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Optimal Control of Two-Phase Flow

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Optimal control of two-phase flow

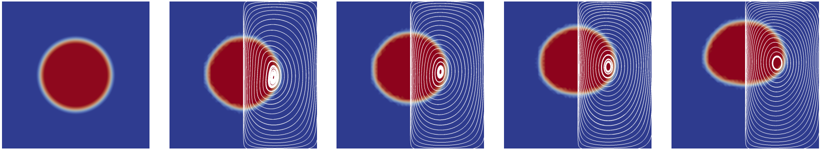

Figure: without control

Optimal control of two-phase flow

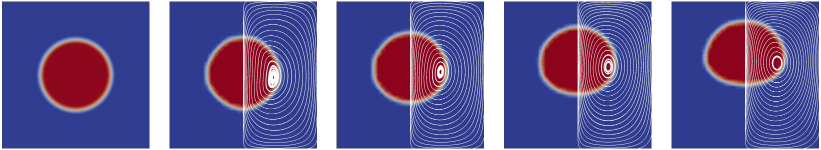

Figure: without control

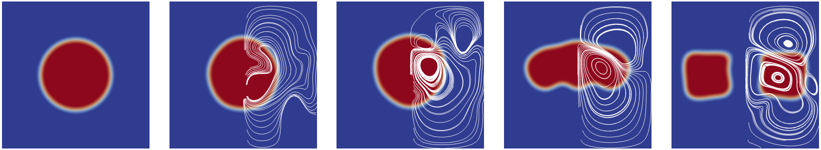

Figure: with control

Outline

Outline

Diffuse interface approach

Setting: Two subdomains Ω_1 and Ω_2 separated by unknown Γ_ϵ.

Assumption: Γ_ϵ of small thickness $\mathcal{O}(\epsilon) > 0$ and components are mixed inside.

Representation: Continuous order parameter φ for Ω_1 and Ω_2.

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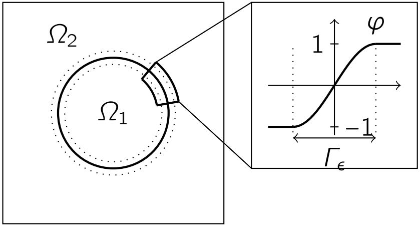

$$\varphi(x) = 1 \Leftrightarrow x \in \Omega_1$$

$$\varphi(x) = -1 \Leftrightarrow x \in \Omega_2$$

$$-1 < \varphi(x) < 1 \Leftrightarrow x \in \Gamma_\epsilon$$

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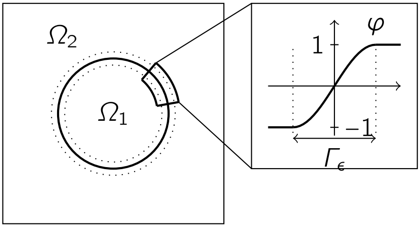

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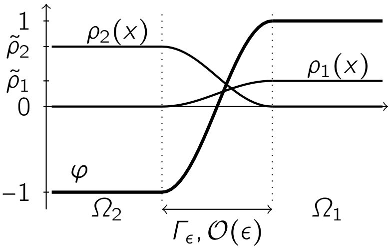
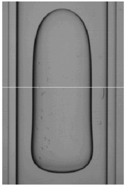


$$\varphi(x) = \frac{\rho_1(x)}{\tilde{\rho}_1} - \frac{\rho_2(x)}{\tilde{\rho}_2}$$

The two-phase flow model [Abels, Garcke, Grün, 2012]



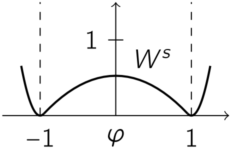
v velocity, p pressure,
φ phase field variable, μ chemical potential

$$\rho\partial_t v + ((\rho v + J)\cdot\nabla)\, v - \operatorname{div}(2\eta D v) + \nabla p = -\varphi\nabla\mu + \rho g,$$
$$\operatorname{div} v = 0,$$
$$\partial_t\varphi + v\cdot\nabla\varphi - \operatorname{div}(m\nabla\mu) = 0,$$
$$-\sigma\epsilon\Delta\varphi + \sigma\epsilon^{-1}W'(\varphi) = \mu,$$

where $2Dv = \nabla v + (\nabla v)^t$, $J = -\rho'(\varphi)m(\varphi)\nabla\mu$.

g gravity,
ϵ interfacial width,
σ surface tension,
$\sigma = c_W \sigma^{phys}$,

$\rho(\varphi)$ density,
$\eta(\varphi)$ viscosity,
$m(\varphi)$ mobility.



The free energy density W

logarithmic:

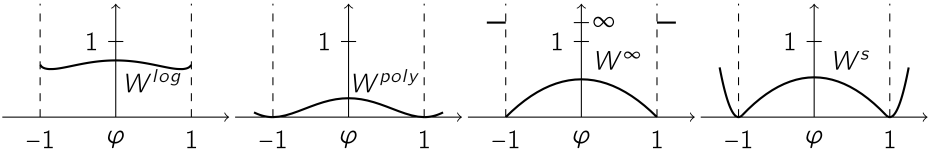
$$W^{log}(\varphi) = \frac{\theta}{2}\left((1+\varphi)\log(1+\varphi) + (1-\varphi)\log(1-\varphi)\right) + \frac{\theta_\varphi}{2}(1-\varphi^2),$$

polynomial: $W^{poly}(\varphi) = \frac{1}{4}\left(1-\varphi^2\right)^2,$

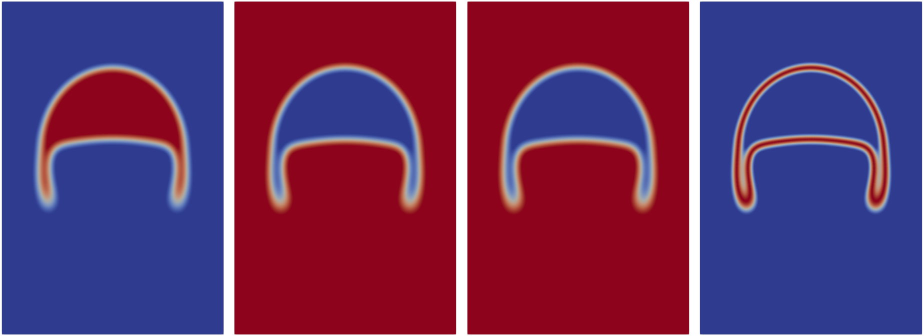
double-obstacle: $W^{\infty}(\varphi) = \frac{1}{2}\left(1-\varphi^2\right)$ iff $|\varphi| \leq 1, \quad \infty$ else,

relaxed double-obstacle:

$$W^{s}(\varphi) = \frac{1}{2}\left(1-(\xi\varphi)^2\right) + \frac{s}{2}\left(\max(0,\xi\varphi-1)^2 + \min(0,\xi\varphi+1)^2\right) + \theta.$$



Functions depending on φ



φ | $\rho(\varphi) = \frac{\rho_1+\rho_2}{2} + \frac{\rho_2-\rho_1}{2}\varphi$ | $\eta(\varphi) = \frac{\eta_1+\eta_2}{2} + \frac{\eta_2-\eta_1}{2}\varphi$ | $W(\varphi)$

The formal energy inequality

Theorem

Let v, φ, μ denote a sufficiently smooth solution (if exists) and let

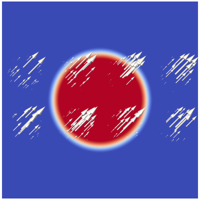
$$E(t) = \int_\Omega \frac{1}{2}\rho(t)|v(t)|^2 \,\mathrm{dx} + \sigma \int_\Omega \frac{\epsilon}{2}|\nabla\varphi(t)|^2 + \frac{1}{\epsilon}W(\varphi(t)) \,\mathrm{dx}$$

denote the energy of the system. Let $v|_{\partial\Omega} = 0$ hold.
Then it holds

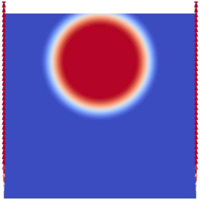
$$\frac{d}{dt}E(t) = -\int_\Omega 2\eta(\varphi)|Dv|^2 \,\mathrm{dx} - \int_\Omega m(\varphi)|\nabla\mu|^2 \,\mathrm{dx} + \int_\Omega gv \,\mathrm{dx},$$

$$E(t_2) + \int_{t_1}^{t_2}\int_\Omega m(\varphi(s))|\nabla\mu(s)|^2 \,\mathrm{dxds} + \int_{t_1}^{t_2}\int_\Omega 2\eta(\varphi(s))|Dv(s)|^2 \,\mathrm{dxds}$$
$$= E(t_1) + \int_{t_1}^{t_2}\int_\Omega gv(s) \,\mathrm{dxds}$$

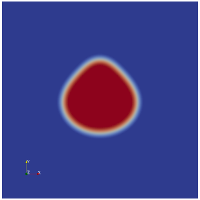
Applied Controls



$\mathcal{B}u_V = \sum_{i=1}^{s_V} f_i(x) u_V[i]$, $f_i \in L^2(\Omega)^n$



$\mathcal{B}u_B = \sum_{i=1}^{s_B} g_i(x) u_B[i]$, $g_i \in H^{1/2}(\partial\Omega)^n$



$\varphi_0 = \mathcal{B}u_I = u_I$

$$u_V \in L^2(0,T;\mathbb{R}^{s_V}) = U_V,$$
$$u_B \in L^2(0,T;\mathbb{R}^{s_B}) = U_B,$$
$$u_I \in \mathcal{K} := \{v \in H^1(\Omega) \cap L^\infty(\Omega) \,|\, |v| \le 1,\ (v,1) = const\} = U_I,$$
$$u = (u_V, u_B, u_I) \in U = U_V \times U_B \times U_I.$$

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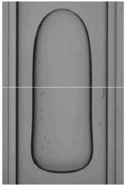
The two-phase flow model with controls

v velocity, p pressure,
φ phase field variable, μ chemical potential



$$
\begin{aligned}
\rho\partial_t v + ((\rho v + J)\cdot\nabla)\, v - \operatorname{div}(2\eta D v) + \nabla p &= -\varphi\nabla\mu + \rho g + \mathcal{B}u_V,\\
\operatorname{div} v &= 0,\\
\partial_t\varphi + v\cdot\nabla\varphi - \operatorname{div}(m\nabla\mu) &= 0,\\
-\sigma\epsilon\Delta\varphi + \sigma\epsilon^{-1}W'(\varphi) &= \mu,
\end{aligned}
$$

where $2Dv = \nabla v + (\nabla v)^t$, $J = -\rho'(\varphi)m(\varphi)\nabla\mu$,
$v|_{\partial\Omega} = \mathcal{B}u_B$, $\varphi(0) = u_I$.

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ϵ interfacial width,
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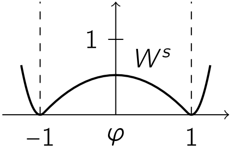

The optimal control problem

The optimal control problem

φ_d: desired distribution,

$\alpha_V + \alpha_B + \alpha_I = 1$

$$
\begin{aligned}
\min J(u_I, u_V, u_B, \varphi) := & \frac{1}{2}\|\varphi(T) - \varphi_d\|^2 \\
& + \frac{\alpha}{2}\Big(\alpha_I \int_\Omega \frac{\epsilon}{2}|\nabla u_I|^2 + \epsilon^{-1} W_u(u_I)\,\mathrm{dx} \\
& \alpha_V \|u_V\|^2_{L^2(0,T;\mathbb{R}^{s_V})} + \alpha_B \|u_B\|^2_{L^2(0,T;\mathbb{R}^{s_B})}\Big) \\
s.t. \quad & \text{two-phase fluid dynamics,} \\
& \text{i.e. } \varphi \equiv \varphi(u_V, u_B, u_I)
\end{aligned}
\qquad (\mathcal{P})
$$

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Outline

A weak formulation

Abbreviate

$$a(u,v,w) := \frac{1}{2}((u\cdot\nabla)v,w) - \frac{1}{2}((u\cdot\nabla)w,v)$$

The model satisfies

$$\partial_t\rho(\varphi) + \operatorname{div}(\rho(\varphi)v + J) = -\nabla\mu\cdot\nabla\rho'(\varphi)$$

If $\rho(\varphi)$ is linear (mass conservation)

$$\begin{aligned}\rho\partial_t v + ((\rho v + J)\cdot\nabla)v - \operatorname{div}(2\eta Dv) &= \mu\nabla\varphi,\\ \partial_t(\rho v) + \operatorname{div}(\rho v\otimes v) + \operatorname{div}(v\otimes J) - \operatorname{div}(2\eta Dv) &= \mu\nabla\varphi.\end{aligned}$$

Then a weak formulation is

$$\frac{1}{2}(\rho\partial_t v + \partial_t(\rho v), w) + a(\rho v + J, v, w) + 2(\eta Dv, Dw) = (\mu\nabla\varphi, w) \quad \forall w\in H_\sigma$$

An energy stable time discretization [Garcke, Hinze, K. 2016]

$u_\star^k := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} u_\star(t)\,\mathrm{dt},\ v^k|_{\partial\Omega} = \mathcal{B}u_B^k,\ \varphi^0 = u_I$

$$
\begin{aligned}
&\frac{1}{\tau}\int_\Omega \left(\frac{\rho^{k-1}+\rho^{k-2}}{2}v^k - \rho^{k-2}v^{k-1}\right)w\,\mathrm{dx}\\
&+a(\rho^{k-1}v^{k-1}+J^{k-1},v^k,w) + \int_\Omega 2\eta^{k-1}Dv^k : Dw\,\mathrm{dx}\\
&+\int_\Omega \varphi^{k-1}\nabla\mu^k\cdot w\,\mathrm{dx} - \int_\Omega \rho^{k-1}g\cdot w\,\mathrm{dx} - \int_\Omega \mathcal{B}u_V^k w\,\mathrm{dx} = 0\,\forall w\in H_\sigma(\Omega),\\
&\frac{1}{\tau}\int_\Omega(\varphi^k-\varphi^{k-1})\Psi\,\mathrm{dx} - \int_\Omega \varphi^{k-1}v^k\cdot\nabla\Psi\,\mathrm{dx}\\
&\qquad+\int_\Omega m\nabla\mu^k\cdot\nabla\Psi\,\mathrm{dx} = 0\,\forall\Psi\in H^1(\Omega),\\
&\sigma\epsilon\int_\Omega \nabla\varphi^k\cdot\nabla\Phi\,\mathrm{dx} - \int_\Omega \mu^k\Phi\,\mathrm{dx}\\
&+\frac{\sigma}{\epsilon}\int_\Omega\left((W_+)'(\varphi^k)+(W_-)'(\varphi^{k-1})\right)\Phi\,\mathrm{dx} = 0\,\forall\Phi\in H^1(\Omega).
\end{aligned}
\qquad (\mathrm{CHNS}_\tau)
$$

Energy inequality

Theorem

Let $k \geq 2$, φ^k, μ^k, v^k be a solution to (CHNS_τ), *and* $u_B \equiv 0$.
Then the following energy inequality holds

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \rho^{k-1}\left|v^k\right|^2 \,\mathrm{dx} + \sigma \int_\Omega \frac{\epsilon}{2}|\nabla\varphi^k|^2 + \frac{1}{\epsilon}W(\varphi^k)\,\mathrm{dx}\\
&+\frac{1}{2}\int_\Omega \rho^{k-2}|v^k - v^{k-1}|^2\,\mathrm{dx} + \frac{\sigma\epsilon}{2}\int_\Omega |\nabla\varphi^k - \nabla\varphi^{k-1}|^2\,\mathrm{dx}\\
&+\tau\int_\Omega 2\eta^{k-1}|Dv^k|^2\,\mathrm{dx} + \tau\int_\Omega m|\nabla\mu^k|^2\,\mathrm{dx}\\
\leq\ &\frac{1}{2}\int_\Omega \rho^{k-2}\left|v^{k-1}\right|^2\,\mathrm{dx} + \sigma\int_\Omega \frac{\epsilon}{2}|\nabla\varphi^{k-1}|^2 + \frac{1}{\epsilon}W(\varphi^{k-1})\,\mathrm{dx}\\
&+\int_\Omega \rho^{k-1} g v^k\,\mathrm{dx} + \int_\Omega (\mathcal{B}u_V^k)v^k\,\mathrm{dx}
\end{aligned}
$$

Existence of a unique solution

Theorem

Let Ω denote a polygonally / polyhedrally bounded Lipschitz domain. Let $v^{k-1} \in H_\sigma(\Omega)$, $\varphi^{k-2} \in H^1(\Omega) \cap L^\infty(\Omega)$, $\varphi^{k-1} \in H^1(\Omega) \cap L^\infty(\Omega)$, and $\mu^{k-1} \in W^{1,3}(\Omega)$ be given data. Further let $\mathcal{B}u_V^k \in L^2(\Omega)^n$, $\mathcal{B}u_B^k \in H^{\frac{1}{2}}(\partial\Omega)$, $\mathcal{B}u_I \in H^1(\Omega) \cap L^\infty(\Omega)$ be given data.
Then there exists a weak solution $\varphi^k \in H^1(\Omega) \cap C(\overline{\Omega})$, $\mu^k \in W^{1,3}(\Omega)$, $v^k \in H_\sigma(\Omega)$ to (CHNS_τ).
Furthermore, it can be found by Newton's method.

Initialization step

For $k = 1$ we solve: $v^1|_{\partial\Omega} = \mathcal{B}u_B^1$, $\varphi^0 = u_I$

$$\frac{1}{\tau}\int_\Omega \left(\frac{\rho^1+\rho^0}{2}v^1 - \rho^0 v^0\right) w \,\mathrm{dx} + a(\rho^1 v^0 + J^1, v^1, w)$$

$$+ \int_\Omega 2\eta^1 Dv^1 : Dw \,\mathrm{dx} - \int_\Omega \mu^1 \nabla\varphi^0 w \,\mathrm{dx} - \int_\Omega \mathcal{B}u_V^1 w \,\mathrm{dx} - \int_\Omega \rho^0 g \cdot w = 0 \; \forall w \in H_\sigma$$

$$\frac{1}{\tau}\int_\Omega (\varphi^1 - \varphi^0)\Psi \,\mathrm{dx} - \int_\Omega \varphi^0 v^0 \cdot \nabla\Psi \,\mathrm{dx}$$

$$+ \int_\Omega m\nabla\mu^1 \cdot \nabla\Psi \,\mathrm{dx} = 0 \; \forall \Psi \in H^1($$

$$\sigma\epsilon \int_\Omega \nabla\varphi^1 \cdot \nabla\Phi \,\mathrm{dx} - \int_\Omega \mu^1 \Phi \,\mathrm{dx}$$

$$+ \frac{\sigma}{\epsilon}\int_\Omega ((W_+)'(\varphi^1) + (W_-)'(\varphi^0))\Phi \,\mathrm{dx} = 0 \; \forall \Phi \in H^1($$

$$(\mathrm{CHNS}_\tau^I)$$

Stability

Theorem

Let Ω denote a polygonally / polyhedrally bounded Lipschitz domain. Let $v^0 \in H_\sigma(\Omega)$, $(u_I, u_V, u_B) \in U$ be given. Then there exist sequences $(v^k)_{k=1}^K \in H_\sigma(\Omega)^K$, $(\varphi^k)_{k=1}^K \in \left(H^1(\Omega) \cap C(\overline{\Omega})\right)^K$, $(\mu^k)_{k=1}^K \in W^{1,3}(\Omega)^K$ such that (v^k, φ^k, μ^k) is the unique solution to (CHNS_τ^I) for $k = 1$ and to (CHNS_τ) for $k = 2, \ldots, K$. Moreover there holds

$$\begin{aligned}
\|(v^k)_{k=1}^K\|_{\ell^\infty(H^1(\Omega))} &\leq C\left(v^0, u_I, u_V, u_B\right),\\
\|(\varphi^k)_{k=1}^K\|_{\ell^\infty(H^1(\Omega)\cap C(\overline{\Omega}))} &\leq C\left(v^0, u_I, u_V, u_B\right),\\
\|(\mu^k)_{k=1}^K\|_{\ell^\infty(W^{1,3}(\Omega))} &\leq C\left(v^0, u_I, u_V, u_B\right).
\end{aligned}$$

Stability in stronger norms

Theorem

Let Ω be polygonally / polyhedrally bounded and convex or of class $C^{1,1}$. Let $v^0 \in H_\sigma(\Omega) \cap L^\infty(\Omega)^n$, $(u_I, u_V, u_B) \in U$ be given. Then there exist sequences $(v^k)_{k=1}^K \in H_\sigma(\Omega)^K$, $(\varphi^k)_{k=1}^K \in H^2(\Omega)^K$, $(\mu^k)_{k=1}^K \in H^2(\Omega)^K$ such that (v^k, φ^k, μ^k) is the unique solution to (CHNS_τ^I) for $k = 1$ and to (CHNS_τ) for $k = 2, \ldots, K$. Moreover there holds

$$\|(v^k)_{k=1}^K\|_{\ell^\infty(H^1(\Omega))} \le C\left(v^0, u_I, u_V, u_B\right),$$
$$\|(\varphi^k)_{k=1}^K\|_{\ell^\infty(H^2(\Omega))} \le C\left(v^0, u_I, u_V, u_B\right),$$
$$\|(\mu^k)_{k=1}^K\|_{\ell^\infty(H^2(\Omega))} \le C\left(v^0, u_I, u_V, u_B\right).$$

The optimal control problem

Theorem

Let Ω be polygonally / polyhedrally bounded and convex or of class $C^{1,1}$. The optimization problem

$$\begin{aligned}\min J(u_I, u_V, u_B, (\varphi^k)_{k=1}^K) :=& \frac{1}{2}\|\varphi^K - \varphi_d\|^2 \\ &+ \frac{\alpha}{2}\Big(\alpha_I \int_\Omega \frac{\epsilon}{2}|\nabla u_I|^2 + \epsilon^{-1} W_u(u_I)\,\mathrm{dx} \\ &\alpha_V \|u_V\|^2_{L^2(0,T;\mathbb{R}^{s_V})} + \alpha_B \|u_B\|^2_{L^2(0,T;\mathbb{R}^{s_B})}\Big)\end{aligned}$$

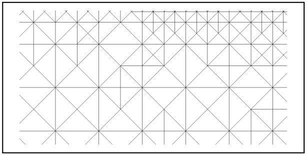
$$s.t. \quad (\mathrm{CHNS}_\tau^I) \text{ and } (\mathrm{CHNS}_\tau) \qquad (\mathcal{P}_\tau)$$

has at least one solution and first order optimality conditions can be derived by Lagrangian calculus.

Outline

Finite element approximation

$\mathcal{T}_h^k$ triangulation of $\overline{\Omega}$ at time instance t_k,



$$V_1^k := \{v \in C(\overline{\Omega}) \,|\, v|_T \in P^1 \;\forall T \in \mathcal{T}_h^k\},$$
$$V_2^k := \{v \in C(\overline{\Omega})^n \,|\, v|_T \in (P^2)^n \;\forall T \in \mathcal{T}_h^k,\, (div(v), q) = 0 \forall q \in V_1^k\},$$

$P^k : H^1(\Omega) \to V_1^k$ prolongation, e.g. H^1-prolongation.

$\varphi_h^k, \mu_h^k \in V_1^k$,
$v_h^k \in V_2^k$

The fully discrete setting

$u_\star^k := \fint_{t_{k-1}}^{t_k} u_\star(t)\,\mathrm{dt},\; v_h^k|_{\partial\Omega} = \Pi_h(\mathcal{B}u_B^k),\; \varphi_h^0 = u_I$

$$
\begin{aligned}
&\frac{1}{\tau}\int_\Omega \left(\frac{\rho_h^{k-1}+\rho_h^{k-2}}{2} v_h^k - \rho_h^{k-2} v_h^{k-1}\right) w \,\mathrm{dx}\\
&+a(\rho_h^{k-1}v_h^{k-1} + J_h^{k-1}, v_h^k, w) + \int_\Omega 2\eta_h^{k-1} D v_h^k : Dw \,\mathrm{dx}\\
&-\int_\Omega \mu_h^k \nabla\varphi_h^{k-1} w \,\mathrm{dx} - \int_\Omega \rho_h^{k-1} g\cdot w \,\mathrm{dx} - \int_\Omega (\mathcal{B}u_V^k) w \,\mathrm{dx} = 0 \;\forall w \in V_2^k,\\
&\frac{1}{\tau}\int_\Omega (\varphi_h^k - P^k\varphi_h^{k-1})\Psi \,\mathrm{dx} + \int_\Omega (v_h^k\cdot\nabla\varphi_h^{k-1})\Psi \,\mathrm{dx}\\
&+\int_\Omega m\nabla\mu_h^k\cdot\nabla\Psi \,\mathrm{dx} = 0 \;\forall \Psi\in V_1^k,\\
&\sigma\epsilon\int_\Omega \nabla\varphi_h^k\cdot\nabla\Phi \,\mathrm{dx} - \int_\Omega \mu_h^k\Phi \,\mathrm{dx}\\
&+\frac{\sigma}{\epsilon}\int_\Omega ((W_+)'(\varphi_h^k) + (W_-)'(P^k\varphi_h^{k-1}))\Phi \,\mathrm{dx} = 0 \;\forall\Phi\in V_1^k.
\end{aligned}
$$

(CHNS_h)

Energy inequality in the fully discrete setting

Theorem

Let $k \geq 2$, $\varphi_h^k, \mu_h^k, v_h^k$ be a solution to (CHNS_h), *and $u_B \equiv 0$. Then the following energy inequality holds*

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \rho_h^{k-1}\left|v_h^k\right|^2 \,\mathrm{dx} + \sigma\int_\Omega \frac{\epsilon}{2}|\nabla\varphi_h^k|^2 + \frac{1}{\epsilon}W(\varphi_h^k)\,\mathrm{dx}\\
&+\frac{1}{2}\int_\Omega \rho_h^{k-2}|v_h^k - v_h^{k-1}|^2\,\mathrm{dx} + \frac{\sigma\epsilon}{2}\int_\Omega |\nabla\varphi_h^k - \nabla P^k\varphi^{k-1_h}|^2\,\mathrm{dx}\\
&+\tau\int_\Omega 2\eta_h^{k-1}|Dv_h^k|^2\,\mathrm{dx} + \tau\int_\Omega m|\nabla\mu_h^k|^2\,\mathrm{dx}\\
\leq\ &\frac{1}{2}\int_\Omega \rho_h^{k-2}\left|v_h^{k-1}\right|^2\,\mathrm{dx} + \sigma\int_\Omega\frac{\epsilon}{2}|\nabla P^k\varphi_h^{k-1}|^2 + \frac{1}{\epsilon}W(P^k\varphi_h^{k-1})\,\mathrm{dx}\\
&+\int_\Omega \rho_h^{k-1} g v_h^k\,\mathrm{dx} + \int_\Omega (\mathcal{B}u_V^k)v_h^k\,\mathrm{dx}
\end{aligned}
$$

Stability in the fully discrete setting

Theorem

Let Ω be polygonally / polyhedrally bounded and convex.
Let $v^0 \in H_\sigma(\Omega) \cap L^\infty(\Omega)$, $u \in U$ be given. Then there exist sequences $(v_h^k)_{k=1}^K \in (V_2^k)_{k=1}^K$, $(\varphi_h^k)_{k=1}^K, (\mu_h^k)_{k=1}^K \in (V_1^k)_{k=1}^K$, such that $(v_h^k, \varphi_h^k, \mu_h^k)$ is the unique solution to (CHNS_h) for $k = 1, \ldots, K$. Moreover it holds

$$\|(v_h^k)_{k=1}^K\|_{\ell^\infty(H^1(\Omega))} \le C\left(v^0, u_I, u_V, u_B\right),$$
$$\|(\varphi_h^k)_{k=1}^K\|_{\ell^\infty(W^{1,4}(\Omega))} \le C\left(v^0, u_I, u_V, u_B\right),$$
$$\|(\mu_h^k)_{k=1}^K\|_{\ell^\infty(W^{1,3}(\Omega))} \le C\left(v^0, u_I, u_V, u_B\right).$$

The optimal control problem in the fully discrete setting

Theorem

The optimization problem

$$\begin{aligned}\min J(u_I, u_V, u_B, (\varphi_h^k)_{k=1}^K) :=& \frac{1}{2}\|\varphi_h^K - \varphi_d\|^2 \\ &+ \frac{\alpha}{2}\Big(\alpha_I \int_\Omega \frac{\epsilon}{2}|\nabla u_I|^2 + \epsilon^{-1} W_u(u_I)\,\mathrm{dx} \\ &\alpha_V \|u_V\|^2_{L^2(0,T;\mathbb{R}^{s_V})} + \alpha_B \|u_B\|^2_{L^2(0,T;\mathbb{R}^{s_B})}\Big) \\ s.t. \quad & (\mathrm{CHNS}_h)\end{aligned} \qquad (\mathcal{P}_h)$$

has at least one solution and first order optimality conditions can be derived by Lagrangian calculus.

The limit $h \to 0$

Theorem

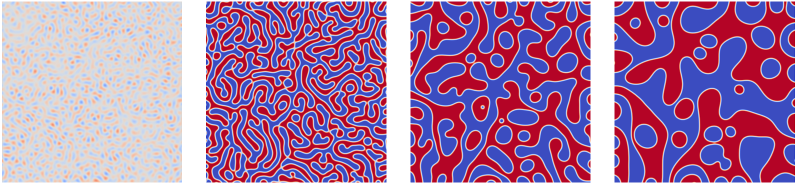
Let $(u_h^\star, v_h^\star, \varphi_h^\star, \mu_h^\star)$ denote a stationary point of $(\mathcal{P}_h)$. Then there exists a stationary point $(u^\star, v^\star, \varphi^\star, \mu^\star)$ of $(\mathcal{P}_\tau)$, such that

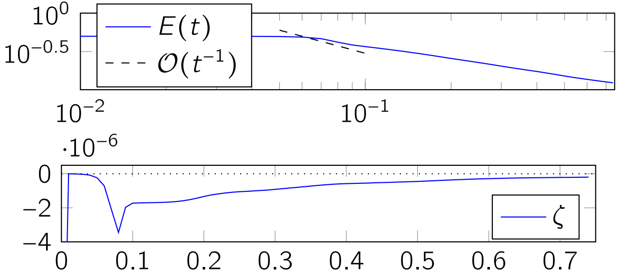
$$u_{V,h}^\star \rightharpoonup u_V^\star \in U_V, \qquad u_{B,h}^\star \rightharpoonup u_B^\star \in U_B, \qquad \varphi_h^{k,\star} \rightharpoonup \varphi^{k,\star} \in W^{1,4}(\Omega),$$

$$u_{I,h}^\star \to u_I^\star \in H^1(\Omega), \qquad \varphi_h^{k,\star} \to \varphi^{k,\star} \in H^1(\Omega),$$
$$\mu_h^{k,\star} \to \mu^{k,\star} \in W^{1,3}(\Omega), \qquad v_h^{k,\star} \to v^{k,\star} \in H_\sigma(\Omega).$$

Outline

Validity of the energy inequality





Rising Bubble, setup

Boundary control, setup from first [Hysing et al, 2009] Benchmark,
$\rho_1 = 1000$, $\rho_2 = 100$, $\eta_1 = 10$, $\eta_2 = 1$, $\sigma = 15.6$, $T = 1.0$

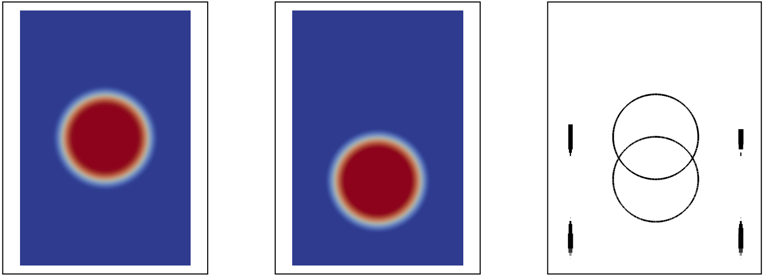
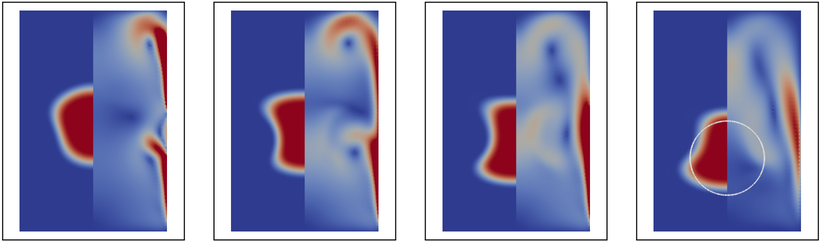
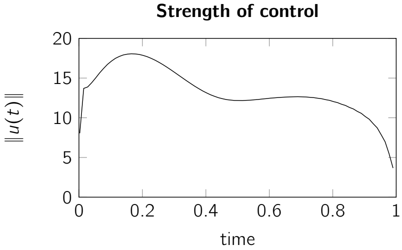
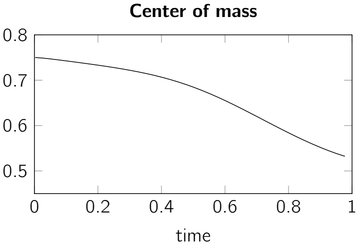


Figure: left to right: φ^0, φ_d, four Ansatzfunctions

Rising Bubble, results







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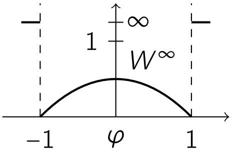
Initial value identification

Optimization with a phase field as control works best with non-smooth free energy densities.

$$W_u(\varphi) = W^\infty(\varphi) = \begin{cases} \frac{1}{2}(1-\varphi^2) & \text{if } |\varphi| \le 1, \\ \infty & \text{else.} \end{cases}$$



Results in constraint minimization problem

$$\min_{u_I \in H^1(\Omega)\cap L^\infty(\Omega), |u_I|\le 1} J(u_I)$$

Solved by VMPT [Blank, Rupprecht, SICON 2017].

Initial value problem, setup

Initial value control, setup from second [Hysing et al] Benchmark,
$\rho_1 = 1000$, $\rho_2 = 1$, $\eta_1 = 10$, $\eta_2 = 0.1$, $\sigma = 1.96$, $T = 1.0$

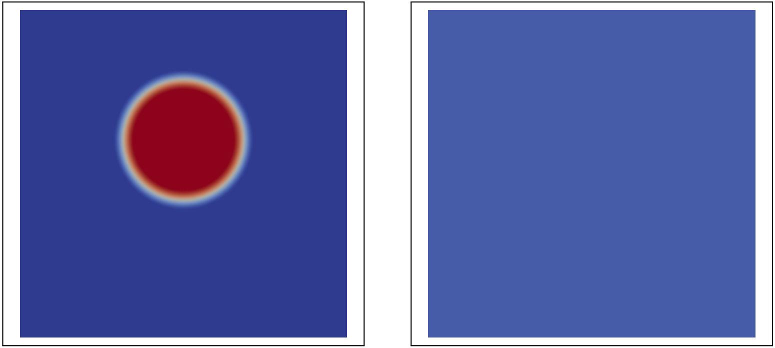


Figure: left to right: φ_d, $\varphi_0 = u_I^0 = -0.8$

Initial value problem, result

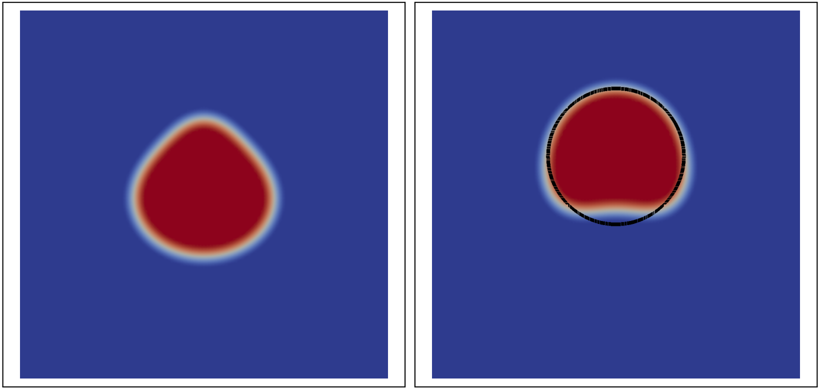


Figure: left to right: u_l^{opt}, $\varphi(u_l^{opt})$ at final time with zero level line of φ_d

Summary

- Energy stable time discretization concept for two-phase flow
  - time discrete
  - fully discrete

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Thank you for your attention.

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