Analysis of control problems of nonmontone semilinear elliptic equations

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A joint work with Mariano Mateos (University of Oviedo, Spain) and Arnd Rösch (University of Duisburg-Essen, Germany)

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The Control Problem

$$(\mathrm{P}) \qquad \min_{u \in \mathcal{U}_{ad}} J(u) := \frac{1}{2} \int_\Omega (y_u(x) - y_d(x))^2 \, dx + \frac{\nu}{2} \int_\Omega u^2(x) \, dx \quad (\nu > 0)$$

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$$(-\infty \leq \alpha < \beta \leq +\infty)$$

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$$\begin{cases} Ay + b(x)\cdot\nabla y + f(x,y) = u \text{ in } \Omega \\ y = 0 \text{ on } \Gamma \end{cases}$$

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- $b \in L^p(\Omega)$ for some p to be fixed later

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- Fredholm Alternative

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Regularity of the Solution

Theorem. Let $b \in L^p(\Omega)^n$ with $p > 2$ if $n = 2$ and $p > 6$ if $n = 3$, $a_0 \in L^q(\Omega)$ with $q > 1$ if $n = 2$ and $q > 2$ if $n = 3$. Let $y \in H_0^1(\Omega)$ satisfy $\mathcal{A}y = u$ for some $u \in L^{\bar{p}}(\Omega)$ with $\bar{p} > \frac{n}{2}$. Then, there exist $\mu \in (0,1)$ and $C_{\mathcal{A},\mu}$ independent of u such that $y \in C^{0,\mu}(\bar{\Omega})$ and

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Theorem. Assume that $a_{ij} \in C^{0,1}(\bar{\Omega})$ for $1 \leq i, j \leq n$, $b \in L^p(\Omega)^n$ for some $p > n$, and $a_0 \in L^2(\Omega)$. We also suppose that Γ is of class $C^{1,1}$ or Ω is convex. Then, $\mathcal{A} : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is an isomorphism.

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The Adjoint Equation

Corollary. The adjoint operator $\mathcal{A}^* : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ given by

$$\mathcal{A}^*\varphi = A^*\varphi - \mathrm{div}[b(x)\varphi] + a_0(x)\varphi$$

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Proof. $\mathcal{A}^*\varphi = A^*\varphi - b(x)\nabla\varphi + (a_0(x) - \operatorname{div} b(x))\varphi$

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$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \leq K_f \Big(\|u\|_{L^{\bar{p}}(\Omega)} + \|f(\cdot,0)\|_{L^{\bar{p}}(\Omega)} + 1 \Big)$$

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A Monotonicity Result

Lemma. Under the assumptions of the above theorem, if $y_1, y_2 \in H_0^1(\Omega) \cap C(\bar{\Omega})$ are solutions of the equations

$$Ay_i + b(x) \cdot \nabla y_i + f(x, y_i) = u_i, \quad i = 1, 2,$$

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• The uniqueness of a solution of the state equation is consequence of the above lemma.

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& Ay + b(x) \cdot \nabla y = u - \phi \\
& A(y - y_k) + b(x) \cdot \nabla (y - y_k) = f_k(x, y_k) - \phi \geq 0
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$$\Rightarrow y_k \leq y \Rightarrow f_k(x, y_k) = f(x, y_k) \text{ if } k \geq \|y\|_\infty$$

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New Trends in PDE Constrained Optimization

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&\Rightarrow z_1 \le y_k \quad \forall k \ge 1 \\
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&A(z_2 - y_k) + b(x)\cdot\nabla(z_2 - y_k) = f_k(x,y_k) - f(x, -\|z_1\|_{C(\bar\Omega)}) \ge 0
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- *Step 3: The general case.*

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Continuity of $u \to y$ and regularity of y

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Theorem. Let $\{u_k\}_{k=1}^{\infty} \subset L^{\bar{p}}(\Omega)$ with $\bar{p} > \frac{n}{2}$ be a sequence weakly converging to u in $L^{\bar{p}}(\Omega)$. Then, $y_{u_k} \to y_u$ strongly in $H_0^1(\Omega) \cap C(\bar{\Omega})$, where y_{u_k} is the solution of the semilinear equation associated to u_k.

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Theorem. Suppose that assumption on f holds with $\bar{p} = 2$, $a_{ij} \in C^{0,1}(\bar{\Omega})$ for $1 \leq i, j \leq n$, and $b \in L^p(\Omega)^n$ with $p > 2$ if $n = 2$ and $p > 6$ if $n = 3$. We also suppose that Γ is of class $C^{1,1}$ or Ω is convex. Then, for every $u \in L^2(\Omega)$ the state equation has a unique solution $y_u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Existence of solution of (P)

- We recall that

$$(\mathrm{P}) \qquad \min_{u \in \mathcal{U}_{ad}} J(u) := \frac{1}{2} \int_\Omega (y_u(x) - y_d(x))^2 \, dx + \frac{\nu}{2} \int_\Omega u^2(x) \, dx \quad (\nu > 0)$$

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Theorem. The control problem (P) has at least one solution $\bar{u}$.

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- $\begin{cases} \forall M > 0 \text{ and } \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that} \\ \left|\dfrac{\partial^2 f}{\partial y^2}(x, y_2) - \dfrac{\partial^2 f}{\partial y^2}(x, y_1)\right| < \varepsilon \text{ if } |y_1|, |y_2| \leq M,\ |y_2 - y_1| \leq \delta \end{cases}$

Back
Close

Differentiability of the Mapping $u \to y_u$

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Given $\hat{p} > \frac{n}{2}$, let us denote $G : L^{\hat{p}}(\Omega) \longrightarrow Y = H_0^1(\Omega) \cap C(\bar{\Omega})$ the mapping associating with each control the state $G(u) = y_u$.

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Theorem. The control-to-state mapping G is of class C^2 and for every $u, v \in L^{\hat{p}}(\Omega)$, we have that $z_v = G'(u)v$ is the solution of

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and for $v, w \in L^{\hat{p}}(\Omega)$, $z_{v,w} = G''(u)(v, w)$ solves the equation

$$\begin{cases} Az + b(x) \cdot \nabla z + \dfrac{\partial f}{\partial y}(x, y_u)z + \dfrac{\partial^2 f}{\partial y^2}(x, y_u)z_v z_w = 0 \text{ in } \Omega \\ z = 0 \text{ on } \Gamma \end{cases}$$

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Theorem. The functional $J : L^2(\Omega) \longrightarrow \mathbb{R}$ is of class C^2. Moreover, given $u, v, v_1, v_2 \in L^2(\Omega)$ we have

$$J'(u)v = \int_\Omega (\varphi_u + \nu u) v \, dx$$

$$J''(u)(v_1, v_2) = \int_\Omega \left[1 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)\right] z_{v_1} z_{v_2} \, dx + \nu \int_\Omega v_1 v_2 \, dx$$

where $\varphi_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ is the unique solution of the adjoint equation

$$\begin{cases} A^*\varphi - \mathrm{div}[b(x)\varphi] + \dfrac{\partial f}{\partial y}(x, y_u)\varphi = y_u - y_d \text{ in } \Omega, \\ \varphi = 0 \text{ on } \Gamma. \end{cases}$$

Local Solutions

Local Solutions

Definition. We say that $\bar{u} \in \mathcal{U}_{ad}$ is an $L^r(\Omega)$-weak local minimum of (P), with $r \in [1, +\infty]$, if there exists some $\varepsilon > 0$ such that

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An element $\bar{u} \in \mathcal{U}_{ad}$ is said a strong local minimum of (P) if there exists some $\varepsilon > 0$ such that

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We say that $\bar{u} \in \mathcal{U}_{ad}$ is a strict (weak or strong) local minimum if the above inequalities are strict for $u \neq \bar{u}$.

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First Order Optimality Conditions

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First Order Optimality Conditions

Theorem. Let $\bar{u}$ be a local solution of (P) in any of the previous senses, then there exist two unique elements $\bar{y}, \bar{\varphi} \in H_0^1(\Omega) \cap C(\bar{\Omega})$ such that

$$\begin{cases} A\bar{y} + b(x) \cdot \nabla \bar{y} + f(x, \bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \Gamma, \end{cases}$$

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$$\int_\Omega (\bar{\varphi} + \nu\bar{u})(u - \bar{u})\,dx \geq 0 \quad \forall u \in \mathcal{U}_{ad}$$

Moreover, $\bar{u} \in H^1(\Omega) \cap C(\bar{\Omega})$ holds.

Second Order Conditions

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•Cone of critical directions:

$$C_{\bar{u}} = \left\{ v \in L^2(\Omega) : J'(\bar{u})v = 0 \text{ and } v(x) \left\{ \begin{array}{l} \geq 0 \text{ if } \bar{u}(x) = \alpha \\ \leq 0 \text{ if } \bar{u}(x) = \beta \end{array} \right. \right\}$$

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Theorem Let $\bar{u} \in \mathcal{U}_{ad}$. Then $J''(\bar{u})v^2 > 0 \ \forall v \in C_{\bar{u}} \setminus \{0\}$ if and only if there exists $\delta > 0$ such that $J''(\bar{u})v^2 \geq \delta \|v\|^2_{L^2(\Omega)} \ \forall v \in C_{\bar{u}}$.

Approximation of the state equation

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$$\left\{ \begin{array}{l} \text{Find } y_h \in Y_h \text{ such that} \\ a(y_h, z_h) + \displaystyle\int_\Omega f(x, y_h(x)) z_h(x)\, dx = \int_\Omega u(x) z_h(x)\, dx \ \ \forall z_h \in Y_h. \end{array} \right.$$

Numerical Analysis of the Linear Equation

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Theorem [Schatz, 1974]. Let $a_0 \in L^2(\Omega)$ be a nonnegative function. There exists $h_{\mathcal{A}} > 0$ depending on $\mathcal{A}$ and $\|a_0\|_{L^2(\Omega)}$ such that the variational problem

$$\left\{\begin{array}{l}\text{Find } y_h \in Y_h \text{ such that} \\ a(y_h, z_h) + \displaystyle\int_\Omega a_0(x) y_h(x) z_h(x)\, dx = \int_\Omega u(x) z_h(x)\, dx \quad \forall z_h \in Y_h\end{array}\right.$$

has a unique solution for every $h \le h_{\mathcal{A}}$ and for every $u \in L^2(\Omega)$. Moreover, there exists a constant $C_{\mathcal{A},a_0}$ such that

$$\|y_h\|_{H_0^1(\Omega)} \le C_{\mathcal{A},a_0} \|\mathcal{A}^{-1} u\|_{H_0^1(\Omega)} \quad \forall h \le h_{\mathcal{A}}.$$

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Theorem. Let us assume that

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Numerical Analysis of the Semilinear Equation

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$\forall M \geq 1+\|y\|_{C(\bar\Omega)}\ \exists h_M>0$ such that for every $h<h_M$ the discrete equation has a unique solution y_h satisfying $\|y_h\|_{C(\bar\Omega)}\leq M$.

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Numerical Analysis of the Semilinear Equation

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$\forall M \ge 1 + \|y\|_{C(\bar{\Omega})}$ $\exists h_M > 0$ such that for every $h < h_M$ the discrete equation has a unique solution y_h satisfying $\|y_h\|_{C(\bar{\Omega})} \le M$. Moreover, there exist constants K_M and $K_{\infty,M}$ independent of u such that

$$\|y - y_h\|_{L^2(\Omega)} + h\|y - y_h\|_{H_0^1(\Omega)} \le K_M\Big(\|u\|_{L^2(\Omega)} + 1\Big)h^2$$

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Further, if there exist other solutions $\{\tilde y_h\}_{h<h_M}$ with $y_h \ne \tilde y_h$ for all h, then $\lim_{h\to 0}\|\tilde y_h\|_{C(\bar\Omega)} = \infty$.

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A Discrete Mapping $u_h \to y_h$

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Theorem. Let $\bar{y} \in Y$ be the solution of state equation corresponding to the control $\bar{u} \in L^2(\Omega)$. Given $\rho > 0$ arbitrary, there exist $\rho^* > 0$ and $h_0 > 0$ such that the discrete equation has a unique solution $y_h(u) \in \bar{B}^Y_{\rho^*}(\bar{y})$ for every $u \in \bar{B}_\rho(\bar{u}) \subset L^2(\Omega)$ and for all $h < h_0$, where

$$B^Y_{\rho^*}(\bar{y}) = \{y \in Y : \|y - \bar{y}\|_Y \le \rho^*\}.$$

Furthermore, there exist constants K and K_∞ such that

$$\|y_u - y_h(u)\|_{L^2(\Omega)} + h\|y_u - y_h(u)\|_{H^1_0(\Omega)} \le K\Big(\|\bar{u}\|_{L^2(\Omega)} + \rho + 1\Big)h^2$$

$$\|y_u - y_h(u)\|_{L^\infty(\Omega)} \le K_\infty\Big(\|\bar{u}\|_{L^2(\Omega)} + \rho + 1\Big)h^{2-\frac{n}{2}} \quad \forall u \in \bar{B}\rho(\bar{u})$$

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Numerical Approximation of (P)

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- Let us define $\mathcal{J} : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$ given by

$$\mathcal{J}(y,u) = \frac{1}{2}\int_\Omega (y(x) - y_d(x))^2\,dx + \frac{\nu}{2}\int_\Omega u^2\,dx$$

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- Let us denote by $\mathcal{U}_h$ one of the following two spaces:

$$\mathcal{U}_h = \mathcal{U}_h^0 := \{u_h \in L^2(\Omega) : \ u_{h|T} \in P_0(T)\ \forall T \in \mathcal{T}_h\}$$

$$\mathcal{U}_h = \mathcal{U}_h^1 := \{u_h \in C(\bar{\Omega}) : \ u_{h|T} \in P_1(T)\ \forall T \in \mathcal{T}_h\}$$

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- We set $\mathcal{U}_{h,ad} = \mathcal{U}_h \cap \mathcal{U}_{ad}$.
- We approximate Problem (P) by the problem

$(\mathcal{P}_h)$ $\min\{\mathcal{J}(y_h, u_h) :\ (y_h, u_h) \in Y_h \times \mathcal{U}_{h,ad}$ satisfies the discrete equation$\}$.

Convergence of $(\mathcal{P}_h)$ to (P)

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Theorem. There exists $h_0 > 0$ such that problem $(\mathcal{P}_h)$ has at least one solution $(\bar{y}_h, \bar{u}_h)$ for all $h < h_0$. Moreover, if $\{(\bar{y}_h, \bar{u}_h)\}_{h<h_0}$ is a sequence of solutions of problems $(\mathcal{P}_h)$, then it is bounded in $H_0^1(\Omega) \times L^2(\Omega)$ and there exist subsequences converging weakly in $H_0^1(\Omega) \times L^2(\Omega)$.

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Error Estimates

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Theorem. Let $\bar{u} \in L^2(\Omega)$ be a local minimizer of (P) satisfying the sufficient second order optimality conditions and let $\{\bar{u}_h\}$ be the sequence of minimizers of the problems $(\mathcal{P}_h)$ described in the above theorem. Then, there exists $h_0 > 0$ such that

- If $\mathcal{U}_{ad} \subsetneq L^2(\Omega)$, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq Ch. \quad \forall h < h_0$$

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- If $\mathcal{U}_{ad} = L^2(\Omega)$ and $\mathcal{U}_h = \mathcal{U}_h^i$, $i = 0, 1$, then

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq Ch^{1+i} \quad \forall h < h_0$$

THANK YOU VERY MUCH FOR YOUR ATTENTION