On the relations between principal eigenvalue and torsional rigidity

Giuseppe Buttazzo
Dipartimento di Matematica
Università di Pisa
buttazzo@dm.unipi.it
http://cvgmt.sns.it

"New Trends in PDE Constrained Optimization"
RICAM (Linz), October 14–18, 2019

Research jointly made with

- Michiel van den Berg
University of Bristol, UK

- Aldo Pratelli
Università di Pisa, Italy

- Ginevra Biondi, Università di Pisa, thesis in preparation.

Our goal is to present some relations between two important quantities that arise in the study of elliptic equations. We always consider the Laplace operator $-\Delta$ with Dirichlet boundary conditions; other elliptic operator can be considered, while considering other boundary conditions (Neumann or Robin) adds to the problem severe extra difficulties, essentially due to the fact that in the Dirichlet case functions in $H^1_0(\Omega)$ can be easily extended to $\mathbf{R}^d$ while this is not in general true in the other cases.

To better understand the two quantities we deal with, let us make the following two measurements.

- Take in Ω an uniform heat source ($f = 1$), fix an initial temperature $u_0(x)$, wait a long time, and measure the average temperature in Ω.

- Consider in Ω no heat source ($f = 0$), fix an initial temperature $u_0(x)$, and measure the decay rate to zero of the temperature in Ω.

The first quantity is usually called torsional rigidity and is defined as

$$T(\Omega) = \int_\Omega u \, dx$$

where u is the solution of

$$-\Delta u = 1 \text{ in } \Omega, \qquad u \in H_0^1(\Omega).$$

In the thermal diffusion model $T(\Omega)/|\Omega|$ is the average temperature of a conducting medium Ω with uniformly distributed heat sources $(f = 1)$.

The second quantity is the first eigenvalue of the Dirichlet Laplacian

$$\lambda(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \ : \ u \in H_0^1(\Omega) \setminus \{0\} \right\}$$

In the thermal diffusion model, by the Fourier analysis,

$$u(t,x) = \sum_{k \geq 1} e^{-\lambda_k t} \langle u_0, u_k \rangle u_k(x),$$

so $\lambda(\Omega)$ represents the decay rate in time of the temperature when an initial temperature is given and no heat sources are present.

If we want, under the measure constraint $|\Omega| = m$, the highest average temperature, or the slowest decay rate, the optimal Ω is the same and is the ball of measure m. Also, it seems consistent to expect a slow (resp. fast) heat decay related to a high (resp. low) temperature. We then want to study if

$$\lambda(\Omega) \sim T^{-1}(\Omega),$$

or more generally

$$\lambda(\Omega) \sim T^{-q}(\Omega),$$

where by $A(\Omega) \sim B(\Omega)$ we mean

$$0 < c_1 \leq A(\Omega)/B(\Omega) \leq c_2 < +\infty \quad \text{for all } \Omega.$$

We further aim to study the so-called Blasche-Santaló diagram for the quantities $\lambda(\Omega)$ and $T(\Omega)$. This consists in identifying the set $E \subset \mathbf{R}^2$

$$E = \Big\{(x, y) \;:\; x = T(\Omega),\; y = \lambda(\Omega)\Big\}$$

where Ω runs among the admissible sets. In this way, minimizing a quantity like

$$F\Big(T(\Omega), \lambda(\Omega)\Big)$$

is reduced to the optimization problem in $\mathbf{R}^2$

$$\min \Big\{F(x, y) \;:\; (x, y) \in E\Big\}.$$

The difficulty consists in the fact that characterizing the set E is hard. Here we only give some bounds by studying the inf and sup of

$$\lambda^\alpha(\Omega)T^\beta(\Omega)$$

when $|\Omega| = m$.

Since the two quantities scale as:

$$T(t\Omega) = t^{d+2}T(\Omega), \qquad \lambda(t\Omega) = t^{-2}\lambda(\Omega)$$

it is not restrictive to reduce ourselves to the case $|\Omega| = 1$, which simplifies a lot the presentation.

For the relations between $T(\Omega)$ and $\lambda(\Omega)$:

- Kohler-Jobin ZAMP 1978;

- van den Berg, Buttazzo, Velichkov in Birkhäuser 2015

- van den Berg, Ferone, Nitsch, Trombetti Integral Equations Operator Theory 2016

- Lucardesi, Zucco paper in preparation;

The Blaschke-Santaló diagram has been studied for other pairs of quantities:

- for $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ by D. Bucur, G. Buttazzo, I. Figueiredo (SIAM J. Math. Anal. 1999);
- for $\lambda_1(\Omega)$ and $\text{Per}(\Omega)$ by M. Dambrine, I. Ftouhi, A. Henrot, J. Lamboley (paper in preparation).

For the inf/sup of

$$\lambda^\alpha(\Omega)T^\beta(\Omega)$$

the case $\beta = 0$ is well-known and reduces to the Faber-Krahn result (B ball with $|B| = 1$)

$$\min\left\{\lambda(\Omega)\ :\ |\Omega| = 1\right\} = \lambda(B),$$

while

$$\sup\left\{\lambda(\Omega)\ :\ |\Omega| = 1\right\} = +\infty$$

(take many small balls or a long thin rectangle).

Similarly, the case $\alpha = 0$ is also well-known through a symmetrization argument (Saint-Venant inequality):

$$\max\Big\{T(\Omega)\ :\ |\Omega| = 1\Big\} = T(B),$$

while

$$\inf\Big\{T(\Omega)\ :\ |\Omega| = 1\Big\} = 0$$

(take many small balls or a long thin rectangle).

The case when α and β have a different sign is also easy, since $T(\Omega)$ is increasing for the set inclusion, while $\lambda(\Omega)$ is decreasing.

So we can reduce the study to the case

$$\lambda(\Omega)T^q(\Omega)$$

with $q > 0$. If we want to remove the constraint $|\Omega| = 1$ the corresponding scaling free shape functional is

$$F_q(\Omega) = \frac{\lambda(\Omega)T^q(\Omega)}{|\Omega|^{(dq+2q-2)/d}}$$

that we consider on various classes of admissible domains.

We start by considering the class of all domains (with $|\Omega| = 1$). The known cases are:

- $q = 2/(d+2)$ in which the minimum of $\lambda(\Omega)T^q(\Omega)$ is reached when Ω is a ball (Kohler-Jobin ZAMP 1978);

- $q = 1$ in which (Pólya inequality)

$$0 < \lambda(\Omega)T(\Omega) < 1.$$

When $0 < q \leq 2/(d+2)$:

$$\begin{cases} \min \lambda(\Omega)T^q(\Omega) = \lambda(B)T^q(B) \\ \sup \lambda(\Omega)T^q(\Omega) = +\infty. \end{cases}$$

For the minimum

$$\begin{aligned} \lambda(\Omega)T^q(\Omega) &= \lambda(\Omega)T(\Omega)^{2/(d+2)}T(\Omega)^{q-2/(d+2)} \\ &\geq \lambda(B)T(B)^{2/(d+2)}T(B)^{q-2/(d+2)} \\ &= \lambda(B)T^q(B), \end{aligned}$$

by Kohler-Jobin and Saint-Venant inequalities.

For the sup take $\Omega = N$ disjoint small balls.

When $2/(d+2) < q < 1$:

$$\begin{cases} \inf \lambda(\Omega)T^q(\Omega) = 0 \\ \sup \lambda(\Omega)T^q(\Omega) = +\infty. \end{cases}$$

For the sup take again $\Omega = N$ disjoint balls.

For the inf take as Ω the union of a fixed ball B_R and of N disjoint balls of radius ε. We have

$$\lambda(\Omega)T^q(\Omega) = R^{-2}\lambda(B_1)T^q(B_1)\Big(R^{d+2}+N\varepsilon^{d+2}\Big)^q$$

and choosing first $\varepsilon \to 0$ and then $R \to 0$ we have that $\lambda(\Omega)T^q(\Omega)$ vanishes.

When $q = 1$:

$$\inf \lambda(\Omega)T(\Omega) = 0, \qquad \sup \lambda(\Omega)T(\Omega) = 1.$$

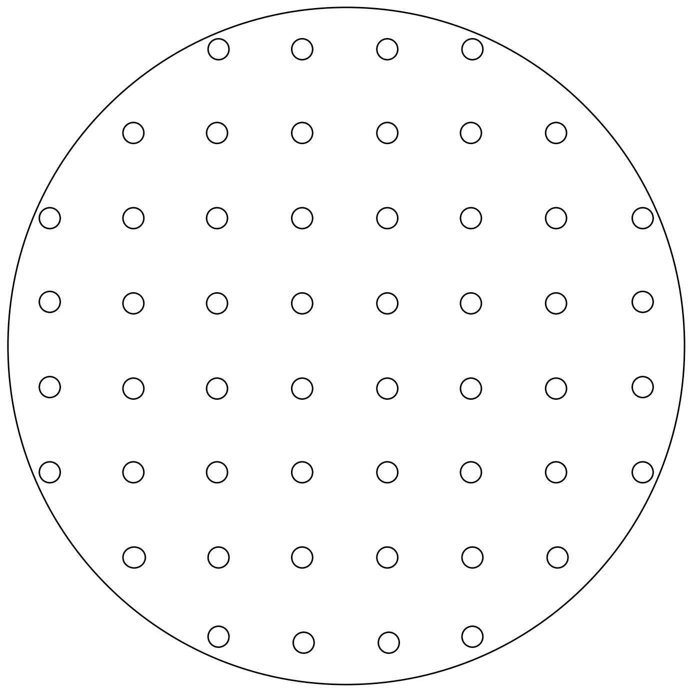
For the inf take as Ω the union of a fixed ball B_R and of N disjoint balls of radius ε, as above.

The sup equality, taking Ω a finely perforated domain, was shown by van den Berg, Ferone, Nitsch, Trombetti [Integral Equations Opera- tor Theory 2016]. A shorter proof can be given using the theory of capacitary measures.

The finely perforated domains:

$\varepsilon =$ distance between holes $\quad r_\varepsilon =$radius of a hole

$r_\varepsilon \sim \varepsilon^{d/(d-2)}$ if $d > 2$, $\quad r_\varepsilon \sim e^{-1/\varepsilon^2}$ if $d = 2$.



When $q > 1$:

$$\inf \lambda(\Omega)T^q(\Omega) = 0, \quad \sup \lambda(\Omega)T^q(\Omega) < +\infty.$$

For the inf take as Ω the union of a fixed ball B_R and of N disjoint balls of radius ε, as above.

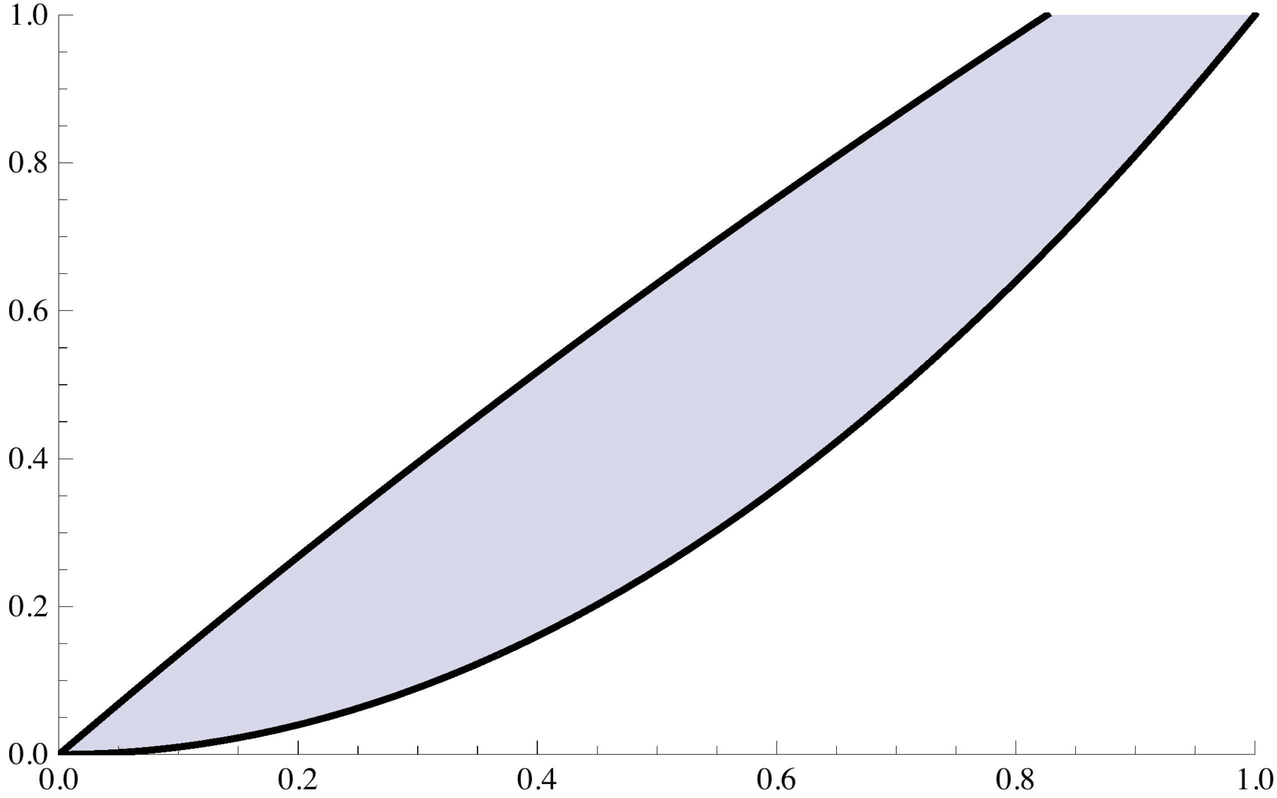
For the sup (using Pólya and Saint-Venant):

$$\begin{aligned}\lambda(\Omega)T^q(\Omega) &= \lambda(\Omega)T(\Omega)T^{q-1}(\Omega) \\ &\le T^{q-1}(\Omega) \le T^{q-1}(B)\end{aligned}$$

It would be interesting to compute explicitly $\sup F_q(\Omega)$ for $q > 1$ (is it attained?).

Summarizing: for general domais we have

	General domains Ω	
$0 < q \leq 2/(d+2)$	$\min F_q(\Omega) = F_q(B)$	$\sup F_q(\Omega) = +\infty$
$2/(d+2) < q < 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) = +\infty$
$q = 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) = 1$
$q > 1$	$\inf F_q(\Omega) = 0$	$\sup F_q(\Omega) < +\infty$



The Blaschke-Santaló diagram with $d = 2$, for $x = \lambda(B)/\lambda(\Omega)$ and $y = T(\Omega)/T(B)$ is contained in the colored region.

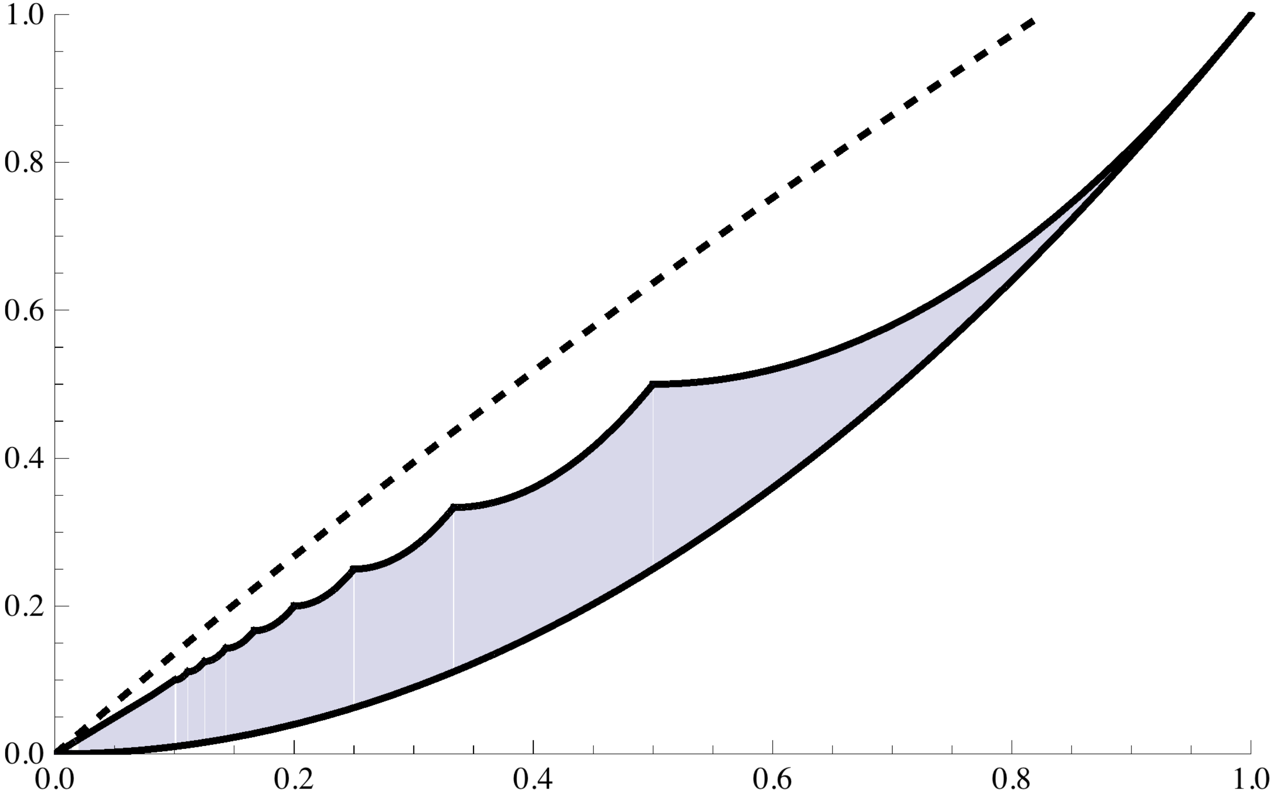
If we limit ourselves to consider only domains Ω that are union of disjoint disks of radii r_k we find

$$x = \frac{\max_k r_k^2}{\sum_k r_k^2}\,, \qquad y = \frac{\sum_k r_k^4}{\left(\sum_k r_k^2\right)^2}\,.$$

It is not difficult to show that in this case we have

$$y \le x^2[1/x] + \Big(1 - x[1/x]\Big)^2$$

where $[s]$ is the integer part of s.
In this way in the Blaschke-Santaló diagram we can reach the colored region in the picture below.



In the Blaschke-Santaló diagram with $d = 2$, the colored region can be reached by domains Ω made by union of disjoint disks.

The case $d = 1$

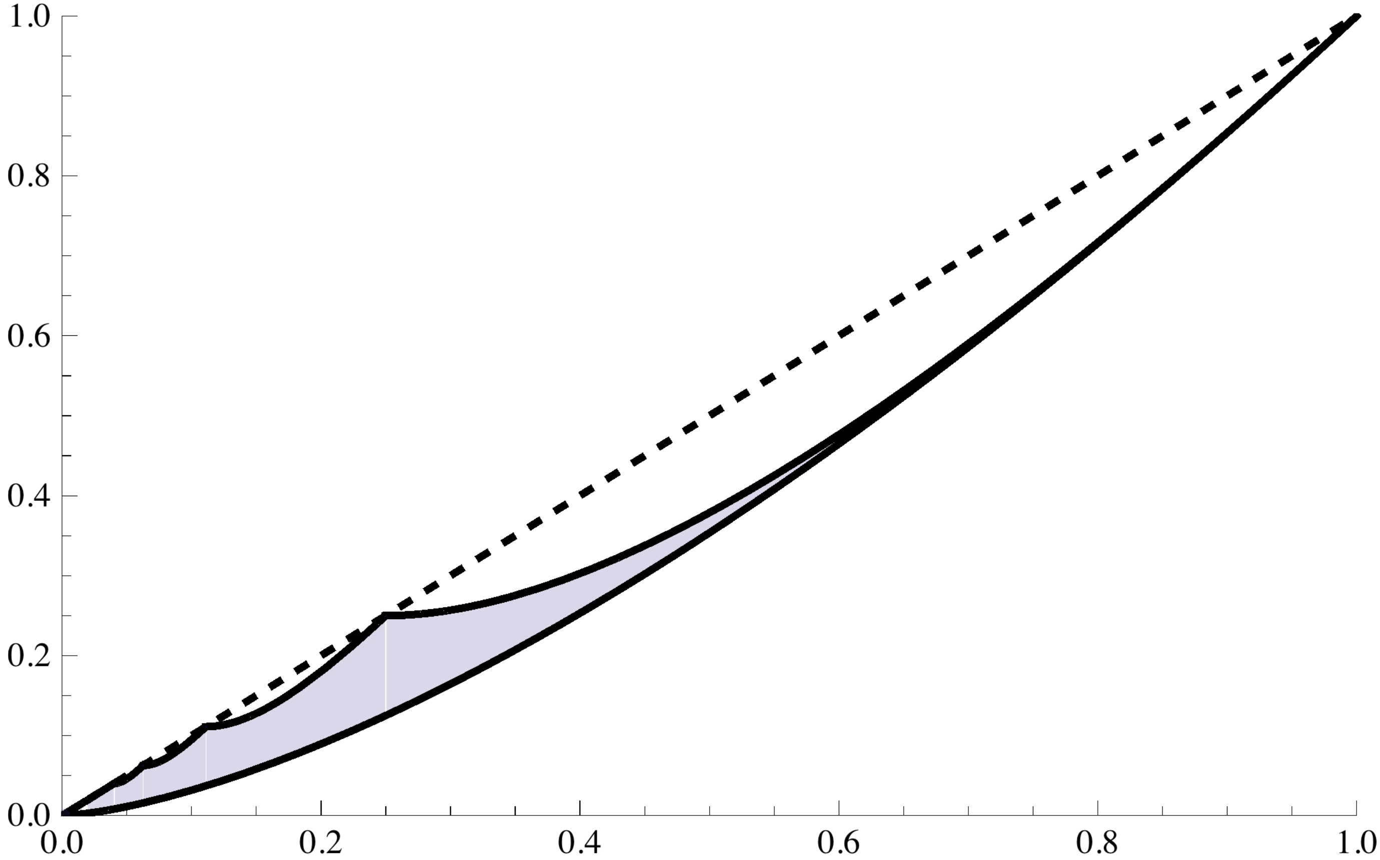
In the one-dimensional case every domain Ω is the union of disjoint intervals of half-length r_k, so that we have

$$x = \frac{\max_k r_k^2}{\left(\sum_k r_k\right)^2}\,, \qquad y = \frac{\sum_k r_k^3}{\left(\sum_k r_k\right)^3}$$

and we deduce

$$y \le x^{3/2}[x^{-1/2}] + \left(1 - x^{1/2}[x^{-1/2}]\right)^3$$

where $[s]$ is the integer part of s.



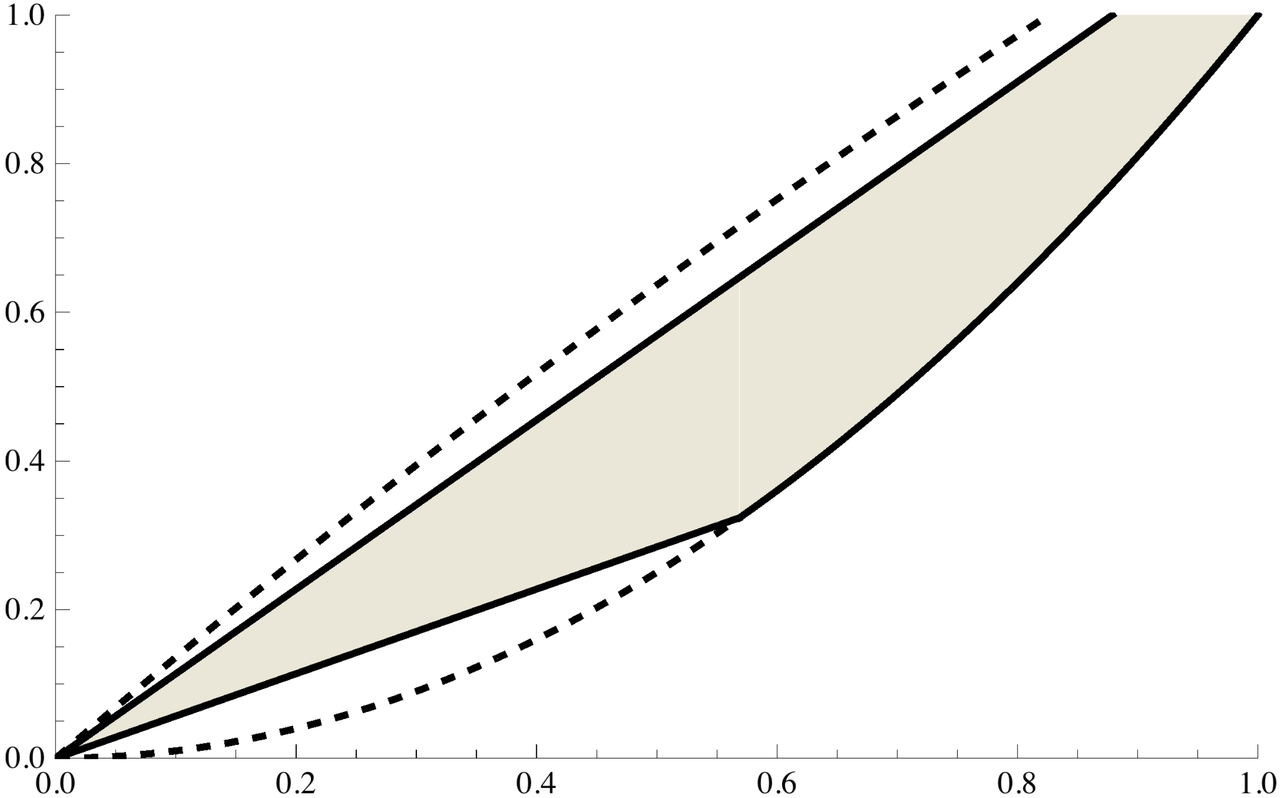
The full Blaschke-Santaló diagram in the case $d = 1$, where $x = \pi^2/\lambda(\Omega)$ and $y = 12\,T(\Omega)$.

The case Ω convex

If we consider only convex domains Ω, the Blaschke-Santaló diagram is clearly smaller. For the dimension $d = 2$ the conjecture is

$$\frac{\pi^2}{24} \leq \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \leq \frac{\pi^2}{12} \qquad \text{for all } \Omega$$

where the left side corresponds to Ω a thin triangle and the right side to Ω a thin rectangle.



If the Conjecture for convex domains is true, the Blaschke-Santaló diagram is contained in the colored region.

At present the only available inequalities are the ones of [BFNT2016]: for every $\Omega \subset \mathbf{R}^2$ convex

$$0.2056 \approx \frac{\pi^2}{48} \leq \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \leq 0.9999$$

instead of the bounds provided by the conjecture, which are

$$\begin{cases} \pi^2/24 \approx 0.4112 & \text{from below} \\ \pi^2/12 \approx 0.8225 & \text{from above.} \end{cases}$$

In dimensions $d \geq 3$ the conjecture is

$$\frac{\pi^2}{2(d+1)(d+2)} \leq \frac{\lambda(\Omega)T(\Omega)}{|\Omega|} \leq \frac{\pi^2}{12}$$

- the right side asymptotically reached by a thin slab

$$\Omega_\varepsilon = \Big\{(x', t) \ : \ 0 < t < \varepsilon\Big\}$$

with $x' \in A_\varepsilon$, being A_ε a $d-1$ dimensional ball of measure $1/\varepsilon$

- the left side asymptotically reached by a thin cone based on A_ε above and with height $d\varepsilon$.

Thin domains

We say that $\Omega_\varepsilon \subset \mathbf{R}^2$ is a thin domain if

$$\Omega_\varepsilon = \Big\{(s,t) \ : \ s \in]0,1[, \ \varepsilon h_-(s) < t < \varepsilon h_+(s)\Big\}$$

where ε is a small positive parameter and h_-, h_+ are two given (smooth) functions. We denote by $h(s)$ the local thickness

$$h(s) = h_+(s) - h_-(s)$$

and we assume that $h(s) \geq 0$.

The following asymptotics hold (as $\varepsilon \to 0$):

$$\lambda(\Omega_\varepsilon) \approx \frac{\varepsilon^{-2}\pi^2}{\|h\|^2_{L^\infty}} \quad \text{[Borisov-Freitas 2010]}$$

$$T(\Omega_\varepsilon) \approx \frac{\varepsilon^3}{12}\int h^3(s)\,ds$$

$$|\Omega_\varepsilon| \approx \varepsilon \int h(s)\,ds.$$

Hence, for a thin domain Ω_ε we have

$$\frac{\lambda(\Omega_\varepsilon)T(\Omega_\varepsilon)}{|\Omega_\varepsilon|} \approx \frac{\pi^2}{12}\frac{\int h^3(s)\,ds}{\|h\|^2_{L^\infty}\int h\,ds}\,.$$

We are able to prove the conjecture above in the class of thin domains. More precisely:

- for every h we have

$$\frac{\int h^3(s)\,ds}{\|h\|^2_{L^\infty}\int h\,ds} \le 1\ ;$$

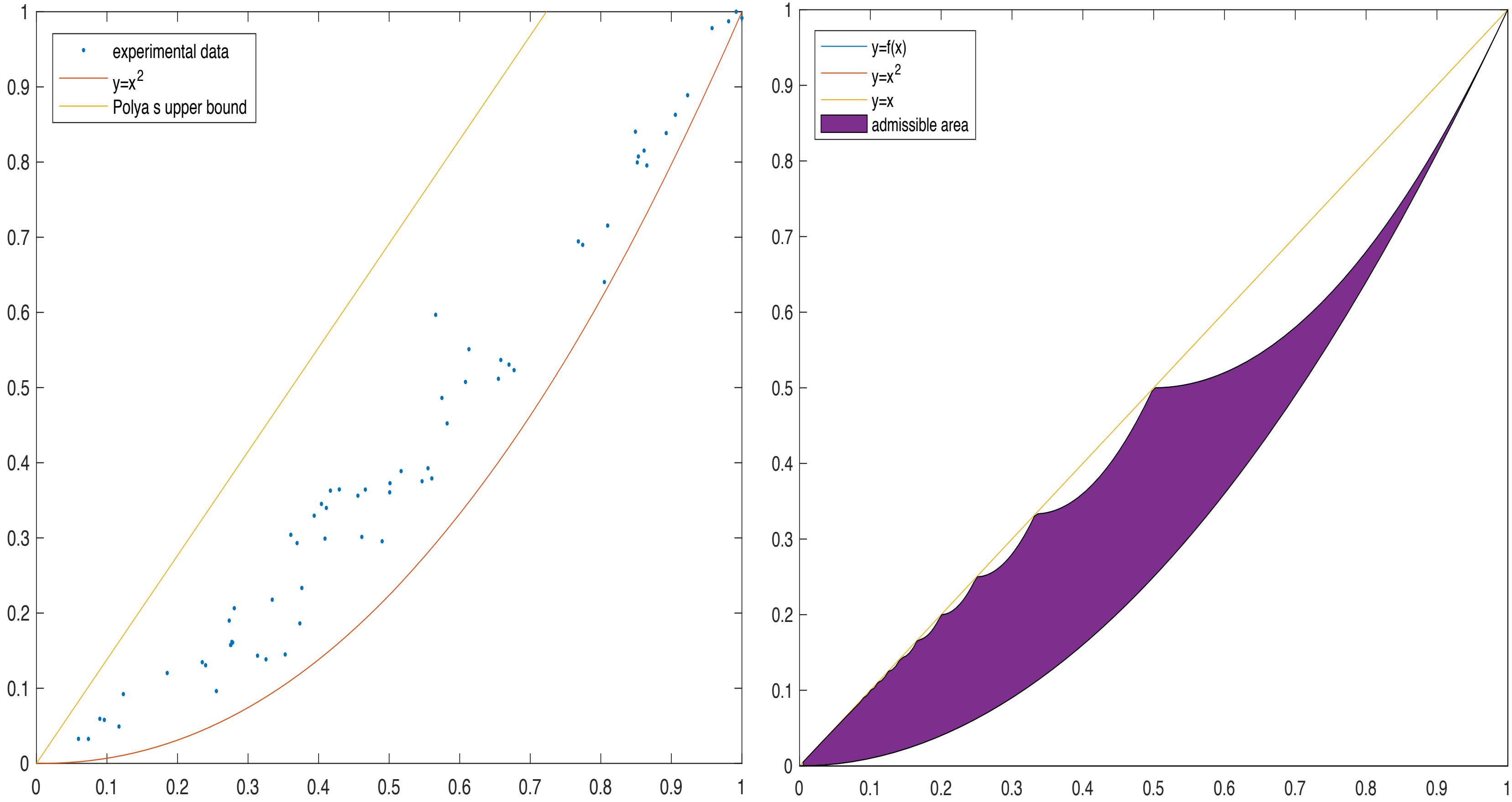
- for every h concave we have

$$\frac{\int h^3(s)\,ds}{\|h\|^2_{L^\infty}\int h\,ds} \ge \frac{1}{2}\ .$$

Hence

$$\frac{\pi^2}{24} \le \lim_{\varepsilon\to 0}\frac{\lambda(\Omega_\varepsilon)T(\Omega_\varepsilon)}{|\Omega_\varepsilon|} \le \frac{\pi^2}{12}$$

where the right inequality holds for all thin domains, while the left inequality holds for convex thin domains.



Plot of 100 experimental domains (left), union of disks (right).

Open questions

- characterize $\sup_{|\Omega|=1} \lambda(\Omega)T^q(\Omega)$ when $q > 1$;

- prove (or disprove) the conjecture for convex sets;

- simply connected domains or star-shaped domains? The bounds may change;

- full Blaschke-Santaló diagram;

- p-Laplacian instead of Laplacian?

- efficient experiments (random domains?).