INSTITUTE OF MATHEMATICS
AND SCIENTIFIC COMPUTING

Nonlinear infinite-horizon control using generalized Lyapunov equations

Tobias Breiten
Karl Kunisch (KFU, Graz), Laurent Pfeiffer (INRIA, Paris)

Workshop on "New trends in PDE constrained optimization"

Special Semester on Optimization 2019 - RICAM Linz

October 14, 2019

Linear quadratic control problems

Consider a linear system

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0 \in Y,$$
$$y_{\mathrm{obs}}(t) = Cy(t)$$

- Hilbert spaces U, Y, Z,
- A generator of an analytic C_0-semigroup e^{At} on Y,
- control operator $B \in \mathcal{L}(U, Y)$, s.t. (A, B) is stabilizable,
- output operator $C \in \mathcal{L}(Y, Z)$, s.t. (A, C) is detectable.

Let us focus on the infinite-horizon control problem

$$\min_{u \in L^2(0,\infty;U)} \mathcal{J}(y_0, u) = \int_0^\infty \frac{1}{2}\|y_{\mathrm{obs}}\|_Z^2 + \frac{\beta}{2}\|u\|_U^2 \,\mathrm{d}t.$$

Optimal feedback $\bar{u} = -\frac{1}{\beta}B^*\Pi\bar{y}$ by algebraic Riccati equation

$$A^*\Pi + \Pi A - \frac{1}{\beta}\Pi BB^*\Pi + C^*C = 0.$$

Linear quadratic control problems

Consider a linear system

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0 \in Y,$$
$$y_{\mathrm{obs}}(t) = Cy(t)$$

- Hilbert spaces U, Y, Z,
- A generator of an analytic C_0-semigroup e^{At} on Y,
- control operator $B \in \mathcal{L}(U, Y)$, s.t. (A, B) is stabilizable,
- output operator $C \in \mathcal{L}(Y, Z)$, s.t. (A, C) is detectable.

Let us focus on the infinite-horizon control problem

$$\min_{u \in L^2(0,\infty;U)} \mathcal{J}(y_0, u) = \int_0^\infty \frac{1}{2}\|y_{\mathrm{obs}}\|_Z^2 + \frac{\beta}{2}\|u\|_U^2 \,\mathrm{d}t.$$

Optimal feedback $\bar{u} = -\frac{1}{\beta}B^*\Pi\bar{y}$ by algebraic Riccati equation

$$A^*\Pi + \Pi A - \frac{1}{\beta}\Pi BB^*\Pi + C^*C = 0.$$

The ARE: feedback, optimal cost

Well-known: $\exists!$ stabilizing $\Pi = \Pi^* \succeq 0 \in \mathcal{L}(Y)$ s.t. $\forall y_1, y_2 \in \mathcal{D}(A)$:

$$\langle Ay_1, \Pi y_2 \rangle_Y + \langle \Pi y_1, Ay_2 \rangle_Y + \langle Cy_1, Cy_2 \rangle_Z - \frac{1}{\beta} \langle B^* \Pi y_1, B^* \Pi y_2 \rangle_U = 0.$$

Minimal value function $\mathcal{V}(y_0) := \min_u \mathcal{J}(y_0, u) = \frac{1}{2} \langle y_0, \Pi y_0 \rangle_Y$.

Note: for $B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$ not obvious that $B^* \Pi \in \mathcal{L}(Y, U)$

Alternative interpretation for $A_\pi := A - \frac{1}{\beta} BB^* \Pi$

$$\mathcal{T}(A_\pi y_1, y_2) + \mathcal{T}(y_1, A_\pi y_2) = \mathcal{R}(y_1, y_2), \qquad (1)$$

where

$$\mathcal{T}(y_1, y_2) := \langle y_1, \Pi y_2 \rangle_Y,$$

$$\mathcal{R}(y_1, y_2) := -\langle Cy_1, Cy_2 \rangle_Z - \frac{1}{\beta} \langle B^* \Pi y_1, B^* \Pi y_2 \rangle_U.$$

The ARE: feedback, optimal cost

Well-known: $\exists!$ stabilizing $\Pi = \Pi^* \succeq 0 \in \mathcal{L}(Y)$ s.t. $\forall y_1, y_2 \in \mathcal{D}(A)$:

$$\langle Ay_1, \Pi y_2\rangle_Y + \langle \Pi y_1, Ay_2\rangle_Y + \langle Cy_1, Cy_2\rangle_Z - \frac{1}{\beta}\langle B^*\Pi y_1, B^*\Pi y_2\rangle_U = 0.$$

Minimal value function $\mathcal{V}(y_0) := \min_u \mathcal{J}(y_0, u) = \frac{1}{2}\langle y_0, \Pi y_0\rangle_Y$.

Note: for $B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$ not obvious that $B^*\Pi \in \mathcal{L}(Y, U)$

Alternative interpretation for $A_\pi := A - \frac{1}{\beta} BB^*\Pi$

$$\mathcal{T}(A_\pi y_1, y_2) + \mathcal{T}(y_1, A_\pi y_2) = \mathcal{R}(y_1, y_2), \qquad (1)$$

where

$$\mathcal{T}(y_1, y_2) := \langle y_1, \Pi y_2\rangle_Y,$$

$$\mathcal{R}(y_1, y_2) := -\langle Cy_1, Cy_2\rangle_Z - \frac{1}{\beta}\langle B^*\Pi y_1, B^*\Pi y_2\rangle_U.$$

The ARE: feedback, optimal cost

Well-known: $\exists!$ stabilizing $\Pi = \Pi^* \succeq 0 \in \mathcal{L}(Y)$ s.t. $\forall y_1, y_2 \in \mathcal{D}(A)$:

$$\langle Ay_1, \Pi y_2\rangle_Y + \langle \Pi y_1, Ay_2\rangle_Y + \langle Cy_1, Cy_2\rangle_Z - \frac{1}{\beta}\langle B^*\Pi y_1, B^*\Pi y_2\rangle_U = 0.$$

Minimal value function $\mathcal{V}(y_0) := \min_u \mathcal{J}(y_0, u) = \frac{1}{2}\langle y_0, \Pi y_0\rangle_Y$.

Note: for $B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$ not obvious that $B^*\Pi \in \mathcal{L}(Y, U)$

Alternative interpretation for $A_\pi := A - \frac{1}{\beta}BB^*\Pi$

$$\mathcal{T}(A_\pi y_1, y_2) + \mathcal{T}(y_1, A_\pi y_2) = \mathcal{R}(y_1, y_2), \tag{1}$$

where

$$\mathcal{T}(y_1, y_2) := \langle y_1, \Pi y_2\rangle_Y,$$

$$\mathcal{R}(y_1, y_2) := -\langle Cy_1, Cy_2\rangle_Z - \frac{1}{\beta}\langle B^*\Pi y_1, B^*\Pi y_2\rangle_U.$$

The ARE: feedback, optimal cost

Well-known: $\exists!$ stabilizing $\Pi = \Pi^* \succeq 0 \in \mathcal{L}(Y)$ s.t. $\forall y_1, y_2 \in \mathcal{D}(A)$:

$$\langle Ay_1, \Pi y_2\rangle_Y + \langle \Pi y_1, Ay_2\rangle_Y + \langle Cy_1, Cy_2\rangle_Z - \frac{1}{\beta}\langle B^*\Pi y_1, B^*\Pi y_2\rangle_U = 0.$$

Minimal value function $\mathcal{V}(y_0) := \min\limits_u \mathcal{J}(y_0, u) = \frac{1}{2}\langle y_0, \Pi y_0\rangle_Y$.

Note: for $B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$ not obvious that $B^*\Pi \in \mathcal{L}(Y, U)$

Alternative interpretation for $A_\pi := A - \frac{1}{\beta}BB^*\Pi$

$$\mathcal{T}(A_\pi y_1, y_2) + \mathcal{T}(y_1, A_\pi y_2) = \mathcal{R}(y_1, y_2), \quad (1)$$

where

$$\mathcal{T}(y_1, y_2) := \langle y_1, \Pi y_2\rangle_Y,$$

$$\mathcal{R}(y_1, y_2) := -\langle Cy_1, Cy_2\rangle_Z - \frac{1}{\beta}\langle B^*\Pi y_1, B^*\Pi y_2\rangle_U.$$

Outline of this talk

Consider the multidimensional analogue of (1)

$$\sum_{i=1}^{k} \mathcal{T}_k(y_1, \ldots, y_{i-1}, A_\pi y_i, y_{i+1}, \ldots, y_k) = \mathcal{R}_k(y_1, \ldots, y_k)$$

where $(y_1, \ldots, y_k) \in \mathcal{D}(A)^k$ and $\mathcal{R}_k \in \mathcal{M}(\mathcal{D}(A)^k, \mathbb{R})$.

Questions:

- Why should we consider these equations?
- Which theoretical results do we have for these equations?
- How should we treat these equations numerically?

Outline of this talk

Consider the multidimensional analogue of (1)

$$\sum_{i=1}^{k} \mathcal{T}_k(y_1, \ldots, y_{i-1}, A_\pi y_i, y_{i+1}, \ldots, y_k) = \mathcal{R}_k(y_1, \ldots, y_k)$$

where $(y_1, \ldots, y_k) \in \mathcal{D}(A)^k$ and $\mathcal{R}_k \in \mathcal{M}(\mathcal{D}(A)^k, \mathbb{R})$.

Questions:

- Why should we consider these equations?
- Which theoretical results do we have for these equations?
- How should we treat these equations numerically?

Outline of this talk

Consider the multidimensional analogue of (1)

$$\sum_{i=1}^{k} \mathcal{T}_k(y_1, \ldots, y_{i-1}, A_\pi y_i, y_{i+1}, \ldots, y_k) = \mathcal{R}_k(y_1, \ldots, y_k)$$

where $(y_1, \ldots, y_k) \in \mathcal{D}(A)^k$ and $\mathcal{R}_k \in \mathcal{M}(\mathcal{D}(A)^k, \mathbb{R})$.

Questions:

- Why should we consider these equations?
- Which theoretical results do we have for these equations?
- How should we treat these equations numerically?

Outline of this talk

Consider the multidimensional analogue of (1)

$$\sum_{i=1}^{k} \mathcal{T}_k(y_1, \ldots, y_{i-1}, A_\pi y_i, y_{i+1}, \ldots, y_k) = \mathcal{R}_k(y_1, \ldots, y_k)$$

where $(y_1, \ldots, y_k) \in \mathcal{D}(A)^k$ and $\mathcal{R}_k \in \mathcal{M}(\mathcal{D}(A)^k, \mathbb{R})$.

Questions:

- Why should we consider these equations?
- Which theoretical results do we have for these equations?
- How should we treat these equations numerically?

Nonlinear ∞-horizon control: $\mathbb{R}^n$

Consider

$$\begin{aligned} \dot{y}(t) &= Ay(t) + f(y(t)) + Bu(t), \quad y(0) = y_0, \\ y_{\mathrm{obs}}(t) &= Cy(t), \end{aligned}$$

- $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$

Associated minimal value function

$$\mathcal{V}(y_0) = \inf_{u\in L^2(0,\infty;\mathbb{R}^m)} \frac{1}{2}\int_0^\infty \|y_{\mathrm{obs}}(t)\|^2 \,\mathrm{d}t + \frac{\beta}{2}\int_0^\infty \|u(t)\|^2 \,\mathrm{d}t.$$

If $\mathcal{V}$ sufficiently smooth, the Hamilton-Jacobi-Bellman equation

$$(Ay + f(y))^\top \nabla\mathcal{V}(y) + \frac{1}{2}\|Cy\|^2 - \frac{1}{2\beta}\|B^\top \nabla\mathcal{V}(y)\|^2 = 0$$

yields optimal feedback law $\bar{u}(\bar{y}) = -\frac{1}{\beta}B^\top\nabla\mathcal{V}(\bar{y})$.

Taylor expansions – basic idea

Idea: Expand $\mathcal{V}$ around 0 as follows

$$\mathcal{V}(y) = \mathcal{V}(0) + D\mathcal{V}(0)(y) + \frac{1}{2!}D^2\mathcal{V}(0)(y,y) + \frac{1}{3!}D^3\mathcal{V}(0)(y,y,y) + \dots$$

and approximate optimal feedback law via

$$u_d(y) = -\frac{1}{\beta}\sum_{k=2}^{d}\frac{1}{(k-1)!}B^\top D^j\mathcal{V}(0)(\cdot,y,\dots,y).$$

Finite-dimensional case:

[Aguilar, Al'brekht, Cebuhar, Costanza, Garrard, Krener, Lukes, ...]

Infinite-dimensional case:

[Thevenet/Buchot/Raymond, B./Kunisch/Pfeiffer]

Taylor expansions – basic idea

Idea: Expand $\mathcal{V}$ around 0 as follows

$$\mathcal{V}(y) = \mathcal{V}(0) + D\mathcal{V}(0)(y) + \frac{1}{2!}D^2\mathcal{V}(0)(y,y) + \frac{1}{3!}D^3\mathcal{V}(0)(y,y,y) + \dots$$

and approximate optimal feedback law via

$$u_d(y) = -\frac{1}{\beta}\sum_{k=2}^{d}\frac{1}{(k-1)!}B^\top D^j\mathcal{V}(0)(\cdot,y,\dots,y).$$

Finite-dimensional case:

[Aguilar,Al'brekht,Cebuhar,Costanza,Garrard,Krener,Lukes,...]

Infinite-dimensional case:

[Thevenet/Buchot/Raymond,B./Kunisch/Pfeiffer]

Taylor expansions – basic idea

Idea: Expand $\mathcal{V}$ around 0 as follows

$$\mathcal{V}(y) = \mathcal{V}(0) + D\mathcal{V}(0)(y) + \frac{1}{2!}D^2\mathcal{V}(0)(y,y) + \frac{1}{3!}D^3\mathcal{V}(0)(y,y,y) + \dots$$

and approximate optimal feedback law via

$$u_d(y) = -\frac{1}{\beta}\sum_{k=2}^{d}\frac{1}{(k-1)!}B^\top D^j\mathcal{V}(0)(\cdot,y,\dots,y).$$

Finite-dimensional case:

[Aguilar,Al'brekht,Cebuhar,Costanza,Garrard,Krener,Lukes,...]

Infinite-dimensional case:

[Thevenet/Buchot/Raymond,B./Kunisch/Pfeiffer]

Taylor expansions – deriving the equations

Consider again

$$D\mathcal{V}(y)(Ay+f(y))+\frac{1}{2}\|Cy\|^2-\frac{1}{2\beta}\|B^\top D\mathcal{V}(y)(\cdot)\|^2=0,$$

$\Rightarrow$ **one differentiation** in direction $z_1 \in \mathcal{D}(A)$ yields

$$D^2\mathcal{V}(y)(Ay+f(y),z_1)+D\mathcal{V}(y)(Az_1+Df(y)(z_1))$$
$$+\langle Cy,Cz_1\rangle-\frac{1}{\beta}\langle B^\top D^2\mathcal{V}(y)(\cdot,z_1),B^\top D\mathcal{V}(y)(\cdot)\rangle=0.$$

$\Rightarrow$ **two differentiations** in directions $z_1,z_2 \in \mathcal{D}(A)$ yield

$$D^3\mathcal{V}(y)(Ay+f(y),z_1,z_2)+D^2\mathcal{V}(y)(Az_2+Df(y)(z_2),z_1)$$
$$+D^2\mathcal{V}(y)(Az_1+Df(y)(z_1),z_2)+D\mathcal{V}(y)(D^2f(y)(z_1,z_2))$$
$$+\langle Cz_1,Cz_2\rangle-\frac{1}{\beta}\langle B^\top D^3\mathcal{V}(y)(\cdot,z_1,z_2),B^\top D\mathcal{V}(y)(\cdot)\rangle$$
$$-\frac{1}{\beta}\langle B^\top D^2\mathcal{V}(y)(\cdot,z_1),B^\top D^2\mathcal{V}(y)(\cdot,z_2)\rangle=0.$$

Taylor expansions – deriving the equations

Consider again

$$D\mathcal{V}(y)(Ay+f(y))+\frac{1}{2}\|Cy\|^2-\frac{1}{2\beta}\|B^\top D\mathcal{V}(y)(\cdot)\|^2=0,$$

$\Rightarrow$ **one differentiation** in direction $z_1 \in \mathcal{D}(A)$ yields

$$D^2\mathcal{V}(y)(Ay+f(y),z_1)+D\mathcal{V}(y)(Az_1+Df(y)(z_1))$$
$$+\langle Cy, Cz_1\rangle-\frac{1}{\beta}\langle B^\top D^2\mathcal{V}(y)(\cdot,z_1), B^\top D\mathcal{V}(y)(\cdot)\rangle=0.$$

$\Rightarrow$ **two differentiations** in directions $z_1, z_2 \in \mathcal{D}(A)$ yield

$$D^3\mathcal{V}(y)(Ay+f(y),z_1,z_2)+D^2\mathcal{V}(y)(Az_2+Df(y)(z_2),z_1)$$
$$+D^2\mathcal{V}(y)(Az_1+Df(y)(z_1),z_2)+D\mathcal{V}(y)(D^2f(y)(z_1,z_2))$$
$$+\langle Cz_1, Cz_2\rangle-\frac{1}{\beta}\langle B^\top D^3\mathcal{V}(y)(\cdot,z_1,z_2), B^\top D\mathcal{V}(y)(\cdot)\rangle$$
$$-\frac{1}{\beta}\langle B^\top D^2\mathcal{V}(y)(\cdot,z_1), B^\top D^2\mathcal{V}(y)(\cdot,z_2)\rangle=0.$$

Taylor expansions – deriving the equations

Consider again

$$D\mathcal{V}(y)(Ay + f(y)) + \frac{1}{2}\|Cy\|^2 - \frac{1}{2\beta}\|B^\top D\mathcal{V}(y)(\cdot)\|^2 = 0,$$

$\Rightarrow$ **one differentiation** in direction $z_1 \in \mathcal{D}(A)$ yields

$$D^2\mathcal{V}(y)(Ay + f(y), z_1) + D\mathcal{V}(y)(Az_1 + Df(y)(z_1))$$
$$+ \langle Cy, Cz_1 \rangle - \frac{1}{\beta}\langle B^\top D^2\mathcal{V}(y)(\cdot, z_1), B^\top D\mathcal{V}(y)(\cdot)\rangle = 0.$$

$\Rightarrow$ **two differentiations** in directions $z_1, z_2 \in \mathcal{D}(A)$ yield

$$D^3\mathcal{V}(y)(Ay + f(y), z_1, z_2) + D^2\mathcal{V}(y)(Az_2 + Df(y)(z_2), z_1)$$
$$+D^2\mathcal{V}(y)(Az_1 + Df(y)(z_1), z_2) + D\mathcal{V}(y)(D^2f(y)(z_1, z_2))$$
$$+ \langle Cz_1, Cz_2 \rangle - \frac{1}{\beta}\langle B^\top D^3\mathcal{V}(y)(\cdot, z_1, z_2), B^\top D\mathcal{V}(y)(\cdot)\rangle$$
$$- \frac{1}{\beta}\langle B^\top D^2\mathcal{V}(y)(\cdot, z_1), B^\top D^2\mathcal{V}(y)(\cdot, z_2)\rangle = 0.$$

Taylor expansions – deriving the equations

Consider again

$$D\mathcal{V}(y)(Ay + f(y)) + \frac{1}{2}\|Cy\|^2 - \frac{1}{2\beta}\|B^\top D\mathcal{V}(y)(\cdot)\|^2 = 0,$$

$\Rightarrow$ **one differentiation** in direction $z_1 \in \mathcal{D}(A)$ yields

$$D^2\mathcal{V}(0)(A0 + f(0), z_1) + D\mathcal{V}(0)(Az_1 + Df(0)(z_1)) + \langle C0, Cz_1\rangle - \frac{1}{\beta}\langle B^\top D^2\mathcal{V}(0)(\cdot, z_1), B^\top D\mathcal{V}(0)(\cdot)\rangle = 0.$$

$\Rightarrow$ **two differentiations** in directions $z_1, z_2 \in \mathcal{D}(A)$ yield

$$\cancel{D^3\mathcal{V}(0)(A0 + f(0), z_1, z_2)} + D^2\mathcal{V}(0)(Az_2 + Df(0)(z_2), z_1) + D^2\mathcal{V}(0)(Az_1 + Df(0)(z_1), z_2) + \cancel{D\mathcal{V}(0)(D^2 f(0)(z_1, z_2))} + \langle Cz_1, Cz_2\rangle - \cancel{\frac{1}{\beta}\langle B^\top D^3\mathcal{V}(0)(\cdot, z_1, z_2), B^\top D\mathcal{V}(0)(\cdot)\rangle} - \frac{1}{\beta}\langle B^\top D^2\mathcal{V}(0)(\cdot, z_1), B^\top D^2\mathcal{V}(0)(\cdot, z_2)\rangle = 0.$$

Taylor expansions – deriving the equations

Consider again

$$D\mathcal{V}(y)(Ay+f(y)) + \frac{1}{2}\|Cy\|^2 - \frac{1}{2\beta}\|B^\top D\mathcal{V}(y)(\cdot)\|^2 = 0,$$

$\Rightarrow$ **three differentiations** and evaluation in zero yields

$$\begin{aligned}0 = {} & D^3\mathcal{V}(0)(Az_3 + Df(0)(z_3), z_1, z_2) - \frac{1}{\beta}\langle B^\top D^3\mathcal{V}(0)(\cdot, z_1, z_2), B^\top D^2\mathcal{V}(0)(\cdot, z_3)\rangle \\ & + D^3\mathcal{V}(0)(Az_2 + Df(0)(z_2), z_1, z_3) - \frac{1}{\beta}\langle B^\top D^3\mathcal{V}(0)(\cdot, z_1, z_3), B^\top D^2\mathcal{V}(0)(\cdot, z_2)\rangle \\ & + D^3\mathcal{V}(0)(Az_1 + Df(0)(z_1), z_2, z_3) - \frac{1}{\beta}\langle B^\top D^3\mathcal{V}(0)(\cdot, z_2, z_3), B^\top D^2\mathcal{V}(0)(\cdot, z_1)\rangle \\ & + D^2\mathcal{V}(0)(D^2f(0)(z_2, z_3), z_1) + D^2\mathcal{V}(0)(D^2f(0)(z_1, z_3), z_2) \\ & \quad + D^2\mathcal{V}(0)(D^2f(0)(z_1, z_2), z_3)\end{aligned}$$

Recall: $\sum_{i=1}^{k} \mathcal{T}_k(y_1, \ldots, y_{i-1}, A_\pi y_i, y_{i+1}, \ldots, y_k) = \mathcal{R}_k(y_1, \ldots, y_k)$

Taylor expansions – deriving the equations

Consider again

$$D\mathcal{V}(y)(Ay+f(y))+\frac{1}{2}\|Cy\|^2-\frac{1}{2\beta}\|B^\top D\mathcal{V}(y)(\cdot)\|^2=0,$$

$\Rightarrow$ **three differentiations** and evaluation in zero yields

$$\begin{aligned}0 &= D^3\mathcal{V}(0)(Az_3+Df(0)(z_3),z_1,z_2)-\frac{1}{\beta}\langle B^\top D^3\mathcal{V}(0)(\cdot,z_1,z_2),B^\top D^2\mathcal{V}(0)(\cdot,z_3)\rangle\\ &\quad + D^3\mathcal{V}(0)(Az_2+Df(0)(z_2),z_1,z_3)-\frac{1}{\beta}\langle B^\top D^3\mathcal{V}(0)(\cdot,z_1,z_3),B^\top D^2\mathcal{V}(0)(\cdot,z_2)\rangle\\ &\quad + D^3\mathcal{V}(0)(Az_1+Df(0)(z_1),z_2,z_3)-\frac{1}{\beta}\langle B^\top D^3\mathcal{V}(0)(\cdot,z_2,z_3),B^\top D^2\mathcal{V}(0)(\cdot,z_1)\rangle\\ &\quad + D^2\mathcal{V}(0)(D^2f(0)(z_2,z_3),z_1)+D^2\mathcal{V}(0)(D^2f(0)(z_1,z_3),z_2)\\ &\qquad + D^2\mathcal{V}(0)(D^2f(0)(z_1,z_2),z_3)\end{aligned}$$

Recall: $\sum_{i=1}^{k}\mathcal{T}_k(y_1,\dots,y_{i-1},A_\pi y_i,y_{i+1},\dots,y_k)=\mathcal{R}_k(y_1,\dots,y_k)$

Nonlinear ∞-horizon control: Navier-Stokes

Given: $\Omega \subset \mathbb{R}^3$ with $C^{1,1}$ boundary Γ, vector valued φ, ψ

Goal: $\tilde{B} \in \mathcal{L}(\mathbb{L}^2(\omega), \mathbb{L}^2(\Omega))$, find control u s.t. solution $(\mathbf{z}, q)$ of

$$
\begin{aligned}
\frac{\partial \mathbf{z}}{\partial t} &= \nu \Delta \mathbf{z} - (\mathbf{z} \cdot \nabla)\mathbf{z} - \nabla q + \varphi + \tilde{B} u && \text{in } \Omega \times (0, T),\\
\operatorname{div} \mathbf{z} &= 0 && \text{in } \Omega \times (0, T),\\
\mathbf{z} &= \psi && \text{on } \Gamma \times (0, T),\\
\mathbf{z}(0) &= \bar{\mathbf{z}} + \mathbf{y}_0
\end{aligned}
$$

s.t. $\lim_{t \to \infty} \mathbf{z}(t) = \bar{\mathbf{z}}$ for *small* $\mathbf{y}_0$, where $(\bar{\mathbf{z}}, \bar{q})$ is a stationary solution of

$$
\begin{aligned}
-\nu \Delta \bar{\mathbf{z}} + (\bar{\mathbf{z}} \cdot \nabla)\bar{\mathbf{z}} + \nabla \bar{q} &= \varphi && \text{in } \Omega,\\
\operatorname{div} \bar{\mathbf{z}} &= 0 && \text{in } \Omega,\\
\bar{\mathbf{z}} &= \psi && \text{on } \Gamma.
\end{aligned}
$$

[Badra, Barbu, Fursikov, Lasiecka, Raymond, Takahashi, Triggiani, ...]

Navier-Stokes - state space formulation

Consider the spaces [Barbu,Boyer,Fabrie,Foias,Temam,...]

$$Y := \left\{ \mathbf{y} \in \mathbb{L}^2(\Omega) \,|\, \operatorname{div} \mathbf{y} = 0,\ \mathbf{y} \cdot \vec{n} = 0 \text{ on } \Gamma \right\},$$
$$V := \left\{ \mathbf{y} \in \mathbb{H}^1_0(\Omega) \,|\, \operatorname{div} \mathbf{y} = 0 \right\}.$$

We have the orthogonal decomposition

$$\mathbb{L}^2(\Omega) = Y \oplus \left\{ \mathbf{z} = \nabla p \,|\, p \in H^1(\Omega) \right\}.$$

Rewrite as abstract Cauchy problem

$$\dot{\mathbf{y}}(t) = A\mathbf{y} - F(\mathbf{y}) + \underbrace{P\tilde{B}}_{=B} u, \quad \mathbf{y}(0) = \mathbf{y}_0,$$

where

$$P \colon \mathbb{L}^2(\Omega) \to Y \quad \text{denotes the Leray projector,}$$

$$A\mathbf{y} = P(\nu\Delta\mathbf{y} - (\mathbf{y}\cdot\nabla)\bar{\mathbf{z}} - (\bar{\mathbf{z}}\cdot\nabla)\mathbf{y}), \quad \mathcal{D}(A) = \mathbb{H}^2(\Omega) \cap V,$$

$$F \colon \mathbb{H}^2(\Omega) \cap V \to Y, \quad F(\mathbf{y}) = P((\mathbf{y}\cdot\nabla)\mathbf{y}).$$

Navier-Stokes - state space formulation

Consider the spaces [Barbu,Boyer,Fabrie,Foias,Temam,...]

$$Y := \left\{ \mathbf{y} \in \mathbb{L}^2(\Omega) \,|\, \operatorname{div} \mathbf{y} = 0,\ \mathbf{y} \cdot \vec{n} = 0 \text{ on } \Gamma \right\},$$
$$V := \left\{ \mathbf{y} \in \mathbb{H}_0^1(\Omega) \,|\, \operatorname{div} \mathbf{y} = 0 \right\}.$$

We have the orthogonal decomposition

$$\mathbb{L}^2(\Omega) = Y \oplus \left\{ \mathbf{z} = \nabla p \,|\, p \in H^1(\Omega) \right\}.$$

Rewrite as abstract Cauchy problem

$$\dot{\mathbf{y}}(t) = A\mathbf{y} - F(\mathbf{y}) + \underbrace{P\tilde{B}}_{=B} u, \quad \mathbf{y}(0) = \mathbf{y}_0,$$

where

$$P \colon \mathbb{L}^2(\Omega) \to Y \ \text{ denotes the Leray projector},$$

$$A\mathbf{y} = P(\nu\Delta\mathbf{y} - (\mathbf{y}\cdot\nabla)\bar{\mathbf{z}} - (\bar{\mathbf{z}}\cdot\nabla)\mathbf{y}), \quad \mathcal{D}(A) = \mathbb{H}^2(\Omega) \cap V,$$

$$F \colon \mathbb{H}^2(\Omega) \cap V \to Y, \quad F(\mathbf{y}) = P((\mathbf{y}\cdot\nabla)\mathbf{y}).$$

Navier-Stokes - the infinite-horizon problem

For $A_\lambda := \lambda I - A, \lambda > 0$ sufficiently large, we introduce

$$W_\infty(\mathcal{D}(A_\lambda), Y) := \left\{ \mathbf{y} \in L^2(0,\infty;\mathcal{D}(A_\lambda)) \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} \in L^2(0,\infty;Y) \right\}$$

with

$$\|\mathbf{y}\|_{W_\infty(\mathcal{D}(A_\lambda),Y)} := \left(\|A_\lambda \mathbf{y}\|^2_{L^2(0,\infty;Y)} + \|\dot{\mathbf{y}}\|^2_{L^2(0,\infty;Y)} \right)^{\frac{1}{2}}.$$

and consider the infinite-horizon optimal control problem

$$\inf_{u \in L^2(0,\infty;U)} \mathcal{J}(\mathbf{y}_0, u) = \frac{1}{2}\int_0^\infty \|\mathbf{y}\|_Y^2 \,\mathrm{d}t + \frac{\alpha}{2}\int_0^\infty \|u(t)\|_U^2 \,\mathrm{d}t,$$

$$\text{s.t.} \quad \dot{\mathbf{y}} = A\mathbf{y} - F(\mathbf{y}) + \underbrace{P\tilde{B}}_{=B} u, \quad \mathbf{y}(0) = \mathbf{y}_0 \in V.$$

Differentiability of $\mathcal{V}$ on V

Smoothness of the value function

There exists $\delta > 0$ s.t. $\mathcal{V}$ is infinitely differentiable on $B_{\delta,V}(0)$.

Proof based on

Differentiability of $\mathcal{V}$ on V

Smoothness of the value function

There exists $\delta > 0$ s.t. $\mathcal{V}$ is infinitely differentiable on $B_{\delta,V}(0)$.

Proof based on

Optimality conditions

There exists $\delta > 0$ s.t. for all $\mathbf{y}_0 \in B_{\delta,V}(0)$, and for all solutions $(\bar{\mathbf{y}}, \bar{u})$, there exists a unique costate $\mathbf{p} \in W_\infty(Y, [\mathcal{D}(A_\lambda)]')$ satisfying

$$-\dot{\mathbf{p}} - A'\mathbf{p} - P((\bar{\mathbf{y}} \cdot \nabla)\mathbf{p} - (\nabla \bar{\mathbf{y}})^T \mathbf{p}) = \bar{\mathbf{y}} \quad (\text{in } L^2(0,\infty; [\mathcal{D}(A_\lambda)]')),$$
$$\beta \bar{u} + B^* \mathbf{p} = 0.$$

Remark: The above equation is satisfied in the sense that

$$\langle \mathbf{p}, \dot{\mathbf{z}} \rangle_{L^2(0,\infty;Y)} - \langle \mathbf{p}, A\mathbf{z} - P((\bar{\mathbf{y}} \cdot \nabla)\mathbf{z} + (\mathbf{z} \cdot \nabla)\bar{\mathbf{y}}) \rangle_{L^2(0,\infty;Y)} = \langle \bar{\mathbf{y}}, \mathbf{z} \rangle_{L^2(0,\infty;Y)},$$

for all $\mathbf{z} \in W^0_\infty(\mathcal{D}(A_\lambda), Y) = \{\mathbf{z} \in W_\infty(\mathcal{D}(A_\lambda), Y) \mid \mathbf{z}(0) = 0\}$.

Differentiability of $\mathcal{V}$ on V

Smoothness of the value function

There exists $\delta > 0$ s.t. $\mathcal{V}$ is infinitely differentiable on $B_{\delta,V}(0)$.

Proof based on

Sensitivity analysis

- $X := V \times L^2(0,\infty;Y) \times L^2(0,\infty;[\mathcal{D}(A_\lambda)]') \times L^2(0,\infty;U)$
- $\Phi\colon W_\infty(\mathcal{D}(A_\lambda),Y) \times L^2(0,\infty;U) \times W_\infty(Y,[\mathcal{D}(A_\lambda)]') \to X$
- $\Phi(\mathbf{y},u,\mathbf{p}) = \begin{pmatrix} \mathbf{y}(0) \\ \dot{\mathbf{y}} - A\mathbf{y} + F(\mathbf{y}) - Bu \\ -\dot{\mathbf{p}} - A'\mathbf{p} - P((\mathbf{y}\cdot\nabla)\mathbf{p} - (\nabla\mathbf{y})^T\mathbf{p}) - \mathbf{y} \\ \beta u + B^*\mathbf{p} \end{pmatrix}$
- $\Phi(\bar{\mathbf{y}},\bar{u},\bar{\mathbf{p}}) = (\mathbf{y}_0,0,0,0)$

A formal HJB equation

Let us formally consider the HJB equation

$$D\mathcal{V}(\mathbf{y})(A\mathbf{y} - F(\mathbf{y})) + \frac{1}{2}\|\mathbf{y}\|_Y^2 - \frac{1}{2\beta}\|B^* D\mathcal{V}(\mathbf{y})\|_U^2 = 0 \text{ for } \mathbf{y} \in \mathcal{D}(A).$$

Problem: not rigorous since we only know $D\mathcal{V}(\mathbf{y}) \in \mathcal{L}(V, \mathbb{R})$.

Remark: in the 2D-case, one can show that $D^k\mathcal{V}(0) \equiv \mathcal{T}_k$ with

$$\sum_{i=1}^{k} \mathcal{T}_k(\mathbf{z}_1, \ldots, \mathbf{z}_{i-1}, A_\pi \mathbf{z}_i, \mathbf{z}_{i+1}, \ldots, \mathbf{z}_k) = \mathcal{R}_k(\mathbf{z}_1, \ldots, \mathbf{z}_k),$$

where the multilinear form $\mathcal{R}_k \colon \mathcal{D}(A)^k \to \mathbb{R}$ is given by

$$\mathcal{R}_k(\mathbf{z}_1, \ldots, \mathbf{z}_k) = \frac{1}{2\beta} \sum_{i=2}^{k-2} \binom{k}{i} \operatorname{Sym}_{i,k-i}\left(\mathcal{C}_i \otimes \mathcal{C}_{k-i}\right)(\mathbf{z}_1, \ldots, \mathbf{z}_k) + \frac{k(k-1)}{2} \operatorname{Sym}_{k-2,2}\left(\mathcal{T}_{k-1} \otimes D^2F(0)\right)(\mathbf{z}_1, \ldots, \mathbf{z}_k)$$

$\mathcal{C}_i(\mathbf{z}_1, \ldots, \mathbf{z}_i) = B^*\mathcal{T}_{i+1}(\cdot, \mathbf{z}_1, \ldots, \mathbf{z}_i)$, $D^2F(0)(\mathbf{z}_1, \mathbf{z}_2) = (\mathbf{z}_1 \cdot \nabla)\mathbf{z}_2 + (\mathbf{z}_2 \cdot \nabla)\mathbf{z}_1$.

A formal HJB equation

Let us formally consider the HJB equation

$$D\mathcal{V}(\mathbf{y})(A\mathbf{y} - F(\mathbf{y})) + \frac{1}{2}\|\mathbf{y}\|_Y^2 - \frac{1}{2\beta}\|B^* D\mathcal{V}(\mathbf{y})\|_U^2 = 0 \text{ for } \mathbf{y} \in \mathcal{D}(A).$$

Problem: not rigorous since we only know $D\mathcal{V}(\mathbf{y}) \in \mathcal{L}(V, \mathbb{R})$.

Remark: in the 2D-case, one can show that $D^k\mathcal{V}(0) \equiv \mathcal{T}_k$ with

$$\sum_{i=1}^{k} \mathcal{T}_k(\mathbf{z}_1, \ldots, \mathbf{z}_{i-1}, A_\pi \mathbf{z}_i, \mathbf{z}_{i+1}, \ldots, \mathbf{z}_k) = \mathcal{R}_k(\mathbf{z}_1, \ldots, \mathbf{z}_k),$$

where the multilinear form $\mathcal{R}_k \colon \mathcal{D}(A)^k \to \mathbb{R}$ is given by

$$\mathcal{R}_k(\mathbf{z}_1, \ldots, \mathbf{z}_k) = \frac{1}{2\beta} \sum_{i=2}^{k-2} \binom{k}{i} \operatorname{Sym}_{i,k-i}\left(\mathcal{C}_i \otimes \mathcal{C}_{k-i}\right)(\mathbf{z}_1, \ldots, \mathbf{z}_k)$$
$$+ \frac{k(k-1)}{2} \operatorname{Sym}_{k-2,2}\left(\mathcal{T}_{k-1} \otimes D^2F(0)\right)(\mathbf{z}_1, \ldots, \mathbf{z}_k)$$

$\mathcal{C}_i(\mathbf{z}_1, \ldots, \mathbf{z}_i) = B^* \mathcal{T}_{i+1}(\cdot, \mathbf{z}_1, \ldots, \mathbf{z}_i)$, $D^2F(0)(\mathbf{z}_1, \mathbf{z}_2) = (\mathbf{z}_1 \cdot \nabla)\mathbf{z}_2 + (\mathbf{z}_2 \cdot \nabla)\mathbf{z}_1$.

A formal HJB equation

Let us formally consider the HJB equation

$$D\mathcal{V}(\mathbf{y})(A\mathbf{y}-F(\mathbf{y}))+\frac{1}{2}\|\mathbf{y}\|_Y^2-\frac{1}{2\beta}\|B^*D\mathcal{V}(\mathbf{y})\|_U^2=0 \text{ for } \mathbf{y}\in\mathcal{D}(A).$$

Problem: not rigorous since we only know $D\mathcal{V}(\mathbf{y})\in\mathcal{L}(V,\mathbb{R})$.

Remark: in the 2D-case, one can show that $D^k\mathcal{V}(0)\equiv\mathcal{T}_k$ with

$$\sum_{i=1}^{k}\mathcal{T}_k(\mathbf{z}_1,\dots,\mathbf{z}_{i-1},A_\pi\mathbf{z}_i,\mathbf{z}_{i+1},\dots,\mathbf{z}_k)=\mathcal{R}_k(\mathbf{z}_1,\dots,\mathbf{z}_k),$$

where the multilinear form $\mathcal{R}_k\colon\mathcal{D}(A)^k\to\mathbb{R}$ is given by

$$\begin{aligned}\mathcal{R}_k(\mathbf{z}_1,\dots,\mathbf{z}_k)=&\frac{1}{2\beta}\sum_{i=2}^{k-2}\binom{k}{i}\operatorname{Sym}_{i,k-i}\left(\mathcal{C}_i\otimes\mathcal{C}_{k-i}\right)(\mathbf{z}_1,\dots,\mathbf{z}_k)\\&+\frac{k(k-1)}{2}\operatorname{Sym}_{k-2,2}\left(\mathcal{T}_{k-1}\otimes D^2F(0)\right)(\mathbf{z}_1,\dots,\mathbf{z}_k)\end{aligned}$$

$\mathcal{C}_i(\mathbf{z}_1,\dots,\mathbf{z}_i)=B^*\mathcal{T}_{i+1}(\cdot,\mathbf{z}_1,\dots,\mathbf{z}_i)$, $D^2F(0)(\mathbf{z}_1,\mathbf{z}_2)=(\mathbf{z}_1\cdot\nabla)\mathbf{z}_2+(\mathbf{z}_2\cdot\nabla)\mathbf{z}_1$.

Regularity properties of $\mathcal{T}_k$

Idea: analyze the same equation in 3D for $\mathcal{T}_k \overset{?}{\neq} D^k\mathcal{V}(0) \in \mathcal{M}(V^k, \mathbb{R})$

Multilinear Lyapunov operator equations

For $k \geq 3$ and $\mathbf{z}_1, \ldots, \mathbf{z}_k \in V$, let

$$\mathcal{T}_k(\mathbf{z}_1, \ldots, \mathbf{z}_k) = -\int_0^\infty \mathcal{R}_k(e^{A_\pi t}\mathbf{z}_1, \ldots, e^{A_\pi t}\mathbf{z}_k)\mathrm{d}t.$$

Then $\mathcal{T}_k \in \mathcal{M}(V^k, \mathbb{R})$ is the unique solution of

$$\sum_{i=1}^k \mathcal{T}_k(\mathbf{z}_1, \ldots, \mathbf{z}_{i-1}, A_\pi \mathbf{z}_i, \mathbf{z}_{i+1}, \ldots, \mathbf{z}_k) = \mathcal{R}_k(\mathbf{z}_1, \ldots, \mathbf{z}_k).$$

Moreover: $\mathcal{T}_k \in \mathcal{S}_k(V, V') := \bigcap_{\ell=1}^k \mathcal{M}(V^{\ell-1} \times V' \times V^{k-\ell}, \mathbb{R})$.

Proof idea: cp. with $A_\pi^* \Pi + \Pi A_\pi = -BB^* \Rightarrow \Pi = -\int_0^\infty e^{A_\pi^* t} BB^* e^{A\pi t}\,\mathrm{d}t$

For $\mathcal{F} \in S_{k-1}(V, V')$, consider $\int_0^\infty |\mathcal{F}(A_0(e^{A_\pi t}\mathbf{z}_1, e^{A_\pi t}\mathbf{z}_2), e^{A_\pi t}\mathbf{z}_3, \ldots, e^{A_\pi t}\mathbf{z}_k)|\mathrm{d}t$

$\Rightarrow$ utilize analyticity of $e^{A_\pi t}$

Regularity properties of $\mathcal{T}_k$

Idea: analyze the same equation in 3D for $\mathcal{T}_k \overset{?}{\neq} D^k\mathcal{V}(0) \in \mathcal{M}(V^k,\mathbb{R})$

Multilinear Lyapunov operator equations

For $k \geq 3$ and $\mathbf{z}_1,\ldots,\mathbf{z}_k \in V$, let

$$\mathcal{T}_k(\mathbf{z}_1,\ldots,\mathbf{z}_k) = -\int_0^\infty \mathcal{R}_k(e^{A_\pi t}\mathbf{z}_1,\ldots,e^{A_\pi t}\mathbf{z}_k)\mathrm{d}t.$$

Then $\mathcal{T}_k \in \mathcal{M}(V^k,\mathbb{R})$ is the unique solution of

$$\sum_{i=1}^{k} \mathcal{T}_k(\mathbf{z}_1,\ldots,\mathbf{z}_{i-1},A_\pi\mathbf{z}_i,\mathbf{z}_{i+1},\ldots,\mathbf{z}_k) = \mathcal{R}_k(\mathbf{z}_1,\ldots,\mathbf{z}_k).$$

Moreover: $\mathcal{T}_k \in \mathcal{S}_k(V,V') := \bigcap_{\ell=1}^{k} \mathcal{M}(V^{\ell-1}\times V'\times V^{k-\ell},\mathbb{R})$.

Proof idea: cp. with $A_\pi^*\Pi + \Pi A_\pi = -BB^* \Rightarrow \Pi = -\int_0^\infty e^{A_\pi^* t}BB^*e^{A\pi t}\,\mathrm{d}t$

For $\mathcal{F} \in S_{k-1}(V,V')$, consider $\int_0^\infty |\mathcal{F}(A_0(e^{A_\pi t}\mathbf{z}_1,e^{A_\pi t}\mathbf{z}_2),e^{A_\pi t}\mathbf{z}_3,\ldots,e^{A_\pi t}\mathbf{z}_k)|\mathrm{d}t$

$\Rightarrow$ utilize analyticity of $e^{A_\pi t}$

Regularity properties of $\mathcal{T}_k$

Idea: analyze the same equation in 3D for $\mathcal{T}_k \overset{?}{\neq} D^k\mathcal{V}(0) \in \mathcal{M}(V^k, \mathbb{R})$

Multilinear Lyapunov operator equations

For $k \geq 3$ and $\mathbf{z}_1, \dots, \mathbf{z}_k \in V$, let

$$\mathcal{T}_k(\mathbf{z}_1, \dots, \mathbf{z}_k) = -\int_0^\infty \mathcal{R}_k(e^{A_\pi t}\mathbf{z}_1, \dots, e^{A_\pi t}\mathbf{z}_k)\mathrm{d}t.$$

Then $\mathcal{T}_k \in \mathcal{M}(V^k, \mathbb{R})$ is the unique solution of

$$\sum_{i=1}^k \mathcal{T}_k(\mathbf{z}_1, \dots, \mathbf{z}_{i-1}, A_\pi \mathbf{z}_i, \mathbf{z}_{i+1}, \dots, \mathbf{z}_k) = \mathcal{R}_k(\mathbf{z}_1, \dots, \mathbf{z}_k).$$

Moreover: $\mathcal{T}_k \in \mathcal{S}_k(V, V') := \bigcap_{\ell=1}^k \mathcal{M}(V^{\ell-1} \times V' \times V^{k-\ell}, \mathbb{R})$.

Proof idea: cp. with $A_\pi^* \Pi + \Pi A_\pi = -BB^* \Rightarrow \Pi = -\int_0^\infty e^{A_\pi^* t} BB^* e^{A\pi t}\, \mathrm{d}t$

For $\mathcal{F} \in S_{k-1}(V, V')$, consider $\int_0^\infty |\mathcal{F}(A_0(e^{A_\pi t}\mathbf{z}_1, e^{A_\pi t}\mathbf{z}_2), e^{A_\pi t}\mathbf{z}_3, \dots, e^{A_\pi t}\mathbf{z}_k)|\mathrm{d}t$

$\Rightarrow$ utilize analyticity of $e^{A_\pi t}$

Regularity properties of $\mathcal{T}_k$

Idea: analyze the same equation in 3D for $\mathcal{T}_k \overset{?}{\neq} D^k\mathcal{V}(0) \in \mathcal{M}(V^k,\mathbb{R})$

Multilinear Lyapunov operator equations

For $k \geq 3$ and $\mathbf{z}_1,\ldots,\mathbf{z}_k \in V$, let

$$\mathcal{T}_k(\mathbf{z}_1,\ldots,\mathbf{z}_k) = -\int_0^\infty \mathcal{R}_k(e^{A_\pi t}\mathbf{z}_1,\ldots,e^{A_\pi t}\mathbf{z}_k)\mathrm{d}t.$$

Then $\mathcal{T}_k \in \mathcal{M}(V^k,\mathbb{R})$ is the unique solution of

$$\sum_{i=1}^k \mathcal{T}_k(\mathbf{z}_1,\ldots,\mathbf{z}_{i-1},A_\pi\mathbf{z}_i,\mathbf{z}_{i+1},\ldots,\mathbf{z}_k) = \mathcal{R}_k(\mathbf{z}_1,\ldots,\mathbf{z}_k).$$

Moreover: $\mathcal{T}_k \in \mathcal{S}_k(V,V') := \bigcap_{\ell=1}^k \mathcal{M}(V^{\ell-1}\times V'\times V^{k-\ell},\mathbb{R})$.

Proof idea: cp. with $A_\pi^*\Pi + \Pi A_\pi = -BB^* \Rightarrow \Pi = -\int_0^\infty e^{A_\pi^* t}BB^*e^{A\pi t}\,\mathrm{d}t$

For $\mathcal{F} \in S_{k-1}(V,V')$, consider $\int_0^\infty |\mathcal{F}(A_0(e^{A_\pi t}\mathbf{z}_1, e^{A_\pi t}\mathbf{z}_2), e^{A_\pi t}\mathbf{z}_3,\ldots,e^{A_\pi t}\mathbf{z}_k)|\mathrm{d}t$

$\Rightarrow$ utilize analyticity of $e^{A_\pi t}$

Error estimates for polynomial system

Based on $\mathcal{T}_k$, let us define

- $u_d(\mathbf{y}) = -\frac{1}{\beta} \sum_{k=2}^{d} \frac{1}{(k-1)!} B^* \mathcal{T}_k(\cdot, \mathbf{y}, \ldots, \mathbf{y})$
- $\mathcal{V}_d \colon V \to \mathbb{R}, \ \ \mathcal{V}_d(\mathbf{y}) := \sum_{k=2}^{d} \frac{1}{k!} \mathcal{T}_k(\mathbf{y}, \ldots, \mathbf{y})$.

Consider then the perturbed cost functional

$$\mathcal{J}_d(\mathbf{y}_0, u) := \frac{1}{2} \int_0^\infty \|\mathbf{y}\|_Y^2 \, \mathrm{d}t + \frac{\beta}{2} \int_0^\infty \|u\|_U^2 \, \mathrm{d}t + \int_0^\infty r_d(\mathbf{y}) \, \mathrm{d}t.$$

where $r_d(\mathbf{y})$ is a polynomial remainder term.

Observation:

- $u_d(\mathbf{y}_d)$ is optimal for $\mathcal{J}_d(\mathbf{y}_0, u)$
- $|\mathcal{V}(\mathbf{y}_0) - \mathcal{V}_d(\mathbf{y}_0)| \leq M \|\mathbf{y}_0\|_V^{d+1} \Rightarrow \mathcal{T}_k = D^k \mathcal{V}(0)$

Error estimates for polynomial system

Based on $\mathcal{T}_k$, let us define

- $u_d(\mathbf{y}) = -\frac{1}{\beta}\sum_{k=2}^{d}\frac{1}{(k-1)!}B^*\mathcal{T}_k(\cdot,\mathbf{y},\ldots,\mathbf{y})$
- $\mathcal{V}_d\colon V\to\mathbb{R},\quad \mathcal{V}_d(\mathbf{y}) := \sum_{k=2}^{d}\frac{1}{k!}\mathcal{T}_k(\mathbf{y},\ldots,\mathbf{y})$.

Consider then the perturbed cost functional

$$\mathcal{J}_d(\mathbf{y}_0,u) := \frac{1}{2}\int_0^\infty \|\mathbf{y}\|_Y^2\,\mathrm{d}t + \frac{\beta}{2}\int_0^\infty \|u\|_U^2\,\mathrm{d}t + \int_0^\infty r_d(\mathbf{y})\,\mathrm{d}t.$$

where $r_d(\mathbf{y})$ is a polynomial remainder term.

Observation:

- $u_d(\mathbf{y}_d)$ is optimal for $\mathcal{J}_d(\mathbf{y}_0,u)$
- $|\mathcal{V}(\mathbf{y}_0)-\mathcal{V}_d(\mathbf{y}_0)| \le M\|\mathbf{y}_0\|_V^{d+1} \Rightarrow \mathcal{T}_k = D^k\mathcal{V}(0)$

Error estimates for polynomial system

Based on $\mathcal{T}_k$, let us define

- $u_d(\mathbf{y}) = -\frac{1}{\beta} \sum_{k=2}^{d} \frac{1}{(k-1)!} B^* \mathcal{T}_k(\cdot, \mathbf{y}, \ldots, \mathbf{y})$
- $\mathcal{V}_d \colon V \to \mathbb{R}, \quad \mathcal{V}_d(\mathbf{y}) := \sum_{k=2}^{d} \frac{1}{k!} \mathcal{T}_k(\mathbf{y}, \ldots, \mathbf{y}).$

Consider then the perturbed cost functional

$$\mathcal{J}_d(\mathbf{y}_0, u) := \frac{1}{2} \int_0^\infty \|\mathbf{y}\|_Y^2 \, \mathrm{d}t + \frac{\beta}{2} \int_0^\infty \|u\|_U^2 \, \mathrm{d}t + \int_0^\infty r_d(\mathbf{y}) \, \mathrm{d}t.$$

where $r_d(\mathbf{y})$ is a polynomial remainder term.

Observation:

- $u_d(\mathbf{y}_d)$ is optimal for $\mathcal{J}_d(\mathbf{y}_0, u)$
- $|\mathcal{V}(\mathbf{y}_0) - \mathcal{V}_d(\mathbf{y}_0)| \leq M \|\mathbf{y}_0\|_V^{d+1} \Rightarrow \mathcal{T}_k = D^k \mathcal{V}(0)$

Structured tensor equations

How to numerically solve

$$\sum_{i=1}^{k} \mathcal{T}_k(y_1, \ldots, y_{i-1}, A_\Pi y_i, y_{i+1}, \ldots, y_k) = \frac{1}{2\beta} \mathcal{R}_k(z_1, \ldots, z_k)$$

Identify $\mathcal{T}_k$ with tensor $T_k \in \mathbb{R}^{n^k}$ of order k

$$\Rightarrow \text{Solve} \quad \underbrace{\left(\sum_{i=1}^{k} I^{\otimes^{k-i}} \otimes A_\Pi^T \otimes I^{\otimes^{i-1}} \right)}_{\mathbf{A}} T_k = \frac{1}{2\beta} R_k(T_2, \ldots, T_{k-1}).$$

Since $\mathbf{A}$ is stable:

$$\mathbf{A}^{-1} = -\int_0^\infty e^{t\mathbf{A}} \,\mathrm{d}t = -\int_0^\infty \bigotimes_{i=1}^{k} e^{tA_\Pi^T} \,\mathrm{d}t \approx -\sum_{\ell=-r}^{r} w_\ell \bigotimes_{i=1}^{k} e^{t_\ell A_\Pi^T}.$$

Use quadrature formula with suitable weights w_ℓ and points t_ℓ.

[GRASEDYCK, HACKBUSCH, STENGER]

Summary and future challenges

- infinite-horizon control by HJB approximations
- smoothness of the value function $\mathcal{V}$ around 0
- $D^k\mathcal{V}(0)$ characterized by Riccati/multilinear Lyapunov equations
- polynomial feedback laws based on Taylor expansion of $D\mathcal{V}$
- boundary control?
- general semilinear PDEs?
- computation of higher order feedback laws?

Summary and future challenges

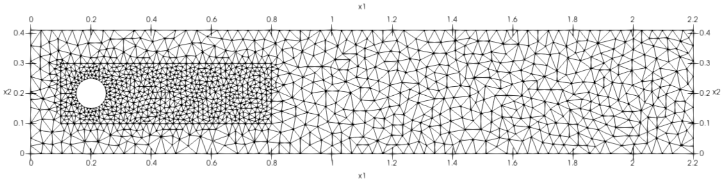
- infinite-horizon control by HJB approximations
- smoothness of the value function $\mathcal{V}$ around 0
- $D^k\mathcal{V}(0)$ characterized by Riccati/multilinear Lyapunov equations
- polynomial feedback laws based on Taylor expansion of $D\mathcal{V}$
- boundary control?
- general semilinear PDEs?
- computation of higher order feedback laws?

Thank you for your attention!

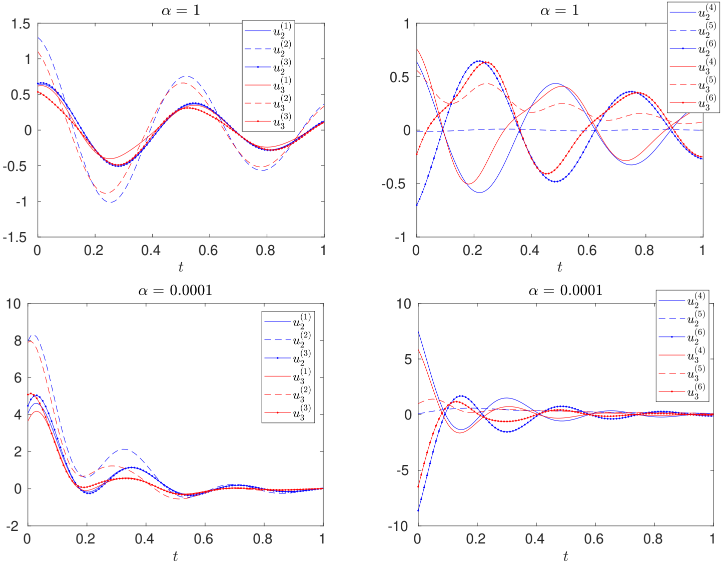
Setup and discretization

[Behr/Benner/Heiland'18]

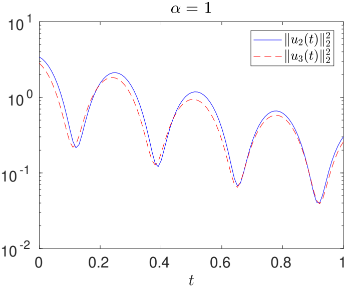
- *Taylor-Hood* P_2-P_1 finite elements
- $\mathrm{Re} := \frac{1}{\nu} = 90$, $n_v = 9356$, $n_p = 1289$
- control domain $\Omega_c := [0.27, 0.32] \times [0.15, 0.25]$
- control operator $Bu = \sum_{\ell=1}^{3} \begin{bmatrix} 0 \\ w_\ell(x_2) \end{bmatrix} u_\ell(t) + \begin{bmatrix} w_\ell(x_2) \\ 0 \end{bmatrix} u_{\ell+3}(t)$,

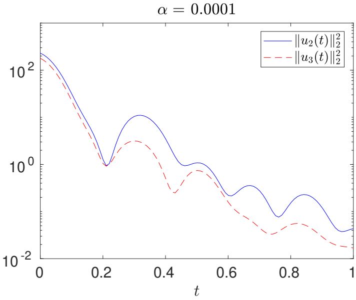


Control laws for $\alpha = 1$ and $\alpha = 10^{-4}$



Dynamics of $\|u(\cdot)\|^2$ for $\alpha = 1$ and $\alpha = 10^{-4}$





The underlying ODE [Heinkenschloss/Sorensen/Sun'08]

Consider the discrete system given by

$$\begin{aligned} E\dot{y}(t) &= Ay(t) + H(y(t)\otimes y(t)) + Bu(t) + Gp(t), \\ 0 &= G^T y(t), \end{aligned}$$

with $E, A \in \mathbb{R}^{n_v\times n_v}, H \in \mathbb{R}^{n_n\times n_v^2}, B \in \mathbb{R}^{n_v\times 6}, G \in \mathbb{R}^{n_v\times n_p}$.

The second equation implies

$$0 = G^T\dot{y}(t) = G^TE^{-1}\left(Ay(t) + H(y(t)\otimes y(t)) + Bu(t) + Gp(t)\right).$$

We can eliminate the pressure via

$$p(t) = -(G^TE^{-1}G)^{-1}G^TE^{-1}\left(Ay(t) + H(y(t)\otimes y(t)) + Bu(t)\right).$$

Using the projection $P = I - G(G^TE^{-1}G)^{-1}G^TE^{-1}$, we obtain

$$E\dot{y}(t) = PAy(t) + PH(y(t)\otimes y(t)) + PBu(t).$$

The underlying ODE [Heinkenschloss/Sorensen/Sun'08]

Consider the discrete system given by

$$\begin{aligned} E\dot{y}(t) &= Ay(t) + H(y(t)\otimes y(t)) + Bu(t) + Gp(t),\\ 0 &= G^T y(t), \end{aligned}$$

with $E, A \in \mathbb{R}^{n_v \times n_v}, H \in \mathbb{R}^{n_n \times n_v^2}, B \in \mathbb{R}^{n_v \times 6}, G \in \mathbb{R}^{n_v \times n_p}$.

The second equation implies

$$0 = G^T \dot{y}(t) = G^T E^{-1}\left(Ay(t) + H(y(t)\otimes y(t)) + Bu(t) + Gp(t)\right).$$

We can eliminate the pressure via

$$p(t) = -(G^T E^{-1} G)^{-1} G^T E^{-1}\left(Ay(t) + H(y(t)\otimes y(t)) + Bu(t)\right).$$

Using the projection $P = I - G(G^T E^{-1} G)^{-1} G^T E^{-1}$, we obtain

$$E\dot{y}(t) = PAy(t) + PH(y(t)\otimes y(t)) + PBu(t).$$

The underlying ODE [Heinkenschloss/Sorensen/Sun'08]

Consider the discrete system given by

$$\begin{aligned} E\dot{y}(t) &= Ay(t) + H(y(t) \otimes y(t)) + Bu(t) + Gp(t), \\ 0 &= G^T y(t), \end{aligned}$$

with $E, A \in \mathbb{R}^{n_v \times n_v}, H \in \mathbb{R}^{n_n \times n_v^2}, B \in \mathbb{R}^{n_v \times 6}, G \in \mathbb{R}^{n_v \times n_p}$.

The second equation implies

$$0 = G^T \dot{y}(t) = G^T E^{-1} \left(Ay(t) + H(y(t) \otimes y(t)) + Bu(t) + Gp(t)\right).$$

We can eliminate the pressure via

$$p(t) = -(G^T E^{-1} G)^{-1} G^T E^{-1} \left(Ay(t) + H(y(t) \otimes y(t)) + Bu(t)\right).$$

Using the projection $P = I - G(G^T E^{-1} G)^{-1} G^T E^{-1}$, we obtain

$$E\dot{y}(t) = PAy(t) + PH(y(t) \otimes y(t)) + PBu(t).$$

The underlying ODE [Heinkenschloss/Sorensen/Sun'08]

For the system

$$(PEP^T)\dot{y}(t) = (PAP^T)y(t) + \left(PHP^T \otimes P^T\right)(y(t) \otimes y(t)) + (PB)u(t)$$

decompose $P = \Theta_\ell \Theta_r^T$ with $\Theta_\ell^T \Theta_r = I$ and project onto $\mathrm{range}(P)$

$$\underbrace{(\Theta_r^T E \Theta_r)}_{\widetilde{E}} \dot{\tilde{y}}(t) = \underbrace{(\Theta_r^T A \Theta_r)}_{\widetilde{A}} \tilde{y}(t) + \underbrace{(\Theta_r^T H \Theta_r \otimes \Theta_r)}_{\widetilde{H}} \tilde{y}(t) \otimes \tilde{y}(t) + \underbrace{(\Theta_r^T B)}_{\widetilde{B}} u(t),$$

where $\tilde{y} = \Theta_\ell^T y(t)$.

Consequence: ODE system instead of DAE system.

Note: avoid explicit computation, in particular, for $\tilde{H}$!

Computing the feedback gain

Consider a third order feedback of the form

$$u_3(\mathbf{y}) = -\frac{1}{\alpha} B^* \mathcal{T}_2(\cdot, \mathbf{y}) - \frac{1}{2\alpha} B^* \mathcal{T}_3(\cdot, \mathbf{y}, \mathbf{y})$$

Computing the feedback gain

Consider a third order feedback of the form

$$u_3(\mathbf{y}) = -\frac{1}{\alpha} B^* \mathcal{T}_2(\cdot, \mathbf{y}) - \frac{1}{2\alpha} B^* \mathcal{T}_3(\cdot, \mathbf{y}, \mathbf{y})$$

Its corresponding discretized version reads

$$u_3(\tilde{y}) = -\frac{1}{\alpha}(\tilde{y}^T \widetilde{E}^T \otimes \widetilde{B}^T)\pi - \frac{1}{2\alpha}(\tilde{y}^T \widetilde{E}^T \otimes \tilde{y}^T \widetilde{E}^T \otimes \widetilde{B}^T)\sigma.$$

Computing the feedback gain

Its corresponding discretized version reads

$$u_3(\tilde{y}) = -\frac{1}{\alpha}(\tilde{y}^T \widetilde{E}^T \otimes \widetilde{B}^T)\pi - \frac{1}{2\alpha}(\tilde{y}^T \widetilde{E}^T \otimes \tilde{y}^T \widetilde{E}^T \otimes \widetilde{B}^T)\sigma.$$

Computing the feedback gain

Its corresponding discretized version reads

$$u_3(\tilde{y}) = -\frac{1}{\alpha}(\tilde{y}^T \widetilde{E}^T \otimes \widetilde{B}^T)\pi - \frac{1}{2\alpha}(\tilde{y}^T \widetilde{E}^T \otimes \tilde{y}^T \widetilde{E}^T \otimes \widetilde{B}^T)\sigma.$$

For obtaining $\pi = \mathrm{vec}(\Pi)$, solve algebraic Riccati equation

$$\widetilde{A}^T \Pi \widetilde{E} + \widetilde{E}^T \Pi \widetilde{A} - \widetilde{E}^T \Pi \widetilde{B} \widetilde{B}^T \Pi \widetilde{E} + \Theta_r^T \Theta_r = 0.$$

Computing the feedback gain

Its corresponding discretized version reads

$$u_3(\tilde{y}) = -\frac{1}{\alpha}(\tilde{y}^T \widetilde{E}^T \otimes \widetilde{B}^T)\pi - \frac{1}{2\alpha}(\tilde{y}^T \widetilde{E}^T \otimes \tilde{y}^T \widetilde{E}^T \otimes \widetilde{B}^T)\sigma.$$

For obtaining $\pi = \mathrm{vec}(\Pi)$, solve algebraic Riccati equation

$$\widetilde{A}^T \Pi \widetilde{E} + \widetilde{E}^T \Pi \widetilde{A} - \widetilde{E}^T \Pi \widetilde{B} \widetilde{B}^T \Pi \widetilde{E} + \Theta_r^T \Theta_r = 0.$$

For obtaining σ, solve linear tensor equation

$$\underbrace{\left(\widetilde{E}^T \otimes \widetilde{E}^T \otimes \widetilde{A}_\pi^T + \widetilde{E}^T \otimes \widetilde{A}_\pi^T \otimes \widetilde{E}^T + \widetilde{A}_\pi^T \otimes \widetilde{E}^T \otimes \widetilde{E}^T\right)}_{\mathcal{A}^T} \sigma = f,$$

where

$$f = -2\left(\widetilde{H}^T \otimes \widetilde{E}^T + \widetilde{E}^T \otimes \widetilde{H}^T + (I \otimes \mathcal{P}^T)(\widetilde{H}^T \otimes \widetilde{E}^T)\right)\pi.$$

Computing the feedback gain cont'd

Problem: how to store/compute solution σ of $\mathcal{A}^T\sigma = f$?
$\Rightarrow$ requires $\approx$ 4 TB of memory!

Remedy 1: only compute the feedback gain $\widetilde{K} = (\widetilde{E}^T \otimes \widetilde{E}^T \otimes \widetilde{B}^T)\sigma$
$\Rightarrow$ requires $\approx$ 4 GB of memory

Remedy 2: approximate $\widetilde{K} = (\widetilde{E}^T \otimes \widetilde{E}^T \otimes \widetilde{B}^T)\mathcal{A}^{-T} f$ via quadrature

$$\mathcal{A}^{-1} = -\int_0^\infty \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \mathrm{d}t$$

$$\approx -\sum_{j=-r}^{r} \frac{2w_j}{\lambda} \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right)$$

and utilizing properties of tensor multiplication.

[Grasedyck'04]

Computing the feedback gain cont'd

Problem: how to store/compute solution σ of $\mathcal{A}^T\sigma = f$?
$\Rightarrow$ requires $\approx$ 4 TB of memory!

Remedy 1: only compute the feedback gain $\widetilde{K} = (\widetilde{E}^T \otimes \widetilde{E}^T \otimes \widetilde{B}^T)\sigma$
$\Rightarrow$ requires $\approx$ 4 GB of memory

Remedy 2: approximate $\widetilde{K} = (\widetilde{E}^T \otimes \widetilde{E}^T \otimes \widetilde{B}^T)\mathcal{A}^{-T} f$ via quadrature

$$\mathcal{A}^{-1} = -\int_0^\infty \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \mathrm{d}t$$

$$\approx -\sum_{j=-r}^{r} \frac{2w_j}{\lambda} \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right)$$

and utilizing properties of tensor multiplication.

[Grasedyck'04]

Computing the feedback gain cont'd

Problem: how to store/compute solution σ of $\mathcal{A}^T\sigma = f$?
$\Rightarrow$ requires $\approx$ 4 TB of memory!

Remedy 1: only compute the feedback gain $\widetilde{K} = (\widetilde{E}^T \otimes \widetilde{E}^T \otimes \widetilde{B}^T)\sigma$
$\Rightarrow$ requires $\approx$ 4 GB of memory

Remedy 2: approximate $\widetilde{K} = (\widetilde{E}^T \otimes \widetilde{E}^T \otimes \widetilde{B}^T)\mathcal{A}^{-T} f$ via quadrature

$$\begin{aligned}\mathcal{A}^{-1} &= -\int_0^\infty \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{t\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \mathrm{d}t \\ &\approx -\sum_{j=-r}^{r} \frac{2w_j}{\lambda} \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right) \otimes \left(e^{\frac{t_j}{\lambda}\widetilde{E}^{-1}\widetilde{A}_\pi}\widetilde{E}^{-1}\right)\end{aligned}$$

and utilizing properties of tensor multiplication.

[GRASEDYCK'04]