UNIVERSITÄT KOBLENZ · LANDAU

Mathematisches Institut

Variational discretization of PDE constrained optimal control problems with measure controls

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joint work with Michael Hinze and Henrik Schumacher

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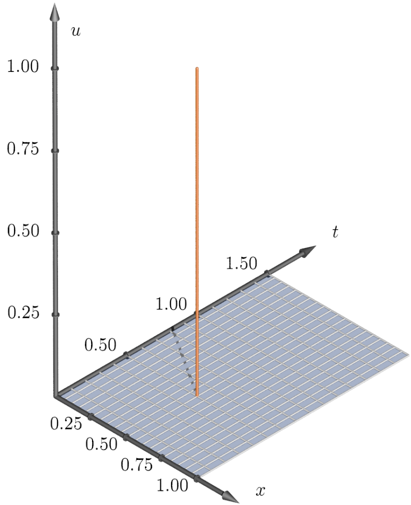
Motivation

Goal: preserve sparsity on the discrete level



Problem formulation

Parabolic control problem (from [2])

$$\min_{(u_0,u)\in\mathcal{M}(\bar{\Omega}_c)\times\mathcal{M}(\bar{Q}_c)} J(u_0,u) = \frac{1}{q}\left\|y - y_d\right\|^q_{L^q(Q)} + \alpha\left\|u\right\|_{\mathcal{M}(\bar{Q}_c)} + \beta\left\|u_0\right\|_{\mathcal{M}(\bar{\Omega}_c)}, \tag{P}$$

where y solves the state equation:

$$\begin{cases} \partial_t y - \triangle y &= u \qquad \text{in } Q = \Omega \times (0,T) \\ y(x,0) &= u_0 \qquad \text{in } \Omega \subset \mathbb{R}^d \\ y(x,t) &= 0 \qquad \text{on } \Sigma = \Gamma \times (0,T) \end{cases}$$

Theorem ([2, Theorem 2.7])

Problem (P) has at least one solution $(\bar{u},\bar{u}_0) \in \mathcal{M}(\bar{Q}_c) \times \mathcal{M}(\bar{\Omega}_c)$ for every $1 \leq q < \min\left\{2, \frac{d+2}{d}\right\}$. If $q > 1$ the solution is unique.

[2] Casas, E., & Kunisch, K. (2016)

Problem formulation

Optimality conditions

Theorem ([2, Theorem 3.1])

Let $(\bar{u}, \bar{u}_0)$ *denote a solution to (P) with associated state* $\bar{y}$. *Then there exists an element* $w \in L^2(0, T; H_0^1(\Omega)) \cap \mathcal{C}(\bar{Q})$ *satisfying*

$$\begin{cases} -\partial_t w - \triangle w & = \bar{g} \qquad \text{in } Q, \\ w(x, T) & = 0 \qquad \text{in } \Omega, \\ w(x, t) & = 0 \qquad \text{on } \Sigma, \end{cases}$$

where

$$\bar{g} \begin{cases} = |\bar{y} - y_d|^{q-2} (\bar{y} - y_d) & , \text{ if } 1 < q < \min\left\{2, \frac{d+2}{d}\right\}, \\ \in sign(\bar{y} - y_d) & , \text{ if } q = 1. \end{cases}$$

[2] Casas, E., & Kunisch, K. (2016)

Problem formulation

Optimality conditions

Theorem (continued)

w also fulfills:

$$\begin{cases} \int_{\bar{Q}_c} w \, d\bar{u} + \alpha \|\bar{u}\|_{\mathcal{M}(\bar{Q}_c)} = 0, \\ \|w\|_{\mathcal{C}(\bar{Q}_c)} \begin{cases} = \alpha & , \text{if } \bar{u} \neq 0, \\ \leq \alpha & , \text{if } \bar{u} = 0 \end{cases} \end{cases}$$

$$\begin{cases} \int_{\bar{\Omega}_c} w(0) d\bar{u}_0 + \alpha \|\bar{u}_0\|_{\mathcal{M}(\bar{\Omega}_c)} = 0, \\ \|w(0)\|_{\mathcal{C}(\bar{\Omega}_c)} \begin{cases} = \beta & , \text{if } \bar{u}_0 \neq 0, \\ \leq \beta & , \text{if } \bar{u}_0 = 0 \end{cases} \end{cases}$$

Furthermore w is unique if $q > 1$.

[2] Casas, E., & Kunisch, K. (2016)

Problem formulation

Sparsity structure

Remark ([2, Corollary 3.2])

Let $(\bar{u}, \bar{u}_0)$ denote a solution to (P) with associated state $\bar{y}$. Then we have the following sparsity structure:

$$Supp(\bar{u}^+) \subset \left\{(x,t) \in \bar{Q}_c : w(x,t) = -\alpha\right\}$$
$$Supp(\bar{u}^-) \subset \left\{(x,t) \in \bar{Q}_c : w(x,t) = +\alpha\right\}$$
$$Supp(\bar{u}_0^+) \subset \left\{x \in \bar{\Omega}_c : w(x,0) = -\beta\right\}$$
$$Supp(\bar{u}_0^-) \subset \left\{x \in \bar{\Omega}_c : w(x,0) = +\beta\right\}$$

where $\bar{u} = \bar{u}^+ - \bar{u}^-$ and $\bar{u}_0 = \bar{u}_0^+ - \bar{u}_0^-$ are the Jordan decompositions.

[2] Casas, E., & Kunisch, K. (2016)

Fenchel predual (based on [3])

- $W := \{w \in L^p(Q) : \partial_t w, \partial_x w, \partial_x^2 w \in L^p(Q), w|_{\bar{\Omega}\times\{T\}} = w|_{\Gamma\times[0,T]} = 0\}$
- $W \hookrightarrow \mathcal{C}(\bar{Q})$, $\Phi : W \to \mathcal{C}(\bar{\Omega}_c) \times \mathcal{C}(\bar{Q}_c)$ embedding and restriction, $\frac{1}{q} + \frac{1}{p} = 1$,
- $L := -(\partial_t + \Delta) : W \to L^p(Q)$ the adjoint operator of the state equation

The **Fenchel predual** of (P) is:

$$\min_{w \in W_q} K(w) := \frac{1}{p}\|Lw\|^p_{L^p(Q)} + \langle Lw, y_d\rangle_{L^p(Q),L^q(Q)} + \ell_{\alpha,\beta}(\Phi w), \qquad (P^*)$$

with $\ell_{\alpha,\beta} : \mathcal{C}(\bar{\Omega}_c) \times \mathcal{C}(\bar{Q}_c) \to \bar{\mathbb{R}}$

$$\ell_{\alpha,\beta}(f_0, f) := \begin{cases} 0, & \text{if } \|f\|_{\mathcal{C}(\bar{Q}_c)} \le \alpha \text{ and } \|f_0\|_{\mathcal{C}(\bar{\Omega}_c)} \le \beta, \\ \infty, & \text{else.} \end{cases}$$

$\longrightarrow$ existence and uniqueness (for $q > 1$) of solutions

[3] Clason, C., & Kunisch, K. (2011)

Variational discretization (based on [6])

$$\min_{(u_0,u)\in\mathcal{M}(\bar{\Omega}_c)\times\mathcal{M}(\bar{Q}_c)} J_\sigma(u_0,u) := \frac{1}{q}\|y_\sigma(u_0,u)-y_d\|^q_{L^q(Q_h)} + \alpha\|u\|_{\mathcal{M}(\bar{Q}_c)} + \beta\|u_0\|_{\mathcal{M}(\bar{\Omega}_c)} \qquad (P_\sigma)$$

Petrov-Galerkin method

- discrete state space:
  - piecewise linear and continuous finite elements in space
  - piecewise constant functions w.r.t. time
- discrete test space:
  - piecewise linear and continuous finite elements in space
  - piecewise linear and continuous functions w.r.t. time
- control space is not discretized
- yields Crank-Nicholson scheme with implicit Euler step[4],[5]

[4] Daniels, N. von, Hinze, M., & Vierling, M. (2015)
[5] Goll, C., Rannacher, R., & Wollner, W. (2015)
[6] Hinze, M. (2005)

Variational discretization

discrete optimality system delivers:

$$
\begin{aligned}
\|\bar{w}_\sigma\|_\infty &\le \alpha, \\
\|\bar{w}_{0,h}\|_\infty &\le \beta, \\
\operatorname{supp}(\bar{u}^+) &\subset \left\{(x,t) \in \bar{Q}_c : \bar{w}_\sigma(x,t) = -\alpha\right\}, \\
\operatorname{supp}(\bar{u}^-) &\subset \left\{(x,t) \in \bar{Q}_c : \bar{w}_\sigma(x,t) = +\alpha\right\}, \\
\operatorname{supp}(\bar{u}_0^+) &\subset \left\{x \in \bar{\Omega}_c : \bar{w}_{0,h}(x) = -\beta\right\}, \\
\operatorname{supp}(\bar{u}_0^-) &\subset \left\{x \in \bar{\Omega}_c : \bar{w}_{0,h}(x) = +\beta\right\}.
\end{aligned}
$$

Variational discretization

discrete optimality system delivers:

$$\begin{aligned}
&\|\bar{w}_\sigma\|_\infty \leq \alpha, \\
&\|\bar{w}_{0,h}\|_\infty \leq \beta, \\
&\operatorname{supp}(\bar{u}^+) \subset \left\{(x,t) \in \bar{Q}_c : \bar{w}_\sigma(x,t) = -\alpha\right\}, \\
&\operatorname{supp}(\bar{u}^-) \subset \left\{(x,t) \in \bar{Q}_c : \bar{w}_\sigma(x,t) = +\alpha\right\}, \\
&\operatorname{supp}(\bar{u}_0^+) \subset \left\{x \in \bar{\Omega}_c : \bar{w}_{0,h}(x) = -\beta\right\}, \\
&\operatorname{supp}(\bar{u}_0^-) \subset \left\{x \in \bar{\Omega}_c : \bar{w}_{0,h}(x) = +\beta\right\}.
\end{aligned}$$

in the generic setting:

$$\operatorname{supp}(\bar{u}) \subset \{(x_j, t_k)\} \qquad \text{and} \qquad \operatorname{supp}(\bar{u}_0) \subset \{(x_j)\}.$$

induced discrete control space: dirac measures in space **and** time $U_h \times \mathcal{U}_\sigma$

Variational discretization

$$\min_{(u_0,u)\in\mathcal{M}(\bar{\Omega}_c)\times\mathcal{M}(\bar{Q}_c)} J_\sigma(u_0,u) := \frac{1}{q}\|y_\sigma(u_0,u)-y_d\|^q_{L^q(Q_h)} + \alpha\|u\|_{\mathcal{M}(\bar{Q}_c)} + \beta\|u_0\|_{\mathcal{M}(\bar{\Omega}_c)}, \qquad (P_\sigma)$$

where $y_\sigma(u_0,u) = L_\sigma^{-*}(\Phi_h^*\Upsilon_h u_0 + \Phi_\sigma^*\Upsilon_\sigma u)$.

$(\Upsilon_h \oplus \Upsilon_\sigma) : \mathcal{M}(\bar{\Omega}_c)\times\mathcal{M}(\bar{Q}_c) \to U_h \times \mathcal{U}_\sigma, \quad (\Phi_h \oplus \Phi_\sigma)^* : U_h \times \mathcal{U}_\sigma \to \mathcal{W}_\sigma^*, \quad L_\sigma^{-*} : \mathcal{W}_\sigma^* \to \mathcal{Y}_\sigma$

Variational discretization

$$\min_{(u_0,u)\in\mathcal{M}(\bar{\Omega}_c)\times\mathcal{M}(\bar{Q}_c)} J_\sigma(u_0,u) := \frac{1}{q}\|y_\sigma(u_0,u)-y_d\|^q_{L^q(Q_h)} + \alpha\|u\|_{\mathcal{M}(\bar{Q}_c)} + \beta\|u_0\|_{\mathcal{M}(\bar{\Omega}_c)}, \qquad (P_\sigma)$$

where $y_\sigma(u_0,u) = L_\sigma^{-*}(\Phi_h^*\Upsilon_h u_0 + \Phi_\sigma^*\Upsilon_\sigma u)$.

$(\Upsilon_h \oplus \Upsilon_\sigma) : \mathcal{M}(\bar{\Omega}_c)\times\mathcal{M}(\bar{Q}_c) \to U_h \times \mathcal{U}_\sigma, \quad (\Phi_h \oplus \Phi_\sigma)^* : U_h \times \mathcal{U}_\sigma \to \mathcal{W}_\sigma^*, \quad L_\sigma^{-*} : \mathcal{W}_\sigma^* \to \mathcal{Y}_\sigma$

- multiple solutions $(\hat{u}_0,\hat{u}) \in \mathcal{M}(\bar{\Omega}_c)\times\mathcal{M}(\bar{Q}_c)$
- unique solution $(\bar{u}_{0,h},\bar{u}_\sigma) \in U_h \times \mathcal{U}_\sigma$ for $q > 1$
- $(\Upsilon_h\hat{u}_0, \Upsilon_\sigma\hat{u}) = (\bar{u}_{0,h},\bar{u}_\sigma)$ for all solutions $(\hat{u}_0,\hat{u}) \in \mathcal{M}(\bar{\Omega}_c)\times\mathcal{M}(\bar{Q}_c)$ (similar to [1])

[1] Casas, E., Clason, C., & Kunisch, K. (2013)

Variational discretization - Convergence

Theorem

Let $(\bar{u}_{0,h}, \bar{u}_\sigma)$ be the unique solution of (P_σ) belonging to $U_h \times \mathcal{U}_\sigma$, Γ of class $\mathcal{C}^{1,1}$, $1 < q < \min\{2, \frac{d+2}{d}\}$ and $\sigma = (\tau, h)$ the discretization parameter consisting of temporal (τ) and spatial (h) gridsizes .
If $\{(\bar{u}_{0,h}, \bar{u}_\sigma)\}_\sigma$ is a sequence of such solutions with associated states $\{\bar{y}_\sigma\}_\sigma$ the following convergence properties hold:

$$\lim_{|\sigma|\to 0} \|\bar{y} - \bar{y}_\sigma\|_{L^q(Q)} = 0,$$

$$(\bar{u}_{0,h}, \bar{u}_\sigma) \stackrel{*}{\rightharpoonup} (\bar{u}_0, \bar{u}) \;\; as\ |\sigma| \to 0 \;\; in\ \mathcal{M}(\bar{\Omega}_c) \times \mathcal{M}(\bar{Q}_c),$$

$$\lim_{|\sigma|\to 0} (\|\bar{u}_{0h}\|_{\mathcal{M}(\bar{\Omega}_c)}, \|\bar{u}_\sigma\|_{\mathcal{M}(\bar{Q}_c)}) = (\|\bar{u}_0\|_{\mathcal{M}(\bar{\Omega}_c)}, \|\bar{u}\|_{\mathcal{M}(\bar{Q}_c)}),$$

where $(\bar{u}_0, \bar{u})$ is the unique solution of (P_σ) and $\bar{y}$ associated state.

proof similar to the convergence proof in [2]

[2] Casas, E., & Kunisch, K. (2016)

Full Discretization (from [2])

$$\min_{(u_{0h},u_\sigma)\in\mathcal{U}_\sigma\times U_h} J_{\text{DG}}(u_{0h},u_\sigma) := \frac{1}{q}\|y_\sigma(u_{0h},u_\sigma)-y_d\|^q_{L^q(Q_h)} + \alpha\|u_\sigma\|_{\mathcal{M}(\bar{Q}_c)} + \beta\|u_{0h}\|_{\mathcal{M}(\bar{\Omega}_c)} \qquad (P_\sigma)$$

Discontinuous Galerkin method

- discrete state and test space:
  - piecewise linear and continuous finite elements in space
  - piecewise constant functions w.r.t. time
- discrete control space:
  - dirac measures concentrated in the finite element nodes in space
  - piecewise constant functions w.r.t. time
- yields Euler time stepping scheme

$\longrightarrow$ similar convergence properties

[2] Casas, E., & Kunisch, K. (2016)

Computational Results

Numerical setup

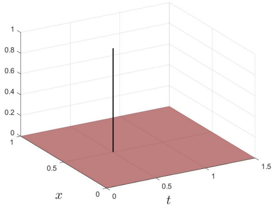
- $\Omega = (0,1)$ and $T = 1,5$
- $Q_c = (\frac{1}{4}, \frac{3}{4}) \times (\frac{1}{4}, \frac{5}{4})$
- $u_0 = 0$, $q = \frac{4}{3}$ and $p = 4$

$$\min_{u \in \mathcal{M}(\bar{Q}_c)} J_\sigma(u) = \frac{1}{q} \|y_\sigma - y_d\|^q_{L^q(Q_h)} + \alpha \|u\|_{\mathcal{M}(\bar{Q}_c)}, \qquad (P_\sigma)$$

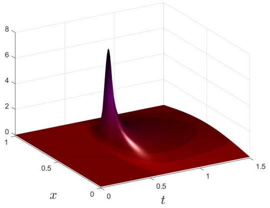
where y_σ solves the state equation:

$$\begin{cases} \partial_t y_\sigma - \triangle y_\sigma &= u & \text{in } Q = \Omega \times (0,T) \\ y_\sigma(x,0) &= 0 & \text{in } \Omega \subset \mathbb{R} \\ y_\sigma(x,t) &= 0 & \text{on } \Sigma = \Gamma \times (0,T) \end{cases}$$

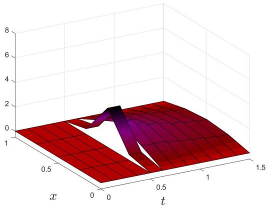
Control
$(x = 0.5, t = 0.5)$



Desired state
(solution of PDE)



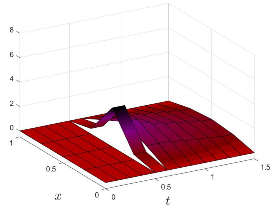
Discrete desired state
$(4 \times 12$ grid, $\tau = \frac{h}{2})$



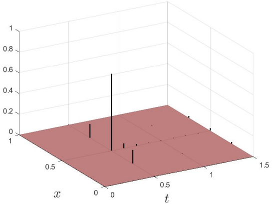
Computational Results

Source Location - Variational Discretization

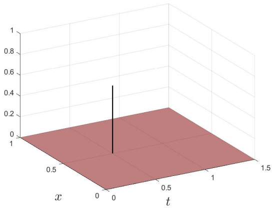
Desired state y_d



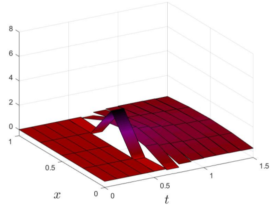
Associated control ($\alpha = 0$)



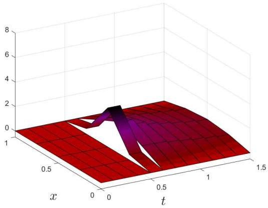
Optimal control for $\alpha = \frac{1}{3}$



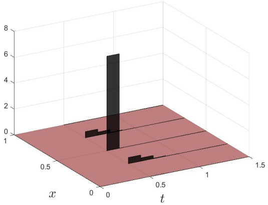
Associated state for $\alpha = \frac{1}{3}$



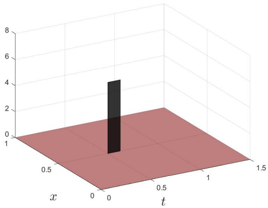
Desired state y_{d}



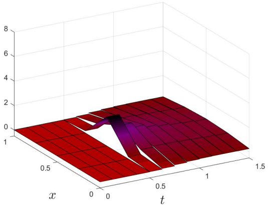
Associated control ($\alpha = 0$)



Optimal control for $\alpha = \frac{1}{3}$



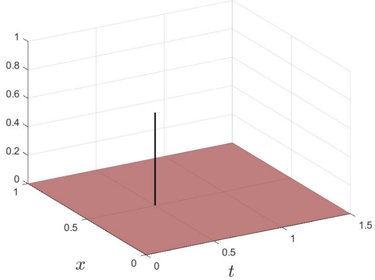
Associated state for $\alpha = \frac{1}{3}$



Computational Results

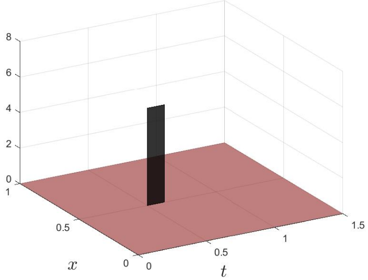
Source Location

Optimal control $\bar{u}_\sigma$ for $\alpha = \frac{1}{3}$ (variational discretization)



$\|\bar{u}_\sigma\|_{\mathcal{M}(\bar{Q}_c)} = 0.6518$ and $\text{supp}(\bar{u}_\sigma) = \{(0.5, 0.5)\}$

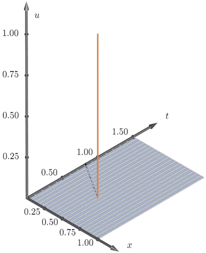
Optimal control $\bar{u}_{\text{DG}}$ for $\alpha = \frac{1}{3}$ (discontinuous Galerkin)



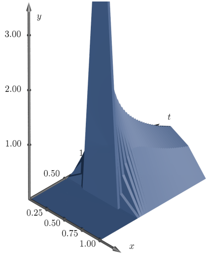
$\|\bar{u}_{\text{DG}}\|_{\mathcal{M}(\bar{Q}_c)} = 0.6845$ and $\text{supp}(\bar{u}_{\text{DG}}) = \{0.5\} \times (0.5, 0.625]$

$$\text{Fenchel duality}: \; y_d = L^{-*}\Phi^* u - |Lw|^{p-2} Lw$$

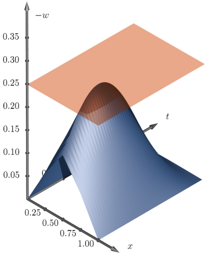
control $u = \delta_{\bar{x},\bar{t}}$



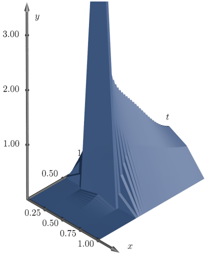
interpolated associated state $y(u) = L^{-*}\Phi^* u$



adjoint state $-w(\bar{x}, \bar{t}) = \alpha$, $|w| < \alpha$ else

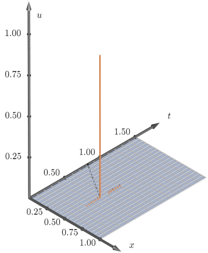


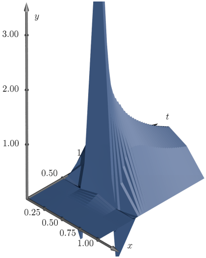
desired state y_d



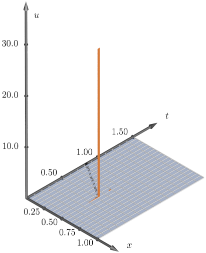
$$(\bar{x}, \bar{t}) = (0.5, 0.5), \alpha = 0.25$$

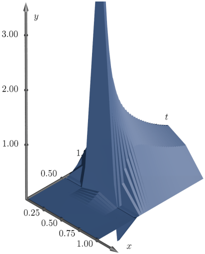
Variational discretization





Discontinuous Galerkin

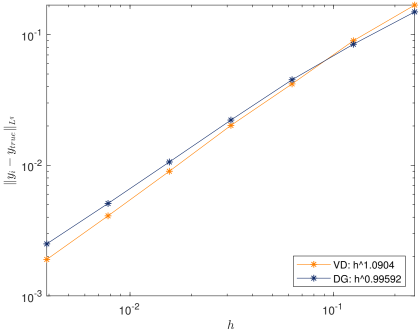




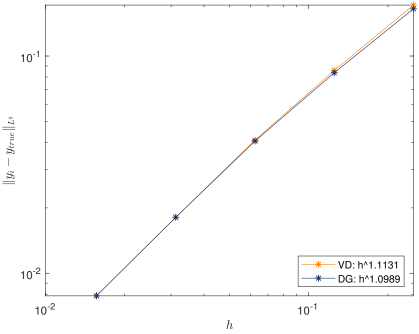
Computational Results

Convergence - state convergence

$$\lim_{|\sigma|\to 0} \|\bar{y} - \bar{y}_\sigma\|_{L^q(Q)} = 0$$

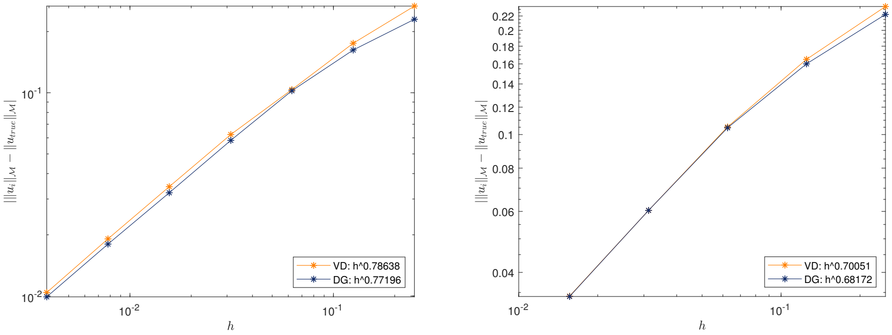


$$\tau = \frac{h}{2}$$



$$\tau = \frac{h^2}{2}$$

$$\lim_{|\sigma|\to 0} \|\bar{u}_\sigma\|_{\mathcal{M}(\bar{Q}_c)} = \|\bar{u}\|_{\mathcal{M}(\bar{Q}_c)}$$



$\tau = \frac{h}{2}$ $\tau = \frac{h^2}{2}$

References

[1] Casas, E., Clason, C., & Kunisch, K. (2013). Parabolic control problems in measure spaces with sparse solutions. SIAM Journal on Control and Optimization, 51(1), 28-63.

[2] Casas, E., & Kunisch, K. (2016). Parabolic control problems in space-time measure spaces. ESAIM: Control, Optimisation and Calculus of Variations, 22(2), 355-370.

[3] Clason, C., & Kunisch, K. (2011). A duality-based approach to elliptic control problems in non-reflexive Banach spaces. ESAIM: Control, Optimisation and Calculus of Variations, 17(1), 243-266.

[4] Daniels, N. von, Hinze, M., & Vierling, M. (2015). Crank–Nicolson Time Stepping and Variational Discretization of Control-Constrained Parabolic Optimal Control Problems. SIAM Journal on Control and Optimization, 53(3), 1182-1198.

[5] Goll, C., Rannacher, R., & Wollner, W. (2015). The damped Crank–Nicolson time-marching scheme for the adaptive solution of the Black–Scholes equation. Journal of Computational Finance, 18(4), 1-37.

[6] Hinze, M. (2005). A variational discretization concept in control constrained optimization: the linear-quadratic case. Computational Optimization and Applications, 30(1), 45-61.

- H., E., Hinze, M., & Schumacher, H. (2018). Maximal discrete sparsity in parabolic optimal control with measures. arXiv: 1804.10549