Sorbonne Université, Laboratoire Jacques-Louis Lions

Turnpike in shape design

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Workshop 1 - Special semester in Optimization - Ricam Linz
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Content

Turnpike phenomenon

Optimal control problem

$$\min_{u \in \mathcal{U}_{ad}} \int_0^T f^0(x(t), u(t))\, dt$$
$$\dot{x}(t) = f(x(t), u(t))$$
$$R(x(0), x(T)) = 0$$

Static problem

$$\min_{u \in \mathcal{U}_{ad}} f^0(x, u)$$
$$f(x, u) = 0$$

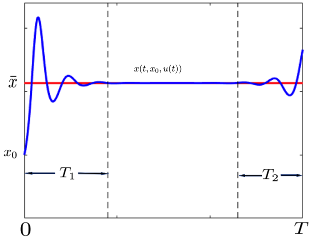
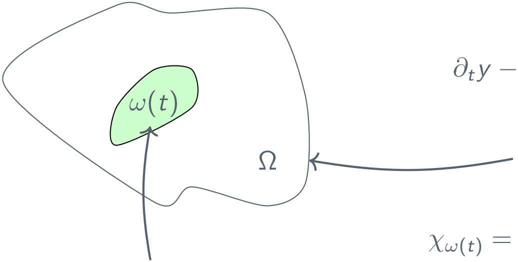


Figure: Turnpike property

Shape design Problem



Shape - source term

$$\partial_t y - \triangle y = \chi_\omega \text{ in } \Omega$$

$$y_{|\partial\Omega} = 0$$

$$\chi_{\omega(t)} = \begin{cases} 1 \text{ if } x \in \omega(t) \\ 0 \text{ else} \end{cases}$$

$$\frac{1}{2T}\int_0^T \|y(t) - y_d\|^2_{L^2(\Omega)} dt + \frac{1}{2}\|y(T) - y_d\|^2_{L^2(\Omega)} \to \min$$

Shape design Problem

- $\Omega \subset \mathbf{R}^N$ bounded, $y_0, y_d \in L^2(\Omega)$, $0 < L < 1$ and $T > 0$
- Set of admissible shapes : $\mathcal{U}_L = \{\omega \subset \Omega \text{ measurable } \mid |\omega| \leq L|\Omega|\}$

Optimal shape design problem (OSD_T)

$$\min_{\omega : (0,T) \to \mathcal{U}_L} \frac{\gamma_1}{2T} \int_0^T \|y(t) - y_d\|_{L^2(\Omega)}^2 dt + \frac{\gamma_2}{2} \|y(T) - y_d\|_{L^2(\Omega)}^2$$

$$\partial_t y - \triangle y = \chi_{\omega(\cdot)}, \quad y_{|\partial\Omega} = 0, \quad y(0) = y_0$$

Static shape design problem (SSD)

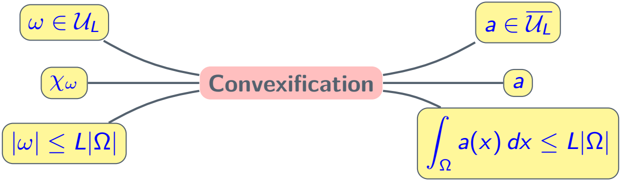
$$\min_{\omega \in \mathcal{U}_L} \frac{\gamma_1}{2} \|y - y_d\|_{L^2(\Omega)}^2$$

$$-\triangle y = \chi_\omega, \quad y_{|\partial\Omega} = 0$$

Shape design Problem→Relaxed problem

Set of admissible shapes : $\mathcal{U}_L = \{\omega \subset \Omega \text{ measurable} \mid |\omega| \leq L|\Omega|\}$

Convexified set : $\overline{\mathcal{U}_L} = \left\{ a \in L^\infty(\Omega; [0,1]) \mid \int_\Omega a(x)\,dx \leq L|\Omega| \right\}$



Optimal control problem (OCP_T)

$$\min_{a:(0,T)\to\overline{\mathcal{U}_L}} \frac{\gamma_1}{2T}\int_0^T \|y(t) - y_d\|^2_{L^2(\Omega)} dt + \frac{\gamma_2}{2}\|y(T) - y_d\|^2_{L^2(\Omega)}$$

$$\partial_t y - \triangle y = a, \quad y_{|\partial\Omega} = 0, \quad y(0) = y_0$$

Mayer Case - Final cost

$$\min_{\omega:(0,T)\to\mathcal{U}_L} \frac{1}{2}\|y(T)-y_d\|^2_{L^2(\Omega)}, \quad \partial_t y - \triangle y = \chi_{\omega(\cdot)}, \quad y_{|\partial\Omega} = 0, \quad y(0) = y_0$$

Theorem (Lance Trélat Zuazua)

- Static and time problems : existence and uniqueness of solution (y_T, p_T, ω_T) and $(\bar{y}, \bar{p}, \bar{\omega})$
- Exponential turnpike :

$$\forall T > 0, \quad d_{\mathcal{H}}\big(\omega_T(t), \bar{\omega}\big) \le Ce^{-\mu(T-t)}$$

Keyword's proof :

- Existence : Pontryagin Maximum principle (Lions - 1971) and Bathtub principle (Lieb & Loss - 2001)
- Turnpike : Adjoint equation, Weyl's law, Hopf lemma

Lagrange Case - Integral cost

$$\min_{\omega:(0,T)\to\mathcal{U}_L} \frac{1}{2T}\int_0^T \|y(t)-y_d\|^2_{L^2(\Omega)}\,dt, \quad \partial_t y - \triangle y = \chi_{\omega(\cdot)}, \quad y_{|\partial\Omega} = 0, \ y(0) = y_0$$

Theorem (Lance Trélat Zuazua, ongoing)

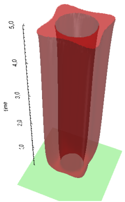
- Existence and uniqueness of solution (y_T, p_T, a_T) and $(\bar{y}, \bar{p}, \bar{a})$ and :
  - $y_d < y_0$ or $y_d > y_1 \implies \bar{a} = \chi_{\bar{\omega}}$ and $a_T = \chi_{\omega_T}$
  - y_d convex $\implies \bar{a} = \chi_{\bar{\omega}}$
- Integral turnpike :

$$\forall T > 0, \quad \int_0^T \left(\|y_T(t) - \bar{y}\|^2_{L^2(\Omega)} + \|p_T(t) - \bar{p}\|^2_{L^2(\Omega)}\right) dt < M$$

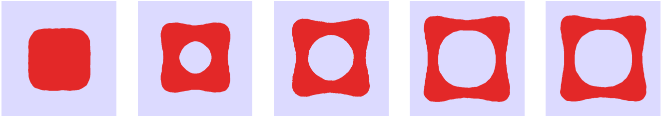
- Keyword's proof of Turnpike : Strict dissipativity (Faulwasser - 2017, Grüne - 2016, Trélat Zhang - 2018) or (Porretta Zuazua - 2013,2016)

Numerical examples

$\Omega = [-1,1]^2$, $L = \frac{1}{8}$, $T \in \{1..5\}$, $y_d = \mathrm{Cst} = 0.1$ and $y_0 = 0$



Time shape



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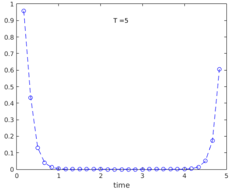

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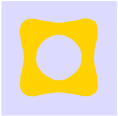

Static shape

(computed with `FreeFem++`and `IpOpt`)

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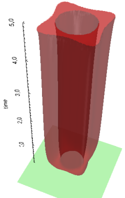
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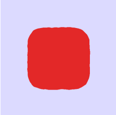
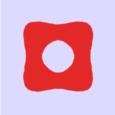

$t = 0$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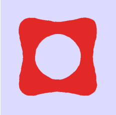
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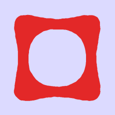
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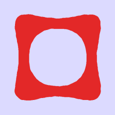
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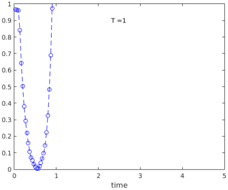

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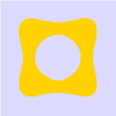
Static shape

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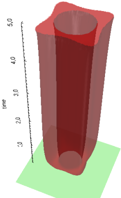

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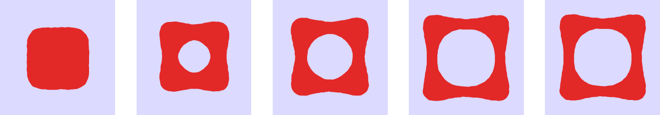

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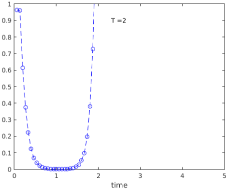

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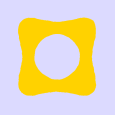

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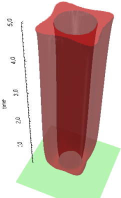

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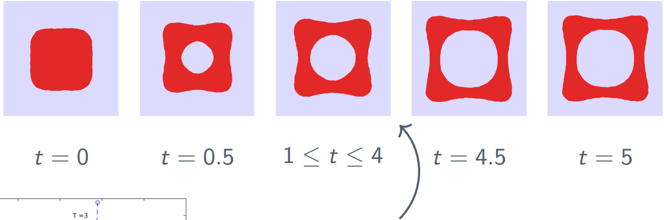

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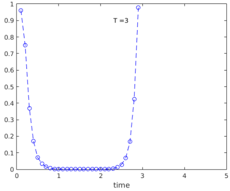

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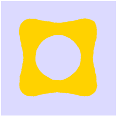

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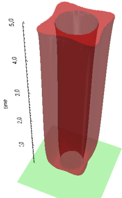

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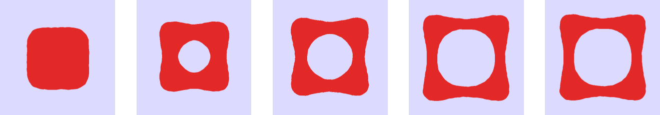

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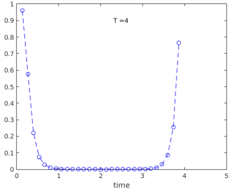

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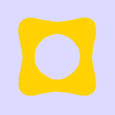

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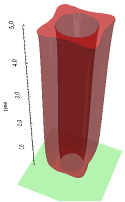

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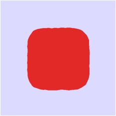
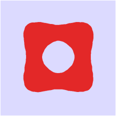
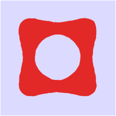


$t = 0$



$t = 0.5$



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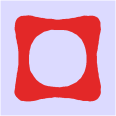

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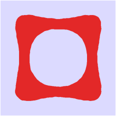

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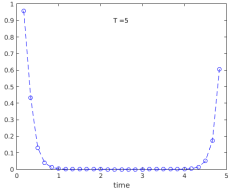

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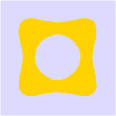
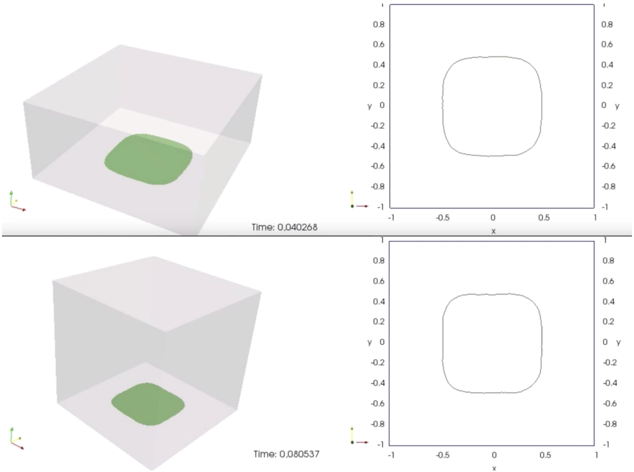


Static shape

(computed with `FreeFem++`and `IpOpt`)

Numerical examples (2D)

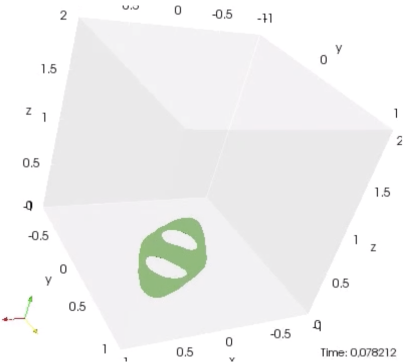
$\Omega = [-1,1]^2$, $L = \frac{1}{8}$, $T = \{1,2\}$, $y_d = \mathrm{Cst} = 0.1$ and $y_0 = 0$

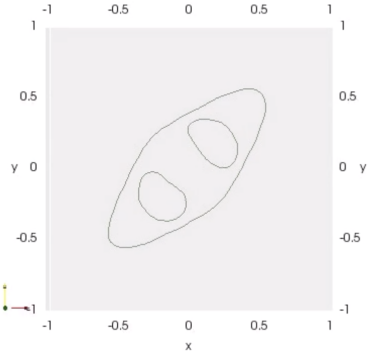


(computed with `FreeFem++`and `IpOpt`)

Numerical examples (2D)

$\Omega = [-1,1]^2,\ L = \frac{3}{16},\ T = 2,\ y_d = \frac{1}{20}(xy+1)$ and $y_0 = 0$





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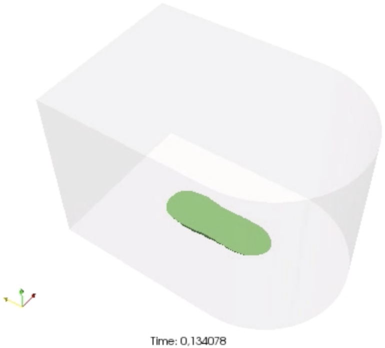
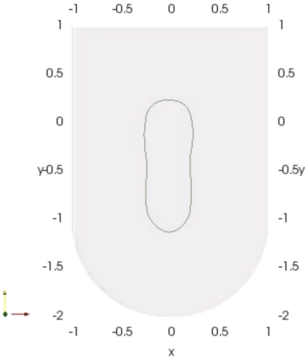
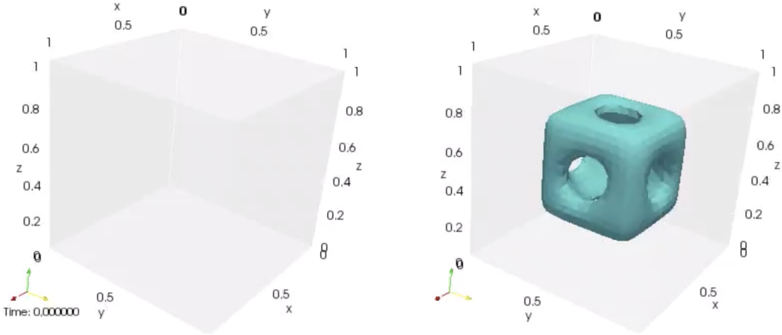




(computed with `FreeFem++`and `IpOpt`)

Numerical examples (3D)

$\Omega = [0,1]^3$, $L = \frac{1}{40}$, $T = 1$, $y_d = 0.1$ and $y_0 = 0$



(computed with `FreeFem++`and `IpOpt`)

Relaxation phenomenon (Lagrange case)

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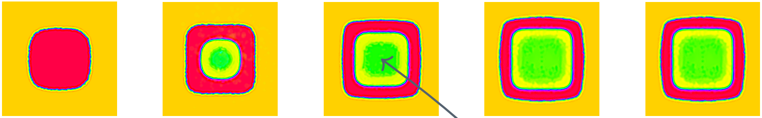

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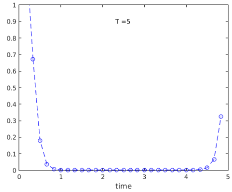

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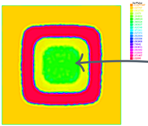

Static control

$(\bar{a}, a_T) \in (0,1)$

(computed with FreeFem++and IpOpt)

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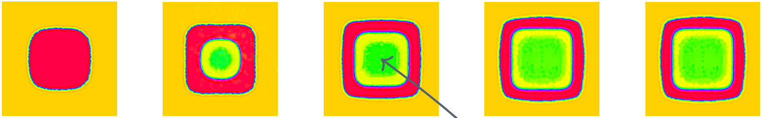

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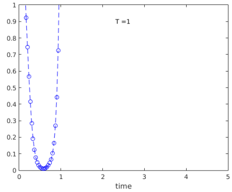

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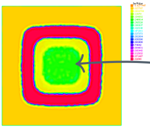

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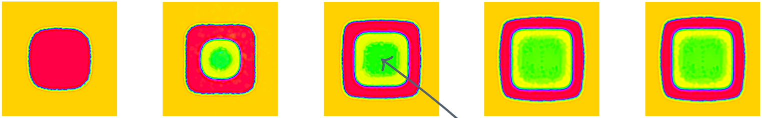
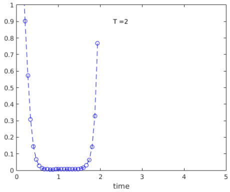
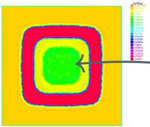


$t = 0$ | $t = 0.5$ | $1 \le t \le 4$ | $t = 4.5$ | $t = 5$



$t \to \|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|a_T(t) - \bar{a}\|$



Static control

$(\bar{a}, a_T) \in (0,1)$

(computed with `FreeFem++`and `IpOpt`)

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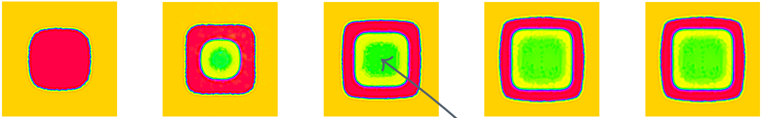

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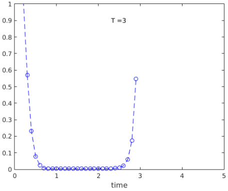

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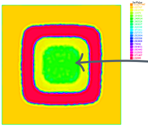

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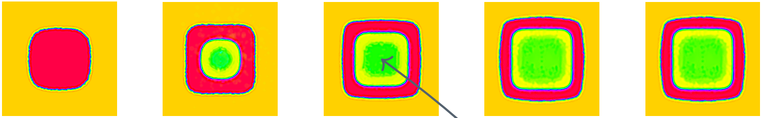

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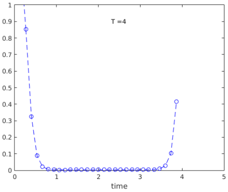
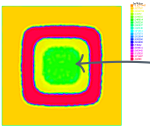


$t \rightarrow \|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|a_T(t) - \bar{a}\|$



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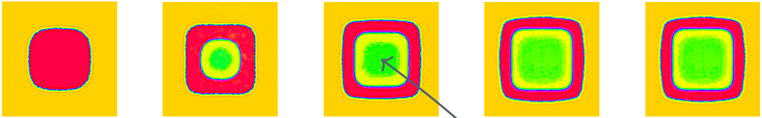

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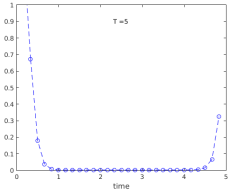

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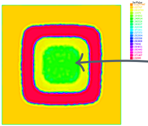

Static control

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Relaxation phenomenon (Lagrange case)

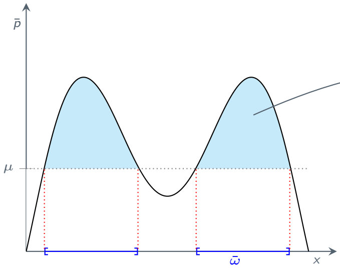
Maximum principle - parabolic equations

$$\partial_t y_T - \triangle y_T = a_T, \quad {y_T}_{|\partial\Omega} = 0, \quad y_T(0) = y_0$$
$$\partial_t p_T + \triangle p_T = y_T - y_d, \quad {p_T}_{|\partial\Omega} = 0, \quad p_T(T) = 0$$
$$\int_\Omega p_T(t,x) a_T(t,x)\, dx = \max_{u \in \overline{\mathcal{U}_L}} \int_\Omega p_T(t,x) u(x)\, dx$$

Maximum principle - elliptic equations

$$-\triangle \bar{y} = \bar{a}, \quad \bar{y}_{|\partial\Omega} = 0$$
$$\triangle \bar{p} = \bar{y} - y_d, \quad \bar{p}_{|\partial\Omega} = 0$$
$$\int_\Omega \bar{p}(x)\bar{a}(x)\, dx = \max_{u \in \overline{\mathcal{U}_L}} \int_\Omega \bar{p}(x) u(x)\, dx$$

$$\overline{\mathcal{U}_L} = \left\{ u \in L^\infty(\Omega),\, 0 \le u(x) \le 1 \mid \int_\Omega a(x)\, dx \le L|\Omega| \right\}$$



$$\bar{a}(x) = \begin{cases} 1 \text{ if } \bar{p}(x) > \mu \\ 0 \text{ if } \bar{p}(x) < \mu \end{cases}$$
$$\mu \text{ s.t. } \int_\Omega \bar{a}(x)\, dx = L|\Omega|$$

$$a_T(t)(x) = \begin{cases} 1 \text{ if } p_T(t)(x) > \mu(t) \\ 0 \text{ if } p_T(t)(x) < \mu(t) \end{cases}$$
$$\mu(t) \text{ s.t. } \int_\Omega a_T(t)(x)(x)\, dx = L|\Omega|$$

Relaxation phenomenon (Lagrange case)

Maximum principle - parabolic equations

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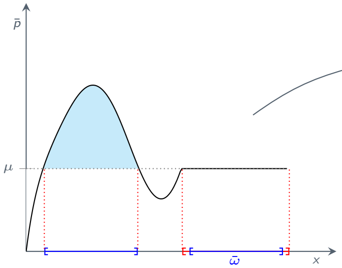

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$$\mu \text{ s.t. } \int_\Omega \bar{a}(x)\,dx = L|\Omega|$$

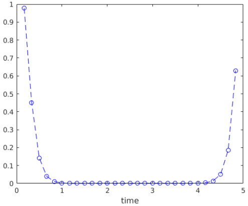
if $|\{\bar{p} = \mu\}| > 0$ then $-\triangle y_d = \bar{a}$

so y_d convex $\implies$ $\bar{a} = \chi_{\{\bar{p} > \mu\}}$

Future prospects

Lagrange case :

- Existence of optimal shapes :
  - sufficient conditions for existence of time shape ?
  - Replacing $|\omega| \leq L|\Omega|$ by $\mathcal{P}(\omega) \leq \alpha \implies$ existence of time and static shape.
- Turnpike on state, adjoint and control:



$t \to \|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|\chi_{\omega_T(t)} - \chi_{\bar{\omega}}\|$

$$\forall T > 0, \int_0^T \left(\|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|\chi_{\omega_T(t)} - \chi_{\bar{\omega}}\| \right) dt < M$$

Future prospects

Lagrange case :

- Existence of optimal shapes :
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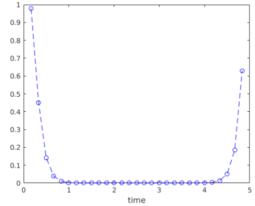


$t \to \|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|\chi_{\omega_T(t)} - \chi_{\bar{\omega}}\|$

$\forall T > 0,\ t \in (0, T)$

$$\|y_T(t) - \bar{y}\| + \|p_T(t) - \bar{p}\| + \|\chi_{\omega_T(t)} - \chi_{\bar{\omega}}\| \leq C\left(e^{-\mu t} + e^{-\mu(T-t)}\right)$$

Future prospects

Mayer case - terminal cost :

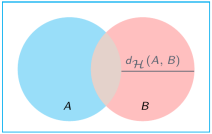


Figure: Hausdorff distance $d_{\mathcal{H}}(A, B)$

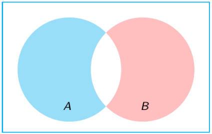


Figure: Symmetric difference $A \triangle B$

We have

$$d_{\mathcal{H}}\left(\omega_T(t), \bar{\omega}\right) \leq Ce^{-\mu(T-t)}$$

We would like

$$|\omega_T(t) \triangle \bar{\omega}| \leq Ce^{-\mu(T-t)}$$

Then state and control Turnpike:

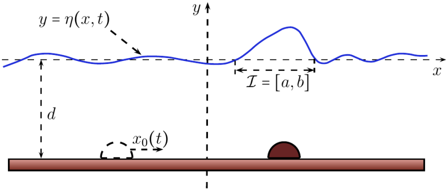
$$\|y_T(t) - \bar{y}\|_{L^2(\Omega)} + \|\chi_{\omega_T(t)} - \chi_{\bar{\omega}}\|_{L^1(\Omega)} \leq Ce^{-\mu(T-t)}$$

Future prospects : Other PDEs

- Generalize to parabolic equations and general semilinear PDEs

$$\partial_t y - Ay = \chi_{\omega(\cdot)} \quad \text{or} \quad \partial_t y - Ay - f(y) = \chi_{\omega(\cdot)}$$

- Wavemaker thanks to shape moving bottom : (Dalphin - 2017, Nersisyan Dutykh Zuazua - 2015)

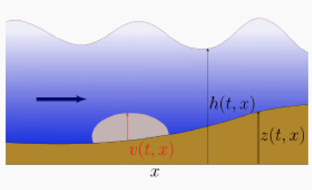


Future prospects : shallow water

- h : water's height
- u : water's velocity
- $q = hu$: flow
- z : shape of the bottom
- S_f : viscous effects

$$\min_z J_T(z) = \int_0^T \|h(t) - h_{obs}\|^2 \, dt$$

$$\begin{aligned} h_t + q_x &= 0 \\ q_t + \left(\frac{g}{2}h^2 + \frac{q^2}{h}\right)_x &= -ghz_x + S_f \\ h(0) = h_0, \quad & \quad q(0) = q_0 \end{aligned}$$



Static wave

Future prospects : shallow water

Kinetic interpretation of shallow water equations (Perthame Simeoni - 2001)

$$\chi : \mathbf{R} \to \mathbf{R}, \chi(\xi) = \chi(-\xi)$$
$$\int_{\mathbf{R}} \chi(\xi)\, d\xi = 1$$
$$\int_{\mathbf{R}} \xi^2 \chi(\xi)\, d\xi = \frac{g}{2}$$

writing $M(h, \xi - u) = \frac{h}{c}\chi\left(\frac{\xi - u}{c}\right)$

$$c = \sqrt{\frac{gh}{2}}$$

$$\begin{pmatrix} h \\ q \\ \frac{g}{2}h^2 + \frac{q^2}{h} \end{pmatrix} = \int_{\mathbf{R}} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \end{pmatrix} M(\xi)\, d\xi \qquad \int_{\mathbf{R}} Q\, d\xi = \int_{\mathbf{R}} \xi Q\, d\xi = 0$$

$\to (h, hu)$ strong solution of the shallow water system iff $M(h, \xi - u)$ satisfies the kinetic equation

$$M_t + \xi . M_x - g z_x . M_\xi = Q$$

Future prospects : shallow water

Perthame Simeoni - 2001

Shallow water description

$$h_t + q_x = 0$$
$$q_t + \left(\frac{g}{2}h^2 + \frac{q^2}{h}\right)_x = -ghz_x$$
$$h(0) = h_0, \quad q(0) = q_0$$

Kinetic description

$$M_t + \xi . M_x - gz_x . M_\xi = Q, \quad M(0) = M_0$$

PMP

Kinetic adjoint

$$P_t + \xi . P_x - gz_x P_\xi = M - M_{obs} + \frac{\partial Q}{\partial M}, \quad P(T) = 0$$

Adjoint representation

$$(p_h)_t - \left(gh - \frac{q^2}{h^2}\right)(p_q)_x - gz_x p_q = h - h_{obs}$$
$$(p_q)_t - (p_h)_x - \frac{2q}{h}(p_q)_x = 0$$
$$p_h(T) = 0, p_q(T) = 0$$

?

Optimization process :p_h, p_q needed to find a gradient descent

(Joint work with Jacques Sainte-Marie)