^1(0, T; \mathcal{H})$ is called H^1**-solution** of the perfect plasticity equation (w.r.t. u_D), if f.a.a. $t \in (0, T)$

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$$\langle \mathbb{A}\partial_t \sigma(t) - \nabla^s \partial_t u_D(t), \tau - \sigma(t)\rangle_{\mathcal{H}} + (\partial_t u(t) - \partial_t u_D(t), \operatorname{div}\tau)_{L^2(\Omega;\mathbb{R}^d)} \geq 0 \quad \forall \tau \in \mathcal{N} \cap \mathcal{K}$$

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(If u satisfies in addition $u = u_D$ a.e. on $\Gamma_D \times (0, T)$, then (u, σ) is a strong solution)

In general an H^1-solution does not exist

Time derivative of the displacement in general only in $L^2_w(0, T; BD(\Omega))$, which is not enough to prove the approximation of optimal solutions, therefore ...

Displacement Tracking via Dirichlet Controls

Optimal control of the displacement

$$\left.\begin{array}{rl} \min & J(u, u_D) := \frac{1}{2}\,\|u - u_d\|^2_{H^1(0,T;\mathcal{H})} + \frac{\alpha}{2}\,\|u_D\|^2_{H^1(0,T;U)} \\ \text{s.t.} & (u,\sigma) \text{ is a } H^1\text{-solution associated with } u_D \\ \text{and} & u_D(0) = u_0 \text{ on } \Gamma_D \end{array}\right\} \qquad (\mathrm{P}_u)$$

- To ease notation, we assume that we can directly control u_D
- Control space $U \hookrightarrow \mathcal{V}$, $\alpha > 0$

Tracking type objective implies boundedness

Theorem

There exists a globally optimal solution of (P_u).

Proof:

- Admissible set is not empty: $(\sigma, u, u_D) \equiv (\sigma_0, u_0, u_0)$ is an H^1-solution
- Continuity properties of H^1-solutions:
  If $u_D^n \rightharpoonup u_D$ in $H^1(0, T; \mathcal{V})$, $u_D^n \to u_D$ in $L^2(0, T; \mathcal{V})$, $u_D^n(T) \to u_D(T)$ in $\mathcal{V}$ **and**, if H^1-solutions (σ_n, u_n) associated with u_D^n exists and $\{u_n\}$ is bounded in $H^1(0, T; \mathcal{V})$, then (for a subsequence)

$$(\sigma_n, u_n) \rightharpoonup (\sigma, u) \quad \text{in } H^1(0, T; \mathcal{H} \times \mathcal{V})$$

  and (σ, u) is an H^1-solution associated with u_D.
- Tracking type objective yields necessary bounds
- Based on continuity properties, proof relies on standard direct method □

Why is the reverse approximation so crucial?

- Even if we restrict to solutions in $H^1(0, T; \mathcal{V})$ (in contrast to displacements with bounded deformation only), the solutions of the perfect plasticity system are in general **not unique**.
- Yosida-regularized plasticity problems however are **uniquely solvable**

$\Rightarrow$ There is no hope that a solution of perfect plasticity can be approximated via Yosida regularization no matter how regular these solutions are!

BUT: in addition to the state variables, we can also vary the controls

Unfortunately, Dirichlet controls are not sufficient for this. We need more ...

For the reverse approximation, we need **loads as additional control functions** in the regularized problems:

Regularized control problem

$$
\left.\begin{aligned}
\min \quad & J_\lambda(u, u_D, \ell) := \tfrac{1}{2}\,\|u - u_d\|^2_{H^1(0,T;\mathcal{H})} + \tfrac{\alpha}{2}\,\|u_D\|^2_{H^1(0,T;U)} \\
& \qquad + \lambda^{-1/2}\|\ell\|^2_{L^2(0,T;H^{-1}(\Omega;\mathbb{R}^d))} \\
& \qquad + \|\ell\|^2_{L^2(0,T;H^{-1/2-\varepsilon}(\Omega;\mathbb{R}^d))} + \|\partial_t \ell\|^2_{L^2(0,T;H^{-1}(\Omega;\mathbb{R}^d))} \\
\text{s.t.} \quad & (\sigma, \nabla^s v)_{\mathcal{H}} = \langle \ell, v\rangle \qquad \forall\, v \in \mathcal{V}_D, \\
& \nabla^s \partial_t u - \mathbb{A}\partial_t \sigma = \partial I_{\mathcal{K}}(\sigma) \qquad \text{in } \Omega \times (0,T), \\
& u = u_D \qquad \text{on } \Gamma_D \times (0,T), \\
& u(0) = u_0, \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega \\
\text{and} \quad & u_D(0) = u_0 \text{ on } \Gamma_D, \quad \ell(0) = 0
\end{aligned}\right\} \qquad (\mathrm{P}_u^\lambda)
$$

A Crucial Regularity and Convergence Result

Lemma

Let $\lambda > 0$, $E \in L^2(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$, and σ denote the solution of

$$E - \mathbb{A}\partial_t\sigma = \partial I_\lambda(\sigma), \quad \sigma(0) = \sigma_0. \tag{$*$}$$

Then there is a constant $C > 0$, independent of λ, such that $\|\sigma\|_{C([0,T];H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \leq C$.

Proof: Time discretization and discrete Gronwall lemma □

Lemma

Let σ_λ be the solution of $(*)$ and $\sigma \in H^1(0,T;\mathcal{H})$ denote the solution of

$$E - \mathbb{A}\partial_t\sigma = \partial I_{\mathcal{K}}(\sigma), \quad \sigma(0) = \sigma_0.$$

Then $\sigma_\lambda \to \sigma$ in $H^1(0,T;\mathcal{H})$ and $\sigma_\lambda \rightharpoonup^* \sigma$ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ as $\lambda \searrow 0$.
Moreover, $\|\sigma_\lambda - \sigma\|_{C([0,T];\mathcal{H})} \lesssim \sqrt{\lambda}$.

Proposition (Reverse approximation property)

Let $(\bar{u}, \bar{\sigma}, \bar{u}_D)$ be an H^1-solution that fulfills

$$\nabla^s \partial_t \bar{u} \in C([0,T];\mathcal{V}) \quad \text{and} \quad \bar{u} = \bar{u}_D \text{ a.e. on } \Gamma_D \times (0,T).$$

Define σ_λ as solution of

$$\nabla^s \partial_t \bar{u} - \mathbb{A}\partial_t \sigma_\lambda = \partial I_\lambda(\sigma_\lambda), \quad \sigma_\lambda(0) = \sigma_0.$$

and define $\ell_\lambda \in H^1(0,T;H^{-1}(\Omega;\mathbb{R}^d))$ by $\langle \ell_\lambda(t), v\rangle := (\sigma_\lambda(t), \nabla^s v)_{\mathcal{H}}$ for all $v \in \mathcal{V}_D$.
Then $(\bar{u}, \sigma_\lambda, \bar{u}_D, \ell_\lambda)$ is feasible for (P_u^λ) and

$$J_\lambda(\bar{u}, \bar{u}_D, \ell_\lambda) \to J(\bar{u}, \bar{u}_D) \quad \text{as } \lambda \searrow 0.$$

Proof: Above convergence result with $E = \nabla^s \partial_t \bar{u}$ □

Theorem

Assume that there is a global minimizer $(\bar u, \bar\sigma, \bar u_D)$ *of* (P_u) *satisfying*

$$\nabla^s \partial_t \bar u \in C([0,T];\mathcal{V}) \quad \text{and} \quad \bar u = \bar u_D \text{ a.e. on } \Gamma_D \times (0,T).$$

Then every sequence $\{(\bar u, \bar\sigma, \bar u_D, \bar\ell_\lambda)\}$ *of global minimizers of* (P_u^λ) *has a weak accumulation point. Each weak accumulation point is actually a strong one and has the form* $(\tilde u, \tilde\sigma, \tilde u_D, 0)$*, where* $(\tilde u, \tilde\sigma, \tilde u_D)$ *is a global minimizer of* (P_u)*.*

Proof:

- Existence of a weak accumulation by norms in the objective
- Feasibility of the weak accumulation point by similar arguments as continuity properties of H^1-solutions
- Optimality of the weak limit by reverse approximation property
- Norm convergence + weak convergence = strong convergence □

Outline

- **Stress tracking**
  Non-smooth optimal control problem that can be treated by standard regularization techniques (e.g. Yosida regularization + smoothing) mainly due to the *uniqueness* of the stress
- **Displacement tracking**
  - Displacement is in general **not** unique

  $\Rightarrow$ For fixed data there is no hope to approximate displacements by regularization
  - **But:** in optimal control we can also vary the controls ($\sim$ data) and, if the control space is rich enough, then optimal solutions can be approximated (at least under additional smoothness assumptions)

To do:

- Numerical solution of the regularized problems + path following
- Weaker regularity assumptions for the displacement tracking

Thank you for your attention!