Optimal control of a semilinear heat equation subject to state and control constraints

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Content

State equation

Control: u, State: y

$\Omega \subset \mathbb{R}^n$, open and bounded with smooth boundary, $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$.

$$\begin{cases} \dot{y}(x,t) - \Delta y(x,t) + \gamma y^3(x,t) = f(x,t) + y(x,t) \sum_{i=0}^{m} u_i(t) b_i(x) & \text{in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases}$$

with $y_0 \in W_0^{1,\infty}(\Omega)$, $f \in L^\infty(Q)$, $b \in W^{1,\infty}(\Omega)^{m+1}$, $\gamma \geq 0$, $u_0 \equiv 1$ is a constant, and $u := (u_1, \dots, u_m) \in L^2(0,T)^m$.

State equation

Lemma

For $i = 0, \dots, m$, *the mapping defined on* $L^2(0,T) \times L^\infty(\Omega) \times L^\infty(0,T;L^2(\Omega))$, *given by* $(u_i, b_i, y) \mapsto u_i b_i y$, *has image in* $L^2(Q)$, *is of class* C^∞, *and satisfies*

$$\|u_i b_i y\|_2 \leq \|u_i\|_2 \|b_i\|_\infty \|y\|_{L^\infty(0,T;L^2(\Omega))}.$$

The state equation has a unique solution in $Y := H^{2,1}(Q)$.

Setting

Cost function

$$J(u,y) := \tfrac{1}{2}\int_Q (y(x,t)-y_d(x))^2 \mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\int_\Omega (y(x,T)-y_{dT}(x))^2 \mathrm{d}x + \sum_{i=1}^m \alpha_i \int_0^T u_i(t)\mathrm{d}t.$$

with $y_d \in L^\infty(Q)$, $y_{dT} \in W_0^{1,\infty}(\Omega)$, $\alpha \in \mathbb{R}^m$.

Optimal control problem

Control constraints $u \in \mathcal{U}_{\mathsf{ad}}$, where

$$\mathcal{U}_{\mathsf{ad}} = \{u \in L^2(0,T)^m;\ \check{u}_i \le u(t) \le \hat{u}_i,\ i = 1,\dots,m\},$$

for some constants $\check{u}_i < \hat{u}_i$, for $i = 1,\dots,m$.

State constraints

$$g_j(y(\cdot,t)) := \int_\Omega c_j(x)y(x,t)\mathrm{d}x + d_j \le 0, \quad \text{for } t \in [0,T],\ \ j = 1,\dots,q,$$

where $c_j \in H^2(\Omega) \cap H_0^1(\Omega)$ for $j = 1,\dots,q$, and $d \in \mathbb{R}^q$.

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Optimal control problem

$$\operatorname*{Min}_{u \in \mathcal{U}_{\textsf{ad}}} J(u, y[u]); \quad \text{subject to the state constraints.} \tag{P}$$

Aim: Second-order analysis

Tools:

- alternative costates
  (Bonnans and Jaisson 2010)
- radiality to derive second order necessary conditions
  (Aronna, Bonnans and Goh 2016)
- Goh transform
  (Goh 1966)

Results

- S. Aronna, F. Bonnans, A.K. State-constrained control-affine parabolic problems I: first and second order necessary optimality conditions 2019, preprint
- S. Aronna, F. Bonnans, A.K. State constrained control-affine parabolic problems II: Second order sufficient optimality conditions 2019, preprint

- J.F. Bonnans, *Singular arcs in the optimal control of a parabolic equation*, 2013, pp. 281-292, proc 11th IFAC Workshop on Adaptation and Learning in Control and Signal Processing (ALCOSP), Caen, F. Giri ed., July 3-5, 2013.
- M. S. Aronna, J.F. Bonnans, A.K., *Optimal Control of Infinite Dimensional Bilinear Systems: Application to the Heat and Wave Equations*, Math. Program. 168 (1) (2018) 717-757, erratum: Math. Programming Ser. A, Vol. 170 (2018).
- M. S. Aronna, J. F. Bonnans, A. K., Optimal control of PDEs in a complex space setting; application to the Schrödinger equation, SIAM J. Control Optim. 57 (2) (2019) 1390–1412.
- M. S. Aronna, J. F. Bonnans, B. S. Goh, *Second order analysis of control-affine problems with scalar state constraint*, Math. Program. 160 (1-2, Ser. A) (2016) 115–147.

Further results

- E. Casas, D. Wachsmuth, G. Wachsmuth, *Second-order analysis and numerical approximation for bang-bang bilinear control problems*, SIAM J. Control Optim. 56 (6) (2018) 4203–4227.
- E. Casas, F. Tröltzsch, A. Unger, *Second order sufficient optimality conditions for a nonlinear elliptic control problem*, J. for Analysis and its Applications (ZAA) 15 (1996) 687–707.
- J. F. Bonnans, *Second-order analysis for control constrained optimal control problems of semilinear elliptic systems*, Appl. Math. Optim. 38 (3) (1998) 303–325.
- E. Casas, M. Mateos, A. Rösch Error estimates for semilinear parabolic control problems in the absence of Tikhonov term, SIAM J. Control Optim., 57(4), 2515–2540, 2019.
- E. Casas, M. Mateos, F. Tröltzsch, *Necessary and sufficient optimality conditions for optimization problems in function spaces and applications to control theory*, in: Proceedings of 2003 MODE-SMAI Conference, Vol. 13 of ESAIM Proceedings, EDP Sciences, 2003, pp. 18–30.

Existence

Compactness

[Lions 1983] and [Edwards 1965]:

$$\left\{ \begin{array}{l} \text{For any } p \in [1,10), \text{ the following injection is compact:} \\ Y \hookrightarrow L^p(0,T;L^{10}(\Omega)), \quad \text{when } n \leq 3. \end{array} \right.$$

The mapping $u \mapsto y[u]$ is sequentially weakly continuous from $L^2(0,T)^m$ into Y.

Content

First order analysis

Implicit function theorem: $u \mapsto y[u]$ is of class C^∞ from $L^2(0,T)^m$ to Y

The generalized Lagrangian of problem (P) is, choosing the multiplier of the state equation to be $(p,p_0) \in L^2(Q) \times H^{-1}(\Omega)$ and taking $\beta \in \mathbb{R}_+$, $\mathrm{d}\mu \in \mathcal{M}_+(0,T)$,

$$\mathcal{L}[\beta, p, p_0, \mathrm{d}\mu](u,y) := \beta J(u,y) - \langle p_0, y(\cdot,0) - y_0 \rangle_{H_0^1(\Omega)}$$
$$+ \int_Q p\Big(\Delta y(x,t) - \gamma y^3(x,t) + f(x,t) + \sum_{i=0}^m u_i(t) b_i(x) y(x,t) - \dot{y}(x,t)\Big)\mathrm{d}x\mathrm{d}t$$
$$+ \sum_{j=1}^q \int_0^T g_j(y(\cdot,t))\mathrm{d}\mu_j(t).$$

Here: $\mathcal{M}_+(0,T)$ positive finite Radon measures; we identify it with the set

$$BV(0,T)^q_{0,+} := \{\mu \in BV(0,T)^q;\ \mu(T) = 0, \mathrm{d}\mu \geq 0\}.$$

First order analysis

For each $z \in L^2(0,T;H^2(\Omega))$ and $(x,t) \in Q$,

$$(Az)(x,t) := -\Delta z(x,t) + 3\gamma\bar{y}(x,t)^2 z(x,t) - \sum_{i=0}^{m} \bar{u}_i(t) b_i(x) z(x,t).$$

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Costate equation: for any $z \in Y$ there exist $p \in L^2(Q)$ with

$$\int_Q p(\dot{z} + Az) \mathrm{d}x\mathrm{d}t + \langle p_0, z(\cdot,0)\rangle_{H_0^1(\Omega)} = \sum_{j=1}^{q} \int_0^T \int_\Omega c_j z \mathrm{d}x \mathrm{d}\mu_j(t)$$
$$+ \beta \int_Q (\bar{y} - y_d) z \mathrm{d}x\mathrm{d}t + \beta \int_\Omega (\bar{y}(x,T) - y_{dT}(x)) z(x,T) \mathrm{d}x.$$

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Alternative costates (Bonnans & Jaisson 2010)

$$p^1 := p + \sum_{j=1}^{q} c_j \mu_j; \quad p_0^1 := p_0 + \sum_{j=1}^{q} c_j \mu_j(0), \qquad \text{(CS)}$$

where $\mu \in BV(0,T)_{0,+}^q$ associated with $\mathrm{d}\mu$.

Lemma

Let $(p, p_0, \mu) \in L^2(Q) \times H^{-1}(\Omega) \times BV(0,T)^q_{0,+}$ *satisfy the weak formulation, and let* (p^1, p^1_0) *be associated costates. Then*

$$p^1 \in Y, \quad p^1(0) = p^1_0,$$

$$-\dot{p}^1 + Ap^1 = \beta(\bar{y} - y_d) + \sum_{j=1}^{q} \mu_j A c_j, \quad p^1(\cdot, T) = \beta(\bar{y}(\cdot, T) - y_{dT}).$$

Moreover, $p(x, 0)$ *and* $p(x, T)$ *are well-defined in* $H^1_0(\Omega)$ *in view of* (CS), *and we have*

$$p(\cdot, 0) = p_0, \quad p(\cdot, T) = \beta(\bar{y}(\cdot, T) - y_{dT}).$$

Proof: Integration by parts

Proof by integration by parts

Remember

$$p^1 := p + \sum_{j=1}^q c_j \mu_j; \quad p_0^1 := p_0 + \sum_{j=1}^q c_j \mu_j(0). \tag{CS}$$

With $\psi = z(\cdot, 0)$ we have

$$\sum_{j=1}^q \int_Q c_j \mu_j \dot{z} \mathrm{d}x \mathrm{d}t + \sum_{j=1}^q \mu_j(0) \langle c_j, \psi \rangle_{L^2(\Omega)} = -\sum_{j=1}^q \int_0^T \int_\Omega c_j z \mathrm{d}x \mathrm{d}\mu_j(t).$$

The latter equation can be rewritten as

$$\int_Q (p^1 - p) \dot{z} \mathrm{d}x \mathrm{d}t + \langle p_0^1 - p_0, \psi \rangle_{H_0^1(\Omega)} = -\sum_{j=1}^q \int_0^T \int_\Omega c_j z \mathrm{d}x \mathrm{d}\mu_j(t). \tag{1.1}$$

Proof

That means, we have

$$\int_Q (p^1 - p)\dot{z}\mathrm{d}x\mathrm{d}t + \langle p_0^1 - p_0, \psi\rangle_{H_0^1(\Omega)} = -\sum_{j=1}^{q} \int_0^T \int_\Omega c_j z \mathrm{d}x \mathrm{d}\mu_j(t).$$

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and together with the costate equation

$$\int_Q p(\dot{z} + Az)\mathrm{d}x\mathrm{d}t + \langle p_0, z(\cdot,0)\rangle_{H_0^1(\Omega)} = \sum_{j=1}^{q} \int_0^T \int_\Omega c_j z \mathrm{d}x \mathrm{d}\mu_j(t)$$
$$+ \beta \int_Q (\bar{y} - y_d) z \mathrm{d}x\mathrm{d}t + \beta \int_\Omega (\bar{y}(x,T) - y_{dT}(x)) z(x,T)\mathrm{d}x.$$

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and

$$\int_Q (p^1 - p)Az = \int_Q \sum_{j=1}^q c_j \mu_j Az$$

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That means, we have

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and

$$\int_Q (p^1 - p)Az = \int_Q \sum_{j=1}^{q} c_j \mu_j Az$$

we obtain, with $\varphi = \dot{z} + Az$, that

$$\int_Q p^1 \varphi \mathrm{d}x\mathrm{d}t + \langle p_0^1, \psi\rangle_{H_0^1(\Omega)}$$
$$= \beta \int_Q (\bar{y} - y_d) z \mathrm{d}x\mathrm{d}t + \beta \int_\Omega (\bar{y}(x,T) - y_{dT}(x)) z(x,T)\mathrm{d}x + \int_Q \sum_{j=1}^{q} c_j \mu_j Az.$$

So, we have

$$\int_Q p^1 \varphi \mathrm{d}x\mathrm{d}t + \langle p_0^1, \psi\rangle_{H_0^1(\Omega)}$$
$$= \beta \int_Q (\bar{y} - y_d) z \mathrm{d}x\mathrm{d}t + \beta \int_\Omega (\bar{y}(x,T) - y_{dT}(x)) z(x,T)\mathrm{d}x + \int_Q \sum_{j=1}^q c_j \mu_j A z.$$

Since A is symmetric, we see that p^1 is solution in Y.

□

This shows the statement of the lemma:

Let $(p, p_0, \mu) \in L^2(Q) \times H^{-1}(\Omega) \times BV(0,T)^q_{0,+}$ satisfy the weak formulation, and let (p^1, p_0^1) be associated costates. Then

$$p^1 \in Y, \quad p^1(0) = p_0^1,$$

$$-\dot{p}^1 + Ap^1 = \beta(\bar{y} - y_d) + \sum_{j=1}^q \mu_j A c_j, \quad p^1(\cdot, T) = \beta(\bar{y}(\cdot, T) - y_{dT}).$$

Moreover, $p(x,0)$ and $p(x,T)$ are well-defined in $H_0^1(\Omega)$ in view of (CS), and we have

$$p(\cdot, 0) = p_0, \quad p(\cdot, T) = \beta(\bar{y}(\cdot, T) - y_{dT}).$$

We know

$$p^1 := p + \sum_{j=1}^{q} c_j\mu_j; \quad p_0^1 := p_0 + \sum_{j=1}^{q} c_j\mu_j(0), \tag{CS}$$

and since p^1 and $c_j\mu_j$ belong to $L^\infty(0,T;H_0^1(\Omega))$ we have

$$p \in L^\infty(0,T;H_0^1(\Omega)).$$

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$$p \in L^\infty(0,T;H_0^1(\Omega)).$$

Corollary

If $\mu \in H^1(0,T)^q$ *then* $p \in Y$ *and*

$$-\dot{p} + Ap = \beta(\bar{y} - y_d) + \sum_{j=1}^{q} c_j \dot{\mu}_j.$$

Proof: This follows directly from the equation for p^1 and (CS).

Reduced problem

Set

$$F(u) := J(u, y[u]),$$
$$G\colon L^2(0,T)^m \to C([0,T])^q, \quad G(u) := g(y[u]).$$

Reduced problem:

$$\min_{u \in \mathcal{U}_{\mathbf{ad}}} F(u); \quad G(u) \in K, \tag{RP}$$

with $K := C([0,T])^q_-$ closed convex cone.

Its interior is the set of functions in $C([0,T])^q$ with negative values.

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We assume that the reduced problem (RP) is qualified at $\bar{u}$ if:

$$\begin{cases} \text{there exists } u \in \mathcal{U}_{\mathbf{ad}} \text{ such that } v := u - \bar{u} \text{ satisfies} \\ G(\bar{u}) + DG(\bar{u})v \in \operatorname{int}(K). \end{cases}$$

Lagrange multiplier

We say that

$$(\beta, p, \mathrm{d}\mu) \text{ is a Lagrange multiplier}$$

if it satisfies the following *first-order optimality conditions:*

- $\mathrm{d}\mu$ is complementary to the state constraint,
- p is the costate,
- $(\beta, \mathrm{d}\mu) \neq 0$.
- Setting

  $$\Psi(t) := \beta\alpha(t) + \int_\Omega b(x)\bar{y}(x,t)p(x,t)\mathrm{d}x$$

  one has:

  $$\int_0^T \Psi(t)(u(t) - \bar{u}(t))\mathrm{d}t \geq 0, \quad \text{for every } u \in \mathcal{U}_{\text{ad}}.$$

Denote the set of Lagrange multipliers $(\beta, p, \mathrm{d}\mu)$ by $\Lambda(\bar{u}, \bar{y})$.

Lemma

Let $(\bar{u}, y[\bar{u}])$ *be an* L^2*-local solution of* (P). *Then:*

- *the associated set* Λ *of multipliers is nonempty,*
- *if in addition the qualification condition holds at* $\bar{u}$*, then there is no singular multiplier, and we call*

$$\Lambda_1 := \{(p, \mathrm{d}\mu) \text{ with } (1, p, \mathrm{d}\mu) \in \Lambda(\bar{u}, \bar{y}).\}$$

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Proof: (i) Set

$$L[\beta, \mathrm{d}\mu](u) := \beta F(u) + \sum_{j=1}^{q} \int_0^T G_j(u)(t)\mathrm{d}\mu_j(t).$$

Let $\bar{u}$ be a local solution of (RP). By, e.g., [Bonnans & Shapiro, Prop. 3.18], since K has nonempty interior, there exists a generalized Lagrange multiplier

$$(\beta, \mathrm{d}\mu) \in \mathbb{R}_+ \times N_K(G(\bar{u}))$$

such that

$$(\beta, \mathrm{d}\mu) \neq 0 \quad \text{and} \quad - D_u L[\beta, \mathrm{d}\mu](\bar{u}) \in N_{\mathcal{U}_{\mathrm{ad}}}(\bar{u}).$$

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Due to the costate equation, the latter condition is equivalent to the variational inequality above.

(ii) Follows by [Bonnans & Shapiro, Prop. 3.16].

Contact sets

In the following let $(\bar{u}, \bar{y})$ be an admissible trajectory.

associated with

- control constraints:

$$\check{I}_i := \{t \in [0,T];\ \bar{u}_i(t) = \check{u}_i\}, \quad \hat{I}_i := \{t \in [0,T];\ \bar{u}_i(t) = \hat{u}_i\}, \quad I_i := \check{I}_i \cup \hat{I}_i.$$

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- jth state constraint, $j = 1, \ldots, q$, is

$$I_j^C := \{t \in [0,T];\ g_j(\bar{y}(\cdot,t)) = 0\}.$$

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- jth state constraint, $j = 1, \ldots, q$, is

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- Given $0 \le a < b \le T$, we call (a, b) a maximal state constrained arc for the jth state constraints, if I_j^C contains (a, b) but it contains no open interval strictly containing (a, b).
- We define in the same way a maximal (lower or upper) control bound constraints arc.

First order optimality condition

$$\Psi_i^p(t) = \alpha_i + \int_\Omega b_i(x)\bar{y}(x,t)p(x,t)\mathrm{d}x, \quad \text{for } i = 1,\dots,m,$$

one has $\Psi^p \in L^\infty(0,T)^m$ and

$$\sum_{i=1}^m \int_0^T \Psi_i^p(t)(u_i(t) - \bar{u}_i(t))\mathrm{d}t \geq 0, \quad \text{for every } u \in \mathcal{U}_{\mathsf{ad}}. \tag{1.2}$$

Corollary

The first order optimality condition is equivalent to

$$\{t \in [0,T];\ \Psi_i^p(t) > 0\} \subseteq \check{I}_i, \qquad \{t \in [0,T];\ \Psi_i^p(t) < 0\} \subseteq \hat{I}_i,$$

for every $(p, \mathrm{d}\mu) \in \Lambda_1$.

Content

Hypothesis

Finite arc property:

$$\left\{ \begin{array}{c} \text{the contact sets for the state and bound constraints are,} \\ \text{up to a finite set, the union of finitely many maximal arcs.} \end{array} \right.$$

There exist junction points

$$0 =: \tau_0 < \cdots < \tau_r := T,$$

such that the intervals (τ_k, τ_{k+1}) are maximal arcs with constant active constraints.

Hypothesis

Finite arc property:

$$\begin{cases} \text{the contact sets for the state and bound constraints are,} \\ \text{up to a finite set, the union of finitely many maximal arcs.} \end{cases}$$

There exist junction points

$$0 =: \tau_0 < \cdots < \tau_r := T,$$

such that the intervals (τ_k, τ_{k+1}) are maximal arcs with constant active constraints.

Definition

For $k = 0, \ldots, r-1$, let $\check{B}_k, \hat{B}_k, C_k$ denote the set of indexes of active lower and upper bound constraints, and state constraints, on the maximal arc (τ_k, τ_{k+1}), and set $B_k := \check{B}_k \cup \hat{B}_k$.

For $v : [0,T] \to X$, X Banach space, we denote (if they exist) its left and right limits at $\tau \in [0,T]$ by $v(\tau\pm)$, with

$$v(0-) := v(0), \quad v(T+) := v(T)$$

and the jump by

$$[v(\tau)] := v(\tau+) - v(\tau-).$$

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We denote the time derivative of the state constraints by

$$g_j^{(1)}(\bar{y}(\cdot,t)) := \frac{\mathrm{d}}{\mathrm{d}t} g_j(\bar{y}(\cdot,t)) = \int_\Omega c_j(x)\dot{\bar{y}}(x,t)\mathrm{d}x, \quad j = 1,\dots,q.$$

Note that $g_j^{(1)}(\bar{y}(\cdot,t))$ is an element of $L^1(0,T)$, for each $j = 1,\dots,q$.

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Note that $g_j^{(1)}(\bar{y}(\cdot,t))$ is an element of $L^1(0,T)$, for each $j = 1,\ldots,q$.

Lemma

Let $\bar{u}$ have left and right limits at $\tau \in (0,T)$. Then

$$[\Psi_i^p(\tau)][\bar{u}_i(\tau)] = [g_j^{(1)}(\bar{y}(\cdot,\tau))][\mu_j(\tau)] = 0, \;\; i = 1,\ldots,m, \quad j = 1,\ldots,q.$$

Local controllability condition

For fixed k in $\{0, \ldots, r-1\}$ and maximal arc (τ_k, τ_{k+1}), setting

$$M_{ij}(t) := \int_\Omega b_i(x) c_j(x) \bar{y}(x,t) \mathrm{d}x, \quad 1 \leq i \leq m, \;\; 1 \leq j \leq q.$$

Let $\bar{M}_k(t)$ (of size $|\bar{B}_k| \times |C_k|$) denote the submatrix of $M(t)$ having rows with index in $\bar{B}_k$ and columns with index in C_k.

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Hypothesis

Assume $|C_k| \leq |\bar{B}_k|$, *for* $k = 0, \ldots, r-1$, *and*

$$\begin{cases} \text{there exists } \alpha > 0, \ \text{such that } |\bar{M}_k(t)\lambda| \geq \alpha|\lambda|, \\ \text{for all } \lambda \in \mathbb{R}^{|C_k|}, \ \text{a.e. on } (\tau_k, \tau_{k+1}), \ \text{for } k = 0, \ldots, r-1. \end{cases} \tag{1.3}$$

This hypothesis was already used in a different setting (i.e. higher-order state constraints in the finite dimensional case) in e.g. [Bonnans, Hermant 2009; Maurer 1979].

Hypothesis

We assume

- discontinuity of the derivative of the state constraints at corresponding junction points,
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Under the hypotheses and the lemma (on the jumps) we obtain

Theorem

(i) *For $u \in L^\infty(0,T)^m$, the associated state $y[u]$ belongs to $C(\bar{Q})$.*

(ii) *For every $(p, \mathrm{d}\mu) \in \Lambda_1$, one has that $\mu \in W^{1,\infty}(0,T)^q$ and p is essentially bounded in Q.*

Content

Second variation

For $(p, \mathrm{d}\mu) \in \Lambda_1$, set

$$\kappa(x,t) := 1 - 6\gamma\bar{y}(x,t)p(x,t),$$

and consider the quadratic form

$$\mathcal{Q}[p, \mathrm{d}\mu](z,v) := \int_Q \left(\kappa z^2 + 2p\sum_{i=1}^m v_i b_i z\right) \mathrm{d}x\mathrm{d}t + \int_\Omega z(x,T)^2 \mathrm{d}x.$$

Let (u, y) be a trajectory, and set

$$(\delta y, v) := (y - \bar{y}, u - \bar{u}).$$

We have

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\delta y + A\delta y = \sum_{i=1}^{m} v_i b_i y - 3\gamma\bar{y}(\delta y)^2 - \gamma(\delta y)^3 \quad \text{in } Q, \\ \delta y = 0 \quad \text{on } \Sigma, \quad \delta y(\cdot, 0) = 0 \quad \text{in } \Omega. \end{cases}$$

Proposition

Let $(p, \mathrm{d}\mu) \in \Lambda_1$, and let (u, y) be a trajectory. Then

$$\mathcal{L}[p, \mathrm{d}\mu](u, y, p) - \mathcal{L}[p, \mathrm{d}\mu](\bar{u}, \bar{y}, p)$$
$$= \int_0^T \Psi^p(t) \cdot v(t)\mathrm{d}t + \tfrac{1}{2}\mathcal{Q}[p, \mathrm{d}\mu](\delta y, v) - \gamma \int_Q p(\delta y)^3 \mathrm{d}x\mathrm{d}t.$$

Critical cone

For $\bar{u} \in L^2$ we define

$$C := \left\{ \begin{array}{l} (z[v], v) \in Y \times L^2(0,T)^m;\ v_i(t)\Psi_i^p(t) = 0 \text{ a.e. on } [0,T], \\ \text{for all } (p, \mathrm{d}\mu) \in \Lambda_1 \\ v_i(t) \geq 0 \text{ a.e. on } \check{I}_i,\ v_i(t) \leq 0 \text{ a.e. on } \hat{I}_i, \text{ for } i = 1, \dots, m, \\ \displaystyle\int_\Omega c_j(x) z[v](x,t)\mathrm{d}x \leq 0 \text{ on } I_j^C, \text{ for } j = 1, \dots, q \end{array} \right\}.$$

Strict critical cone

Imposing that the linearization of active constraints is zero

$$C_{\mathrm{s}} := \left\{ \begin{array}{l} (z[v], v) \in Y \times L^2(0,T)^m;\ v_i(t) = 0 \ \text{a.e. on } I_i, \ \text{for } i = 1, \dots, m, \\ \displaystyle\int_\Omega c_j(x) z[v](x,t) \mathrm{d}x = 0 \ \text{on } I_j^C, \ \text{for } j = 1, \dots, q \end{array} \right\}.$$

Hence, clearly $C_{\mathrm{s}} \subseteq C$, and C_{s} is a closed subspace of $Y \times L^2(0,T)^m$.

Radiality of critical directions

Hypothesis: uniform distance to control bounds whenever they are not active,

Aronna *et al.* 2016: a critical direction (z, v) is quasi radial if there exists $\tau_0 > 0$ such that, for $\tau \in [0, \tau_0]$, the following conditions are satisfied:

$$\max_{t\in[0,T]} \left\{ g_j(\bar{y}(\cdot,t)) + \tau g_j'(\bar{y}(\cdot,t))z(t) \right\} = o(\tau^2), \quad \text{for } j = 1, \ldots, q,$$

$$\check{u}_i \leq \bar{u}_i(t) + \tau v_i(t) \leq \hat{u}_i, \quad \text{a.e. on } [0, T], \quad \text{for } i = 1, \ldots, m.$$

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$$\check{u}_i \le \bar{u}_i(t) + \tau v_i(t) \le \hat{u}_i, \quad \text{a.e. on } [0,T], \quad \text{for } i = 1,\ldots,m.$$

Corollary

The set of quasi radial critical directions of C_s is dense in C_s.

Theorem (Second order necessary condition)

Let the admissible trajectory $(\bar{u}, \bar{y})$ *be an* L^∞*-local solution of* (P)*. Then*

$$\max_{(p,\mathrm{d}\mu)\in\Lambda_1} \mathcal{Q}[p,\mathrm{d}\mu](z,v) \geq 0, \qquad \textit{for all } (z,v) \in C_{\mathrm{s}}.$$

Content

Goh transform

Given a critical direction (z, v), set

$$w(t) := \int_0^t v(s)\mathrm{d}s; \quad B(x,t) := \bar{y}(x,t)b(x); \quad \zeta(x,t) := z(x,t) - B(x,t) \cdot w(t),$$

based on [Goh 1966].

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based on [Goh 1966]. We have

$$\dot{\zeta} + A\zeta = \underbrace{\left(\dot{z} + Az - \sum_{i=1}^m v_i B_i\right)}_{=0} - \sum_{i=1}^m w_i(AB_i + \dot{B}_i), \quad \zeta(\cdot,0) = 0.$$

Since $\dot{B}_i = b_i\dot{\bar{y}}$ it follows that

$$\dot{\zeta}(x,t) + (A\zeta)(x,t) = B^1(x,t)\cdot w(t), \quad \zeta(\cdot,0) = 0, \tag{1.4}$$

where

$$B_i^1 := -fb_i + 2\nabla\bar{y}\cdot\nabla b_i + \bar{y}\Delta b_i - 2\gamma\bar{y}^3 b_i, \quad \text{for } i = 1,\dots,m.$$

Lemma (Transformed second variation)

We can define a quadratic form $\widehat{\mathcal{Q}}$ *such that for* $v \in L^2(0,T)^m$, *and* $w \in AC([0,T])^m$ *given by the Goh transform, and for all* $(p, \mathrm{d}\mu) \in \Lambda_1$, *we have*

$$\mathcal{Q}[p, \mathrm{d}\mu](z[v], v) = \widehat{\mathcal{Q}}[p, \mathrm{d}\mu](\zeta[w], w, w(T)).$$

Goh transform of the critical cone

Set of primitives of strict critical direction

$$PC := \left\{ \begin{array}{l} (\zeta, w, w(T)) \in Y \times H^1(0,T)^m \times \mathbb{R}^m; \\ (\zeta, w) \text{ is given by the Goh transform for some } (z,v) \in C_{\mathrm{s}} \end{array} \right\},$$

and let

$$PC_2 := \text{closure of } PC \text{ in } Y \times L^2(0,T)^m \times \mathbb{R}^m.$$

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$\longrightarrow$ We can give a characterization of a superset PC_2' which coincides with PC_2 for scalar controls (i.e. $m = 1$).

$\longrightarrow$ We will formulate the second-order sufficient optimality condition on a superset $PC_2 \subset PC_2^*$.

We take a closer look.

We recall

For fixed k in $\{0, \ldots, r-1\}$ and maximal arc (τ_k, τ_{k+1}), setting

$$M_{ij}(t) := \int_\Omega b_i(x) c_j(x) \bar{y}(x,t) \mathrm{d}x, \quad 1 \le i \le m, \;\; 1 \le j \le q.$$

For any $(\zeta, w, h) \in PC$, it holds

$$w_{B_k}(t) = \frac{1}{\tau_{k+1} - \tau_k} \int_{\tau_k}^{\tau_{k+1}} w_{B_k}(s)\mathrm{d}s, \quad \text{for } k = 0, \ldots, r-1. \tag{1.5}$$

For any $(\zeta, w, h) \in PC$, it holds

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Take $(z, v) \in C_\mathrm{s}$, and $(w, \zeta[w])$ given by the Goh transform.

Let $k \in \{0, \dots, r-1\}$ and $j \in C_k$. Then $0 = \int_\Omega c_j(x) z(x,t)\mathrm{d}x$ on (τ_k, τ_{k+1}). Therefore, letting $M_j(t)$ denote the jth column of the matrix $M(t)$, one has

$$M_j(t) \cdot w(t) = -\int_\Omega c_j(x)\zeta[w](x,t)\mathrm{d}t, \quad \text{on } (\tau_k, \tau_{k+1}), \text{ for } j \in C_k. \tag{1.6}$$

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We can rewrite (1.5)-(1.6) in the form

$$\mathcal{A}^k(t) w(t) = \left(\mathcal{B}^k w\right)(t), \quad \text{on } (\tau_k, \tau_{k+1}), \tag{1.7}$$

where $\mathcal{A}^k(t)$ is an $m_k \times m$ matrix with $m_k := |B_k| + |C_k|$, and
$\mathcal{B}^k \colon L^2(0,T)^m \to H^1(\tau_k, \tau_{k+1})^{m_k}$.

Let $c_{k+1} \in \mathbb{R}^m$ be such that, for some ν^{k+i},

$$c_{k+1} = \mathcal{A}^{k+i}(\tau_{k+1})^\top \nu^{k+i}, \quad \text{for } i = 0, 1, \tag{1.8}$$

meaning that c_{k+1} is a linear combination of the rows of $\mathcal{A}^{k+i}(\tau_{k+1})$ for both $i = 0, 1$.

Lemma

Let $k = 0, \ldots, r-1$, and let c_{k+1} satisfy (1.8). Then, the junction condition

$$c_{k+1} \cdot \big(w(\tau_{k+1}^+) - w(\tau_{k+1}^-)\big) = 0, \tag{1.9}$$

holds for all $(\zeta, w, h) \in PC_2$.

Set

$$PC_2' := \{(\zeta[w], w, h);\ w \in \mathrm{Ker}(\mathcal{A} - \mathcal{B}), \text{(1.9) holds, for all } c \text{ satisfying (1.8)}\}.$$

We have proved that

$$PC_2 \subseteq PC_2'.$$

In the case of a scalar control $(m = 1)$ we can show that these two sets coincide.

Proposition

If the control is scalar, then

$$PC_2 = \left\{ \begin{array}{l} (\zeta[w], w, h) \in Y \times L^2(0,T) \times \mathbb{R};\ \ w \in \mathrm{Ker}(\mathcal{A} - \mathcal{B}); \\ w \text{ is continuous at BB, BC, CB junctions} \\ \lim_{t \downarrow 0} w(t) = 0 \text{ if the first arc is not singular} \\ \lim_{t \uparrow T} w(t) = h \text{ if the last arc is not singular} \end{array} \right\}.$$

Second order necessary condition in transformed variables

Theorem

If $(\bar{u}, \bar{y})$ is an L^∞-local solution of problem (P)*, then*

$$\max_{(p,\mathrm{d}\mu)\in\Lambda_1} \widehat{\mathcal{Q}}[p,\mathrm{d}\mu](\zeta, w, h) \geq 0, \quad \textit{on } PC_2.$$

Content

Pontryagin minimum

(i) An admissible trajectory $(\bar{u}, \bar{y})$ is said to be a *Pontryagin minimum* if for all $N > 0$, there exists $\varepsilon_N > 0$ such that, $(\bar{u}, \bar{y})$ is optimal among all the admissible trajectories (u, y) verifying

$$\|u - \hat{u}\|_\infty < N \quad \text{and} \quad \|u - \hat{u}\|_1 < \varepsilon_N.$$

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(ii) A sequence $(v_\ell) \subset L^\infty(0,T)^m$ is said to *converge to* 0 *in the Pontryagin sense* if it is bounded in $L^\infty(0,T)^m$ and $\|v_\ell\|_1 \to 0$.

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(iii) We say that $(\bar{u}, \bar{y})$ is a *Pontryagin minimum satisfying the weak quadratic growth condition* if there exists $\rho > 0$ such that, for every sequence of admissible variations $(v_\ell, \delta y_\ell)$ having (v_ℓ) convergent to 0 in the Pontryagin sense, one has

$$F(u_\ell) - F(\bar{u}) \geq \rho(\|w_\ell\|_2^2 + |w_\ell(T)|^2),$$

for ℓ sufficiently large and where $w_\ell(t) = \int_0^t v_\ell(s)\mathrm{d}s$.

Consider the condition

$$g_j'(\bar{y}(\cdot,T))(\zeta(\cdot,T)+B(\cdot,T)h)=0, \text{ if } T\in I_j^C \text{ and } [\mu_j(T)]>0, \text{ for } j=1,\dots,q. \tag{1.10}$$

We define

$$PC_2^* := \left\{ \begin{array}{l} (\zeta[w],w,h)\in Y\times L^2(0,T)^m\times\mathbb{R}^m;\ w_{B_k} \text{ is constant on each arc;} \\ (1.4),(1.6),(1.11)(\text{i})\text{-}(\text{ii}),(1.10) \text{ hold.} \end{array} \right\}.$$

PC_2^* is a superset of PC_2.

We recall that $(\zeta[w],w,h)$ in PC satisfy

$$\left\{ \begin{array}{lll} (\text{i}) & w_i=0 \text{ a.e. on } (0,\tau_1), \text{ for each } i\in B_0, \\ (\text{ii}) & w_i=h_i \text{ a.e. on } (\tau_{r-1},T), \text{ for each } i\in B_{r-1}, \\ (\text{iii}) & g_j'(\bar{y}(\cdot,T))[\zeta(\cdot,T)+B(\cdot,T)\cdot h]=0 \text{ if } j\in C_{r-1}. \end{array} \right. \tag{1.11}$$

and

$$\dot{\zeta}(x,t)+(A\zeta)(x,t)=B^1(x,t)\cdot w(t), \quad \zeta(\cdot,0)=0, \tag{1.4}$$

$$M_j(t)\cdot w(t)=-\int_\Omega c_j(x)\zeta[w](x,t)\mathrm{d}t, \quad \text{on } (\tau_k,\tau_{k+1}), \text{ for } j\in C_k. \tag{1.6}$$

Theorem (Sufficient conditions)

a) Assume additional that

(i) $(\bar{u}, \bar{y})$ is a feasible trajectory with nonempty associated set of multipliers Λ_1;

(ii) strict complementarity for control and state constraints;

(iii) for each $(p, \mathrm{d}\mu) \in \Lambda_1$, $\widehat{\mathcal{Q}}[p, \mathrm{d}\mu](\cdot)$ is a Legendre form on

$$\{(\zeta[w], w, h) \in Y \times L^2(0,T)^m \times \mathbb{R}^m\};$$

(iv) the uniform positivity: there exists $\rho > 0$ with

$$\max_{(p,\mathrm{d}\mu)\in\Lambda_1} \widehat{\mathcal{Q}}[p, \mathrm{d}\mu](\zeta[w], w, h) \geq \rho(\|w\|_2^2 + |h|^2), \quad \text{for all } (w,h) \in PC_2^*.$$

Then $(\bar{u}, \bar{y})$ is a Pontryagin minimum satisfying the weak quadratic growth condition.

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   Then $(\bar{u}, \bar{y})$ is a Pontryagin minimum satisfying the weak quadratic growth condition.

b) Conversely, for an admissible trajectory $(\bar{u}, y[\bar{u}])$ satisfying a (certain) quadratic growth condition, it holds

$$\max_{(p,\mathrm{d}\mu)\in\Lambda_1} \widehat{\mathcal{Q}}[p, \mathrm{d}\mu](\zeta[w], w, h) \geq \rho(\|w\|_2^2 + |h|^2), \quad \text{for all } (w,h) \in PC_2.$$

Aronna, Bonnans, K., preprint, 2019.

Summary

- Second-order analysis for semilinear parabolic equations with
  - state constraints,
  - several controls.
- Techniques:
  - alternative costates,
  - radiality,
  - Goh transformation.
- Result:
  - Second-order sufficient optimality condition with gap.

Thank you for your attention.