Numerical Analysis of initial data identification of parabolic problems

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Joint work with Boris Vexler (TU München) and Daniel Walter, (RICAM)

New trends in PDE constrained optimization
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Outline

Model Problem

Homogeneous Parabolic PDEs

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{in } I \times \Omega, \\ u &= 0 \quad \text{on } I \times \partial\Omega, \\ u(0) &= u_0 \quad \text{in } \Omega. \end{aligned}$$

- $I = (0, T)$
- $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ convex and polygonal/polyhedral

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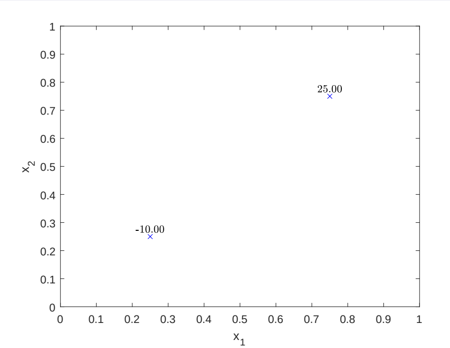
Goal:

Recover initial data u_0 from the final time observation $u(T)$

Homogeneous Parabolic PDEs

Main difficulty: backward parabolic problem is exponentially ill-posed!



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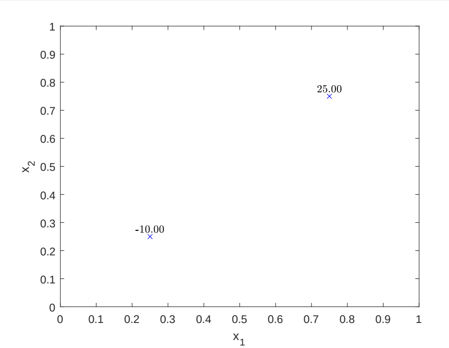

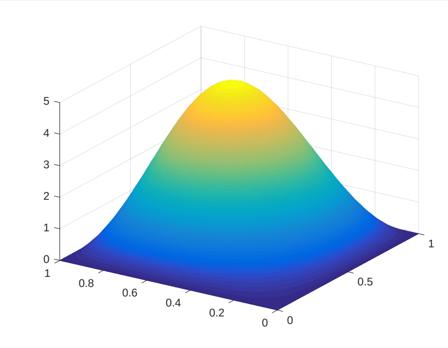

Homogeneous Parabolic PDEs

Difficulty

Backward parabolic is exponentially ill-posed!

Easy to see, thus in 1D ($\Omega = (0,1)$) simple separation of variables $u(x,t) = V(t)W(x)$ gives

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\pi^2 n^2 t} \sin(\pi n x),$$

where c_n has to be find out from final time condition

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Additional assumption

Let u_0 be sparse, i.e. $u_0 = \sum_{j=1}^{M} a_j \delta_{x_j}(x)$,

where $\delta_{x_j}(x)$ are delta functions, i.e. $\int_\Omega f(x)\delta_{x_j}(x)\, dx = f(x_j)$.

Initial value control of parabolic PDEs

Optimal control problem

Minimize $J(q,u) = \frac{1}{2}\|u(T) - u_d\|^2_{L^2(\Omega)} + \alpha\|q\|_X, \quad q \in X, \; u \in V$

subject to the state equation

$$\begin{aligned} \partial_t u - \Delta u &= 0 \;\text{ in } I \times \Omega, \\ u &= 0 \;\text{ on } I \times \partial\Omega \\ u(0) &= q \;\text{ in } \Omega. \end{aligned}$$

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  We can not recover accurately q from $u(T)$

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- Setting I $\rightarrow$ $\boldsymbol{X = L^2(\Omega)}$
  We can not recover accurately q from $u(T)$
- Setting II $\rightarrow$ $\boldsymbol{X = \mathcal{M}(\Omega)}$
  q has a sparse structure and we can (hopefully) recover q from $u(T)$

Look for q in the space of regular Borel measures $\mathcal{M}(\Omega) = (C_0(\Omega))^*$

Optimal control problem

$$\min_{q \in \mathcal{M}(\Omega)} \; J(q,u) = \frac{1}{2}\|u(T) - u_d\|^2_{L^2(\Omega)} + \alpha \|q\|_{\mathcal{M}},$$

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Equivalent problem

$$\min_{q \in \mathcal{M}} \; J(q) = \|q\|_{\mathcal{M}},$$

subject to

$$\partial_t u - \Delta u = 0 \;\text{ in } I \times \Omega,$$
$$u(0) = q \;\text{ in } \Omega$$

and

$$\|u(T) - u_d\|_{L^2(\Omega)} \leq \varepsilon$$

Literature

- Measure valued control for elliptic equations:
  - C. Clason and K. Kunisch, 2011
  - E. Casas, C. Clason, and K. Kunisch, 2012
  - K. Pieper and B. Vexler, 2013
  - E. Casas and K. Kunisch, 2013
- Measure valued control for parabolic equations:
  - E. Casas, and K. Kunisch, 2016
  - E. Casas, C. Clason, and K. Kunisch, 2013
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  - E. Casas, B. Vexler, and E. Zuazua 2015

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Existence and uniqueness

Optimal control problem

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- $I=(0,T)$, $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ convex and polygonal/polyhedral
- $u_d\in L^2(\Omega)$

$\Rightarrow$ For each $q\in\mathcal{M}(\Omega)$ there is $u=S(q)$ with $u\in L^r(I;W^{1,p}(\Omega))$ with $r,p\in[1;2)$ and $\frac{2}{r}+\frac{d}{p}>d+1$ and $u(T)\in L^2(\Omega)$

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Theorem

The above optimal control problem has a unique solution $\bar{q}\in\mathcal{M}(\Omega)$ and $\bar{u}\in L^r(I;W^{1,p}(\Omega))$

Optimality system

State Equation: $\bar{u} \in L^r(I; W^{1,p}(\Omega)) \Rightarrow \quad \bar{u}(T) \in L^2(\Omega)$

$$\begin{aligned} \partial_t \bar{u} - \Delta \bar{u} &= 0 \ \text{ in } I \times \Omega, \\ \bar{u} &= 0 \ \text{ on } I \times \partial\Omega, \\ \bar{u}(0) &= q \ \text{ in } \Omega. \end{aligned}$$

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Adjoint equation: $\bar{z}(0) \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow C_0(\Omega)$

$$\begin{aligned} -\partial_t \bar{z} - \Delta \bar{z} &= 0 && \text{in } I \times \Omega, \\ \bar{z} &= 0 && \text{on } I \times \partial\Omega, \\ \bar{z}(T) &= \bar{u}(T) - u_d && \text{in } \Omega. \end{aligned}$$

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Optimality condition

$$-\langle \bar{z}(0), q - \bar{q} \rangle \leq \alpha \left(\|q\|_{\mathcal{M}} - \|\bar{q}\|_{\mathcal{M}} \right) \quad \forall q \in \mathcal{M}(\Omega).$$

Consequences

From the optimality condition

$$-\langle \bar{z}(0), q - \bar{q} \rangle \le \alpha \left(\|q\|_{\mathcal{M}} - \|\bar{q}\|_{\mathcal{M}} \right) \quad \forall q \in \mathcal{M}(\Omega),$$

taking $q = 0$ and $q = 2\bar{q}$ we get

$$\langle \bar{z}(0), \bar{q} \rangle \le -\alpha \|\bar{q}\|_{\mathcal{M}}, \quad \text{and} \quad \langle \bar{z}(0), \bar{q} \rangle \ge -\alpha \|\bar{q}\|_{\mathcal{M}}.$$

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Consequence for $\bar{z}(0)$

$$\|\bar{z}(0)\|_{C_0(\Omega)} \le \alpha \quad \text{and} \quad \|\bar{z}(0)\|_{C_0(\Omega)} = \alpha \quad \text{if} \quad \bar{q} \ne 0.$$

Consequences

Using Jordan decomposition $\bar{q} = \bar{q}^+ - \bar{q}^-$

Consequence for $\bar{q}$

$$\operatorname{supp} \bar{q}^+ \subset \Omega^+ = \{x \in \Omega \mid \ \bar{z}(0,x) = -\alpha\}$$
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Sparsity

Since $\bar{z}(0)$ is analytic in Ω_0 we have $|\Omega^+| = |\Omega^-| = 0$

In 1D, we have that $|\Omega^+|$ and $|\Omega^-|$ consist of finite number of points.

Finite element discretization in space

Spatial FEM

$\mathcal{T}$ be a quasi-uniform triangulation of Ω with a mesh size h, i.e.,

$$\operatorname{diam}(\tau) \le h \le C|\tau|^{\frac{1}{N}}, \quad \forall \tau \in \mathcal{T}.$$

$$V_h = \{\chi \in H_0^1(\Omega) : \ \chi\,|_\tau \in \mathcal{P}_1\},$$

where $\mathcal{P}_1$ is the set of piecewise linear polynomials.
(V_h is the usual space of Lagrangian finite elements of degree 1).

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We will also need discrete Laplacian $\Delta_h : V_h \to V_h$

$$(-\Delta_h v_h, \chi) = (\nabla v_h, \nabla \chi), \quad \forall \chi \in V_h$$

and Ritz projection $R_h : H^1 \to V_h$

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \forall \chi \in V_h.$$

Discontinuous Galerkin discretization in time.

Partition $I = [0, T]$ into subintervals $I_m = (t_{m-1}, t_m]$ of length $k_m = t_m - t_{m-1}$, where $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$.

Assumptions:

(i) There are constants $c, \beta > 0$ independent on k such that

$$k_{\min} \geq c k^{\beta}.$$

(ii) There is a constant $\kappa > 0$ independent on k such that for all $m = 1, 2, \ldots, M-1$

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The semidiscrete space X_k^q of piecewise polynomial functions in time is defined by

$$X_k^q = \{u_k \in L^2(I; H_0^1(\Omega)) : \; u_k|_{I_m} \in \mathcal{P}_q(H_0^1(\Omega)), \; m = 1, 2, \ldots, M\},$$

where $\mathcal{P}_q(V)$ is the space of polynomial functions of degree q in time with values in a Banach space V.

Bilinear form:

$$B(u,\varphi) = \sum_{m=1}^{M} \langle u_t, \varphi \rangle_{I_m \times \Omega} + (\nabla u, \nabla \varphi)_{I \times \Omega} + \sum_{m=2}^{M} ([u]_{m-1}, \varphi_{m-1}^{+})_{\Omega} + (u_0^{+}, \varphi_0^{+})_{\Omega},$$

where

$$u_m^{+} = \lim_{\varepsilon \to 0^{+}} u(t_m + \varepsilon), \quad u_m^{-} = \lim_{\varepsilon \to 0^{+}} u(t_m - \varepsilon), \quad [u]_m = u_m^{+} - u_m^{-}.$$

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dG(p) semidiscrete (in time) approximation

Find $u_k \in X_k^p$ such that

$$B(u_k, \varphi_k) = \langle q, \varphi_{k,0}^{+} \rangle_{\Omega} \quad \text{for all} \quad \varphi_k \in X_k^p.$$

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Galerkin orthogonality

$$B(u-u_k,\varphi_k) = 0 \quad \text{for all} \quad \varphi_k \in X_k^p.$$

Why dG?

There is a number of important properties making the dG schemes attractive for temporal discretization of parabolic equations:

1. For homogeneous autonomous problems, dG schemes are just subdiagonal Padé schemes.

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5. For optimal control problems, optimize-then-discretize and discretize-then-optimize approaches commute.
6. Different spatial discretizations on each time step can be naturally incorporated.

Bilinear form:

$$B(u,\varphi) = \sum_{m=1}^{M} \langle u_t, \varphi\rangle_{I_m\times\Omega} + (\nabla u, \nabla\varphi)_{I\times\Omega} + \sum_{m=2}^{M} ([u]_{m-1}, \varphi_{m-1}^{+})_\Omega + (u_0^+, \varphi_0^+)_\Omega,$$

where

$$u_m^+ = \lim_{\varepsilon\to 0^+} u(t_m+\varepsilon), \quad u_m^- = \lim_{\varepsilon\to 0^+} u(t_m-\varepsilon), \quad [u]_m = u_m^+ - u_m^-.$$

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dG(p)cG(r) fully discrete approximation

Find $u_{kh} \in X_{k,h}^{p,r}$ such that

$$B(u_{kh}, \varphi_{kh}) = \langle q, \varphi_{kh,0}^+ \rangle_\Omega \quad \text{for all } \; \varphi_{kh} \in X_{k,h}^{p,r}.$$

Bilinear form:

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Galerkin orthogonality

$$B(u-u_{kh}, \varphi_{kh}) = 0 \quad \text{for all} \;\; \varphi_{kh} \in X^{p,r}_{k,h}.$$

Finite element discretization in space and time

Discretization

- conformal P_1 elements in space, quasi-uniform mesh
- dG(q) in time, non-uniform time steps
- maximal time step k, maximal spacial mesh size h, no coupling.

Optimization problem

$$\text{Minimize} \quad J(q_{kh}, u_{kh}) = \frac{1}{2}\|u_{kh}(T) - u_d\|^2_{L^2(\Omega)} + \alpha\|q_{kh}\|_{\mathcal{M}},$$

subject to $q_{kh} \in M_h$ and $u_{kh} \in X^{p,1}_{k,h}$ with

$$B(u_{kh}, \varphi_{kh}) = \langle q_{kh}, \varphi^+_{kh,0}\rangle_\Omega \quad \text{for all} \quad \varphi_{kh} \in X^{p,1}_{k,h},$$

where

$$M_h = \{\mu_h \in \mathcal{M}(\Omega) : \quad \mu_h = \sum_i \beta_i \delta_{x_i}, \quad \beta_i \in \mathbb{R}\} \subset \mathcal{M}(\Omega)$$

Discrete optimality system

State equation

$$\bar{u}_{kh} \in X_{k,h}^{p,1}: \quad B(\bar{u}_{kh}, \varphi_{kh}) = (\bar{q}_{kh}, \varphi_{kh}(0)), \quad \forall \varphi_{kh} \in X_{k,h}^{p,1}$$

Adjoint equation

$$\bar{z}_{kh} \in X_{k,h}^{p,1}: \quad B(\varphi_{kh}, \bar{z}_{kh}) = (\bar{u}_{kh}(T) - u_d, \psi_{kh}(T)) \quad \forall \psi_{kh} \in X_{k,h}^{p,1}$$

Optimality condition

$$\bar{q}_{kh} \in M_h: \quad -\langle \bar{z}_{kh}(0), q - \bar{q}_{kh} \rangle \leq \alpha \left(\|q\|_{\mathcal{M}} - \|\bar{q}_{kh}\|_{\mathcal{M}} \right) \quad \forall q \in \mathcal{M}(\Omega).$$

Discrete optimality system

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Optimality condition

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Consequences:

$$\|\bar{z}_{kh}(0)\|_{C_0(\Omega)} \leq \alpha$$

$$\operatorname{supp} \bar{q}_{kh} \subset \{x \in \Omega : \ |\bar{z}_{kh}(0,x)| = \alpha\} \subset \Omega_0 \subset\subset \Omega.$$

Main result

Theorem (Convergence rates for the state)

There exists a constant C independent of k and h such that

$$\|(\bar{u} - \bar{u}_{kh})(T)\|_{L^2(\Omega)} \leq C(T)\ell_{kh} \left(k^{p+1/2} + h\right).$$

Idea of the proof

- From optimality systems: $\langle \bar{q}_{kh} - \bar{q}, \bar{z}(0) - \bar{z}_{kh}(0)\rangle \geq 0$
- Using state and adjoint equations and $\operatorname{supp} \bar{q}, \bar{q}_{kh} \subset \Omega_0$

$$\begin{aligned}\|(\bar{u} - \bar{u}_{kh})(T)\|^2_{L^2(\Omega)} &\leq \langle \bar{q}_{kh} - \bar{q}, \bar{z}(0) - \hat{z}_{kh}(0)\rangle + \frac{1}{2}\|(\bar{u} - \hat{u}_{kh})(T)\|^2_{L^2(\Omega)} \\ &\leq C\|\bar{z}(0) - \hat{z}_{kh}(0)\|_{L^\infty(\Omega_0)} + \frac{1}{2}\|(\bar{u} - \hat{u}_{kh})(T)\|^2_{L^2(\Omega)}.\end{aligned}$$

Homogeneous equation

$$\begin{aligned} \partial_t v - \Delta v &= 0 \;\text{ in } I \times \Omega, \\ v &= 0 \;\text{ on } I \times \partial\Omega \\ v(0) &= v_0 \;\text{ in } \Omega. \end{aligned}$$

Parabolic Smoothing

$$\|\Delta v(t)\|_{L^p(\Omega)} \leq \frac{C}{t}\|v_0\|_{L^p(\Omega)} \quad t > 0, \quad 1 \leq p \leq \infty.$$

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Theorem (Semidiscrete Smoothing)

Let v_k be the dG(q) solution of the homogenous parabolic problem. Then there exists a constant C independent of k such that

$$\|\partial_t v_k\|_{L^\infty(I_m;L^p(\Omega))} + \|\Delta v_k\|_{L^\infty(I_m;L^p(\Omega))} + \left\|\frac{[v_k]_{m-1}}{k_m}\right\|_{L^p(\Omega)} \leq \frac{C}{t_m}\|v_0\|_{L^p(\Omega)},$$

for $m = 1, \ldots, M$ and $1 \leq p \leq \infty$.

Homogeneous equation

$$
\begin{aligned}
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\end{aligned}
$$

Parabolic Smoothing

$$\|\Delta v(t)\|_{L^p(\Omega)} \le \frac{C}{t}\|v_0\|_{L^p(\Omega)} \quad t > 0.$$

Theorem (Fully discrete Smoothing)

Let v_{kh} be the dG(q)cG(r) solution of the homogenous parabolic problem. Then there exists a constant C independent of k and h such that

$$\|\partial_t v_{kh}\|_{L^\infty(I_m;L^p(\Omega))} + \|\Delta_h v_{kh}\|_{L^\infty(I_m;L^p(\Omega))} + \left\| \frac{[v_{kh}]_{m-1}}{k_m} \right\|_{L^p(\Omega)} \le \frac{C}{t_m}\|v_0\|_{L^p(\Omega)},$$

for $m = 1, \ldots, M$ and $1 \le p \le \infty$.

Pointwise smoothing error estimate

Theorem (Pointwise smoothing)

Let v be the solution to the homogeneous heat equation and v_{kh} be its dG(p)cG(1) approximation. Then there exists a constant C independent of k such that

$$|(v - v_{kh})(T, x_0)| \le C(T)\ell_{kh} \left(k^{2p+1} + h^2\right) \|v_0\|_{L^2(\Omega)},$$

Pointwise smoothing error estimate

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Idea of the proof:

$$|(v - v_{kh})(T, x_0)| \le \|(v - v_k)(T)\|_{L^\infty(\Omega)} + |(v_k - R_h v_k)(T, x_0)| + \|(R_h v_k - v_{kh})(T)\|_{L^\infty(}$$

where $R_h \colon H^1_0(\Omega) \to V_h$ is the Ritz projection:

Definition (Ritz projection)

$$(\nabla R_h v, \nabla \chi)_\Omega = (\nabla v, \nabla \chi)_\Omega \quad \forall \chi \in V_h.$$

Pointwise smoothing error estimate

First estimate

$$\|(v - v_k)(T)\|_{L^\infty(\Omega)} \le C \frac{k^{2p+1}}{T^{2p+1+\frac{N}{4}}} \|v_0\|_{L^2(\Omega)}.$$

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By elliptic interior pointwise error estimates

Third estimate

$$|(v_k - R_h v_k)(T, x_0)| \leq C \left(|\ln h| \|(v_k - I_h v_k)(T)\|_{L^\infty(B_d)} + \|(v_k - R_h v_k)(T)\|_{L^2(\Omega)} \right)$$

Idea of the proof

Supplementary function

$$B(\varphi_{kh}, \hat{z}_{kh}) = (\varphi_{kh}(T), \bar{u}(T) - u_d) \quad \forall \varphi_{kh} \in X_{k,h}^{p,1}.$$

Idea of the proof

Supplementary function

$$B(\varphi_{kh}, \hat{z}_{kh}) = (\varphi_{kh}(T), \bar{u}(T) - u_d) \quad \forall \varphi_{kh} \in X^{p,1}_{k,h}.$$

Initial Error Estimate

$$\|(\bar{u}-\bar{u}_{kh})(T)\|^2_{L^2(\Omega)} \leq 2\left(\|\bar{q}_{kh}\|_M + \|\bar{q}\|_M\right)\|\bar{z}(0)-\hat{z}_{kh}(0)\|_{L^\infty(\Omega_0)}+\|(\bar{u}-\hat{u}_{kh})(T)\|^2_{L^2(}$$

Idea of the proof

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Using Pointwise Smoothing theorem

$$\|\bar{z}(0) - \hat{z}_{kh}(0)\|_{L^\infty(\Omega_0)} \leq C(T)(k^{2p+1} + h^2)\|\bar{u}(T) - u_d\|_{L^2(\Omega)}$$

Idea of the proof

Supplementary function

$$B(\varphi_{kh}, \hat{z}_{kh}) = (\varphi_{kh}(T), \bar{u}(T) - u_d) \quad \forall \varphi_{kh} \in X_{k,h}^{p,1}.$$

Initial Error Estimate

$$\|(\bar{u} - \bar{u}_{kh})(T)\|^2_{L^2(\Omega)} \leq 2\left(\|\bar{q}_{kh}\|_M + \|\bar{q}\|_M\right)\|\bar{z}(0) - \hat{z}_{kh}(0)\|_{L^\infty(\Omega_0)} + \|(\bar{u} - \hat{u}_{kh})(T)\|^2_{L^2(\Omega)}$$

Using Pointwise Smoothing theorem

$$\|\bar{z}(0) - \hat{z}_{kh}(0)\|_{L^\infty(\Omega_0)} \leq C(T)(k^{2p+1} + h^2)\|\bar{u}(T) - u_d\|_{L^2(\Omega)}$$

Additional result

$$\|(\bar{u} - \hat{u}_{kh})(T)\|_{L^2(\Omega)} \leq C(T)\ell_{kh}\left(k^{2p+1} + h^2\right)\|\bar{q}\|_{\mathcal{M}}.$$

Estimates for the control

Theorem (Casas, Vexler, ZuaZua, 2015)

$$\bar{q}_{kh} \overset{*}{\rightharpoonup} \bar{q} \quad \textit{and} \quad \|\bar{q}\|_{\mathcal{M}} - \|\bar{q}_{kh}\|_{\mathcal{M}} \to 0 \quad \textit{as} \quad (k,h) \to (0,0).$$

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No convergence in the norm

We can not expect

$$\|\bar{q} - \bar{q}_{kh}\|_{\mathcal{M}} \to 0 \quad \text{as} \quad (k,h) \to (0,0).$$

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Simple example:

$$\bar{q} = \delta_{x_0}, \quad \bar{q}_{kh} = \delta_{x_{0,kh}}$$

wiht

$$x_{0,kh} \to x_0 \quad \text{as} \quad (k,h) \to (0,0),$$

but

$$\|\bar{q} - \bar{q}_{kh}\|_{\mathcal{M}} = 2,$$

as long as $x_{0,kh} \neq x_0$.

Error estimates for control

Structure of control

$$\operatorname{supp} \bar{q} \subset \{x \in \Omega : \quad |\bar{z}(0,x)| = \alpha\}$$

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Structural assumption (cf. Merino, Neitzel, Tröltzsch 2011)

1. $$\operatorname{supp} \bar{q} = \{x \in \Omega : \quad |\bar{z}(0,x)| = \alpha\} = \{x_1, x_2, \ldots, x_N\}$$
2. For x_i with $\bar{z}(0,x_i) = -\alpha, \quad \nabla_x^2 \bar{z}(0,x_i) > 0$ (positive definite)
3. For x_i with $\bar{z}(0,x_i) = \alpha, \quad \nabla_x^2 \bar{z}(0,x_i) < 0$ (negative definite)

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2. For x_i with $\bar{z}(0,x_i) = -\alpha$, $\quad \nabla_x^2 \bar{z}(0,x_i) > 0 \quad$ (positive definite)
3. For x_i with $\bar{z}(0,x_i) = \alpha$, $\quad \nabla_x^2 \bar{z}(0,x_i) < 0 \quad$ (negative definite)

Direct consequence

$$\bar{q} = \sum_i \beta_i \delta_{x_i}$$

with

$$\beta_i > 0 \quad \text{for} \quad \bar{z}(0,x_i) = -\alpha \quad \text{and} \quad \beta_i < 0 \quad \text{for} \quad \bar{z}(0,x_i) = \alpha$$

Error estimates for control

Theorem

There are $\varepsilon > 0$, $k_0, h_0 > 0$ *such that for all* $k < k_0$ *and* $h < h_0$

1. $$\operatorname{supp} \bar{q}_{kh} \cap B_\varepsilon(x_i) \neq \emptyset \quad i = 1, 2, \ldots, N$$

2. $$\operatorname{supp} \bar{q}_{kh} \subset \cup_i B_\varepsilon(x_i)$$

Error estimates

Theorem (Error estimates for the control)

$$\|\bar{u}(T) - \bar{u}_{kh}(T)\|_{L^2(\Omega)} \leq C(T) \left(k^{2p+1} + \ell_{kh} h\right)$$

Error estimates

Theorem (Error estimates for the control)

$$\|\bar{u}(T) - \bar{u}_{kh}(T)\|_{L^2(\Omega)} \leq C(T)\left(k^{2p+1} + \ell_{kh} h\right)$$

Theorem (Error estimates for locations and coefficients)

For

$$x_{i,kh} \in \bar{q}_{kh} \cap B_\varepsilon(x_i)$$

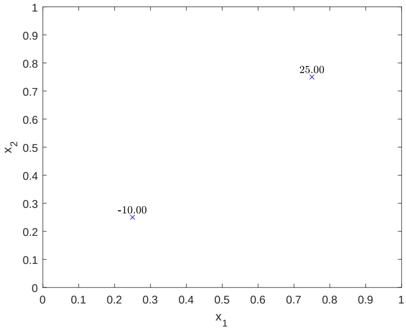
there holds

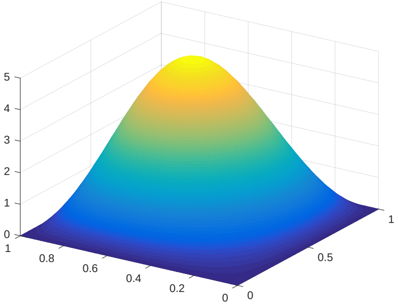
1. $$\max_i |x_i - x_{i,kh}| \leq C(T)\left(k^{2p+1} + \ell_{kh} h\right)$$

2. $$\max_i |\beta_i - \sum_{j=1}^{n_i} \beta_{kh,ij}| \leq C(T)\left(k^{2p+1} + \ell_{kh} h\right)$$

Numerical examples I

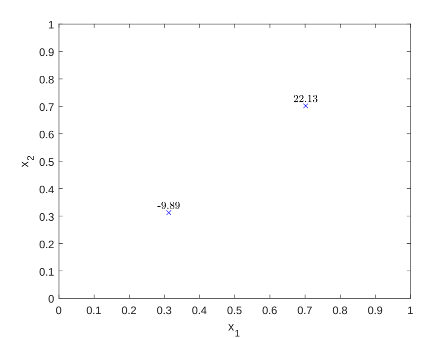
Reference initial state $q^\dagger$ and final $u_d^\dagger$

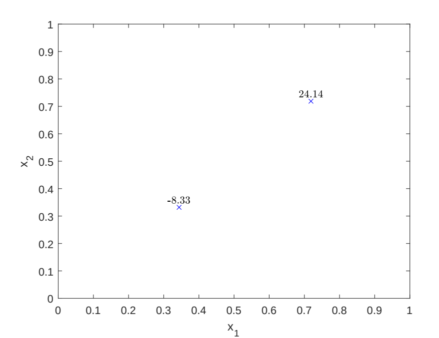




Numerical examples II

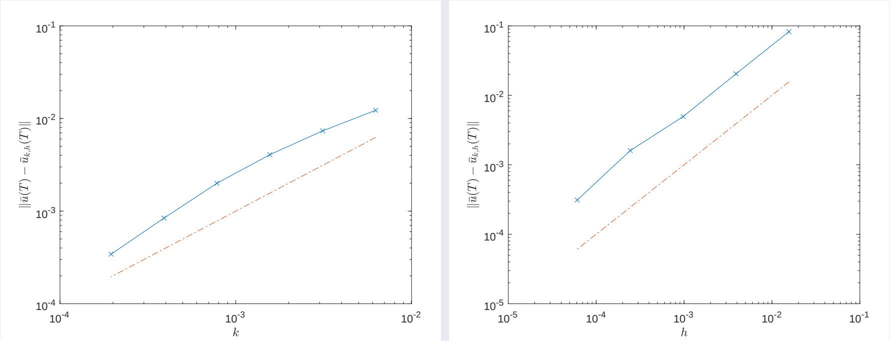
Computed initial state $\bar{q}_{kh}$ without and with noise on u_d





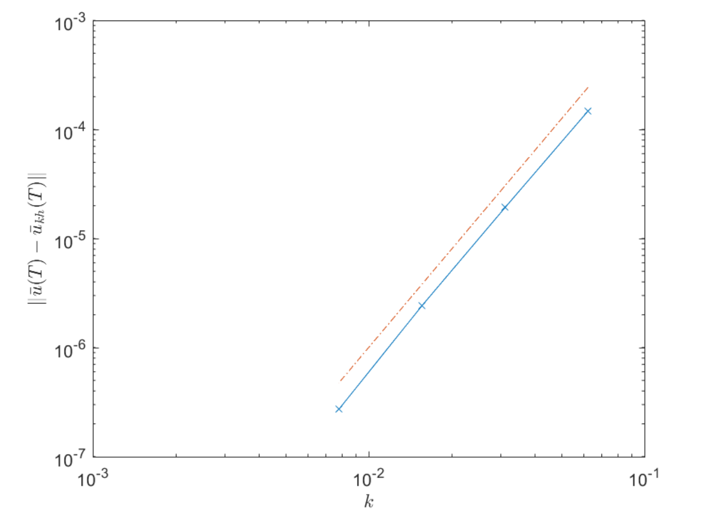
Numerical examples: Convergence plots

Convergence for $\|\bar{u}(T) - \bar{u}_{kh}(T)\|_{L^2(\Omega)}$



Numerical examples: Convergence plots

Convergence for $\|\bar{u}(T) - \bar{u}_{kh}(T)\|_{L^2(\Omega)}$, dG(1)



Conclusions

- Motivation: Identification of sparse initial conditions
- Formulation with control from $\mathcal{M}(\Omega)$
- FEM discretization and error analysis with precise smoothing type error estimates
- Under a structural assumption: Convergence of support points

THANK YOU