Optimal shape of the p-Laplacian eigenvalue

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Outline

Problem (A)

- Let Ω be a bounded domain,
- Given numbers $\gamma > 0, p > 1$ and a measurable subset D of Ω.
- Consider

$$\begin{cases} -\Delta_p u + \gamma \chi_D |u|^{p-2} u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

- $\Delta_p u := \text{div}\left(|\nabla u|^{p-2} \nabla u\right)$,
- χ_D is characteristic function of D.

Problem (A)

- The first eigenvalue has variational form:

$$\lambda(\gamma, D) = \inf\left\{ \frac{\int_\Omega |\nabla v|^p\,dx + \gamma \int_\Omega \chi_D |u|^p\,dx}{\int_\Omega |v|^p\,dx},\ v \in W_0^{1,p}(\Omega), v \neq 0 \right\} \tag{1.2}$$

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- Fix $A \in (0, |\Omega|)$, and define

$$\Lambda(\gamma, D) := \inf\left\{\lambda(\gamma, D) :\ D \subset \Omega,\ |D| = A\right\}$$

- Here $|D|$: Lebesgue measure of a subset D.
- Any minimizer is called optimal configuration. The pair (u, D) is said to be an optimal pair(optimal solution).

S. Chanillo, D. Grieser, M. Imai, K. Kurata and I. Ohnishi, Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes, Commun. Math. Phys. 214 (2000) 315–337.

W. Pielichowski, The optimization of eigenvalue problems involving the p-Laplacian. Univ. Iagel. Acta Math. 42 (2004) 109–122.

Problem (B)

$$\begin{cases} -\Delta_p u = \lambda \varrho |u|^{p-2} u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (B)

$$\begin{cases} -\Delta_p u = \lambda \varrho |u|^{p-2} u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

- The density function ϱ belongs to

$$\mathcal{R} = \{\varrho : \ \varrho(x) = \alpha\chi_D + \beta\chi_{D^c}, \ D \subset \Omega, \ |D| = A\},$$

with $A \in (0, |\Omega|)$ and $\alpha > \beta > 0$,

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S.A. Mohammadi, F. Bozorgnia, H. Voss, Optimal shape design for the p-Laplacian eigenvalue problem, J. Sci. Comput. 78, (2019) 1231–1249.

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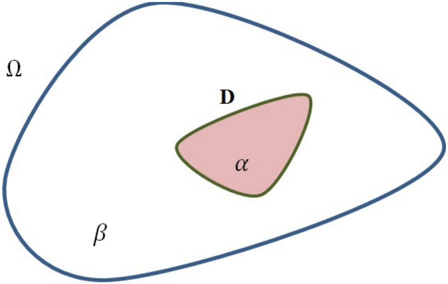
- We rewrite (2.1) as

$$\inf_{D \subset \Omega, \, |D| = A} \lambda(D), \tag{2.2}$$

- The minimum in (2.2) is denoted by $\hat{\lambda}_{\hat{\varrho}}$.
- The pair $(\hat{u}, \hat{\varrho})$ is minimzer for problem (B) with parameters (α, β) iff $(\hat{u}, \hat{D})$ is an optimal pair of problem (A) with

$$\gamma = (\beta - \alpha)\hat{\lambda}_{\hat{\varrho}}$$

Physical Motivation



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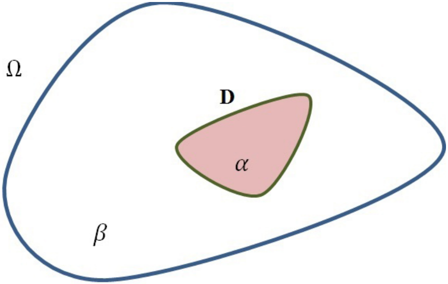

Let $p = 2$ and $N = 2$ and assume that we want to build a membrane with fixed boundary of prescribed shape consisting of given two different materials with densities α and β. The body has prescribed mass $M = \alpha A + \beta(|\Omega| - A)$. Our aim is to distribute these materials in a such a way that the basic frequency of the resulting membrane is as small as possible.

Existence and Qualitative Properties of the Optimaizer

- It has been proved that problem (2.1) admits a solution [Cuccu, Emamizadeh, Porru, (2009)].

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- Some qualitative properties such as Steiner symmetrization and connectivity of the optimal domain have been investigated [Anedda, Cuccu, (2009)-Pielichowski, (2004)].
- For $p = 2$, Chanillo *et al.* have investigated a more general problem and obtained several interesting geometric attributes of the optimal shape [Chanillo, Grieser, Imai, Kurata, Ohnishi, (2000)].

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Theorem

Let $p \in (1, +\infty)$ and $A \in (0, |\Omega|)$, then

(a) *there is a number $\hat{t} > 0$ such that $\hat{D} = \{x \in \Omega : \hat{u}(x) \geq \hat{t}\}$, and so $\hat{D}^c$ contains a tubular neighborhood of the boundary $\partial\Omega$,*

(b) *every connected component $\mathcal{D}_0$ of the interior of $\hat{D}^c$ hits the boundary, i.e., $\bar{\mathcal{D}}_0 \cap \partial\Omega \neq \emptyset$. In particular, if Ω is simply connected, then $\hat{D}$ is connected.*

(c) *if Ω is Steiner symmetric with respect to a hyperplane T, then every minimizer $\hat{D}$ is Steiner symmetric relative to T.*

F. Cuccu, B. Emamizadeh, G. Porru, Optimization of the first eigenvalue in problems involving the p-Laplacian. Proc. Amer. Math. Soc. 137 (2009) 1677–1687.

C. Anedda, F. Cuccu, Steiner symmetry in the minimization of the first eigenvalue in problems involving the p-Laplacian, Proc. Amer. Math. Soc. 144 (2016) 3431–3440.

The following Lemma shows that spectrum is closed:

Lemma

Let $\{\varrho_n\}_1^\infty$ be a sequence of functions in $L^\infty(\Omega)$ uniformly bounded by a constant α_0 and $\{\lambda_{\varrho_n}\}_1^\infty$, $\{u_n\}_1^\infty$ be the corresponding principle eigenvalues and positive eigenfunctions of (1.3) such that $\lambda_{\varrho_n} \to \hat{\lambda}$ as n goes to infinity. Then, there exists a function η in $L^\infty(\Omega)$ so that

$$-\Delta_p \hat{u} = \hat{\lambda}\eta\hat{u}^{p-1}, \quad \text{in } \Omega, \quad \hat{u} = 0 \text{ on } \partial\Omega,$$

and

$$\lim_{n\to\infty} \|u_n - \hat{u}\|_{W_0^{1,p}(\Omega)} = 0.$$

Low Contrast regime

Assume that α and β are close. Let λ be the first eigenvalue of

$$-\Delta_p u = \lambda u^{p-1}, \qquad \text{in} \quad \Omega, \quad u = 0 \qquad \text{on} \quad \partial\Omega, \tag{3.1}$$

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and $\psi(x)$ be the corresponding eigenfunction such that $\|\nabla\psi(x)\|_{L^p} = 1$. For $s > 0$ the superlevel set of ψ

$$E_s = \{x \in \Omega : \ \psi(x) \geq s\}.$$

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Theorem

Let $\beta = 1$ and $\alpha = 1 + \epsilon$. Then, we have

$$\left(\frac{\epsilon A}{|\Omega| + \epsilon A}\right)\lambda \leq \lambda - \hat{\lambda}_{\hat{\varrho}_\epsilon} \leq \left(\frac{\epsilon}{1+\epsilon}\right)\lambda. \tag{3.2}$$

In the case of low contrast the optimal domain is squeezed between two super level sets.

Theorem

Let $\beta = 1$ and $\alpha = 1 + \epsilon$, choose τ such that $|E_\tau| = A$ and assume that $p > N$. Then for every $\delta > 0$ there is ϵ_0 such that whenever $\epsilon < \epsilon_0$ and

$$\hat{\varrho}_\epsilon = 1 + \epsilon \chi_{\hat{D}_\epsilon}, \quad \hat{D}_\epsilon = \{x \in \Omega : \hat{u}_\epsilon(x) \geq \hat{t}_\epsilon\},$$

is an optimal solution, then $|\hat{t}_\epsilon - \tau| < \delta$ and

$$E_{\tau+\delta} \subset \hat{D}_\epsilon \subset E_{\tau-\delta}.$$

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Theorem

$$\lim_{p\to\infty} \hat{\lambda}_p^{\frac{1}{p}} = \Lambda_\infty .$$

Theorem

A function u_∞ obtained as a limit of a subsequence $\{\hat{u}_p\}_1^\infty$ is a viscosity solution of the equation

$$\min\{|\nabla u| - \Lambda_\infty u,\ -\Delta_\infty u\} = 0, \tag{3.3}$$

where

$$-\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j},$$

is the ∞-Laplacian operator.

For $s > 0$ we define

$$E_s^\infty = \{x \in \Omega : u_\infty(x) \geq s\}.$$

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Theorem

Choose τ such that $|E_\tau^\infty| = A$. Then for any $\delta > 0$ there is p_0 such that whenever $p > p_0$ and

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Example. When $\Omega = \{x \in \mathbb{R}^N : \ 0 < a < |x| < a + R\}$ then we observe that

$$E_\tau^\infty = \{x \in \Omega : \ \frac{R}{2} - r < |x| < \frac{R}{2} + r\},$$

where $r = \frac{A}{2\pi R}$ since u_∞ is the distance function $\delta(x) = d(x, \partial\Omega)$ [Y. Yu, 2007].

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- Such rearrangement algorithms have been applied successfully to optimize eigenvalues of biharmonic equations appearing in frequency control based on the density distribution of composite rods and thin plates [Chen, Chou, Kao, (2016)- Kang, Kao, (2017)], to derive stationary and stable flows of an ideal fluid [Mohammadi, (2017)] and to obtain minimum ground state energy in quantum dot nanostructures [Mohammadi, Voss, (2016)], [Antunes, Mohammadi, Voss, (2018)].

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Bathtub Principle

Lemma

Let $f(x)$ be a nonnegative function in $L^1(\Omega)$ such that its level sets have measure zero. Then the maximization problem

$$\sup_{\varrho\in\mathcal{R}} \int_\Omega \varrho f(x)dx,$$

is uniquely solvable by (x)= $\alpha\chi_{\hat{D}} + \beta\chi_{\hat{D}^c}$ *where* $|\hat{D}| = A$ *and*

$$\hat{D} = \{x \in \Omega : \; f(x) \geq t\},$$

$$t = \sup\{s \in \mathbb{R} : \; |\{x \in \Omega : f(x) \geq s\}| \geq A\}.$$

Theorem

For every $\varrho_0 \in \mathcal{R}$ *there exists a function* $\varrho_1 \in \mathcal{R}$ *such that*

$$\lambda_{\varrho_0} \geq \lambda_{\varrho_1}.$$

Particularly, we have

$$\lambda_{\varrho_0} > \lambda_{\varrho_1},$$

if

$$\int_\Omega \varrho_0 u_0^p dx < \int_\Omega \varrho_1 u_0^p dx,$$

where u_0 *is the eigenfunction of* (1.3) *corresponding to* ϱ_0*.*

Proof.

Set $f(x) = u_0^p(x)$ in the bathtub principle (maximization), then one can achieve function ϱ_1 uniquely in $\mathcal{R}$ such that

$$\int_\Omega \varrho_0 u_0^p dx \le \int_\Omega \varrho_1 u_0^p \, dx.$$

Hence, we observe that

$$\frac{\int_\Omega |\nabla u_0|^p dx}{\int_\Omega \varrho_0 u_0^p dx} \ge \frac{\int_\Omega |\nabla u_0|^p dx}{\int_\Omega \varrho_1 u_0^p dx},$$

and applying (1.4)

$$\lambda_{\varrho_0} \ge \lambda_{\varrho_1}.$$

□

Numerical Algorithm

- Given $\varrho_n \in \mathcal{P}$ use the inverse power method to obtain λ_{ϱ_n} and u_n.
- Based upon the level sets of the eigenfunction u_n we extract a new density function $\varrho_{n+1} \in \mathcal{P}$ such that $\lambda_{\varrho_n} \geq \lambda_{\varrho_{n+1}}$.
- Identify $\varrho_{n+1} = \alpha\chi_{D_{n+1}} + \beta\chi_{D^c_{n+1}}$ by setting $f(x) = u_n^p(x)$.
- Recall that $D_{n+1} = \{x \in \Omega : \ f(x) \geq t\}$, and the problem is to determine the parameter t
- introduce the function $F(s) = |\{x \in \Omega : \ f(x) \geq s\}|$ for all $s \geq 0$. Applying the idea of the bisection method for $F(s)$

Algorithm 1. Eigenvalue minimization

Data: An initial density function ϱ_0

Result: Densities $\{\varrho_n\}_1^\infty$ and decreasing eigenvalues $\{\lambda_{\varrho_n}\}_1^\infty$

1. Set $n = 0$;
2. Insert ϱ_n in (1.3) and compute u_n and λ_{ϱ_n} invoking the algorithm in [F. Bozorgnia, (2016)];
3. Compute ϱ_{n+1} applying bathtub principle (maximization);
4. If $(\lambda_{\varrho_n} - \lambda_{\varrho_{n+1}}) < TOL$ then stop;
   else
   Set $n = n + 1$;
   Go to step 2;

Theorem

Consider the sequence of eigenvalues $\{\lambda_{\varrho_n}\}_1^\infty$ *generated by Algorithm 1. We have*

$$\lim_{n\to\infty} \lambda_{\varrho_n} = \lambda_{\hat{\varrho}}, \quad \textit{and} \quad \lim_{n\to\infty} \|\varrho_n - \hat{\varrho}\|_{L^p(\Omega)} = 0,$$

where $\hat{\varrho}$ *is a step function in* $\mathcal{R}$*. Moreover,* $\hat{\varrho}$ *is a local minimizer of the function* λ_ϱ *with respect to* $\mathcal{R}$*.*

Example

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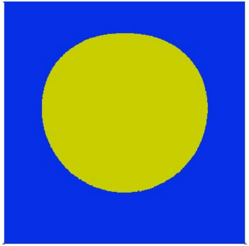
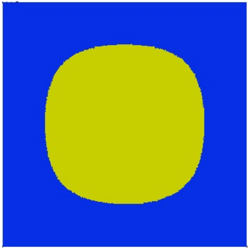
$$\varrho_0 = \begin{cases} 2 & 0 < x_1 < 1, \\ 1 & 1 < x_1 < 2, \end{cases}$$

$\mathrm{TOL} = 5 \times 10^{-3}$, $\alpha = 2$ and $\beta = 1$.

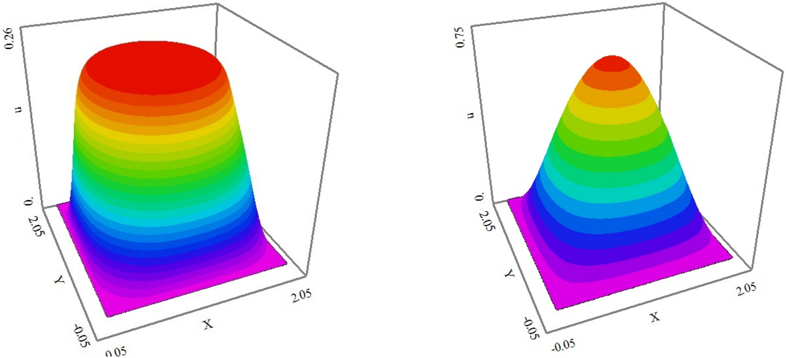


(a) $p = 1.1,\ \hat{\lambda} = 1.40$



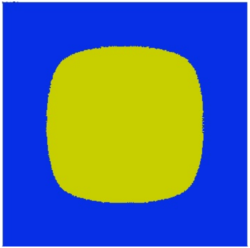
(b) $p = 2,\ \hat{\lambda} = 2.55$

Figure: The minimizer sets corresponding to different values of p are in yellow and $\hat{\lambda}$ is the associated optimal eigenvalue.

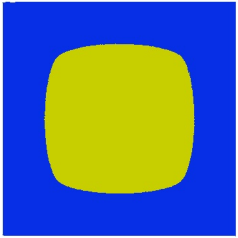


(a) $p = 1.1$ (b) $p = 2$

Figure: The eigenfunctions $\hat{u}$ corresponding to the optimal sets for different values of p.



(a) $p = 5,\ \hat{\lambda} = 7.25$



(b) $p = 10,\ \hat{\lambda} = 17.48$

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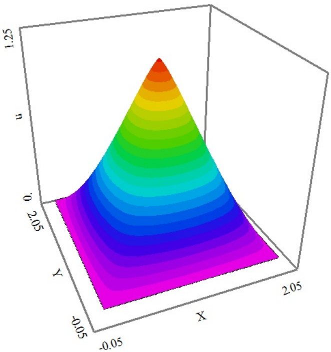
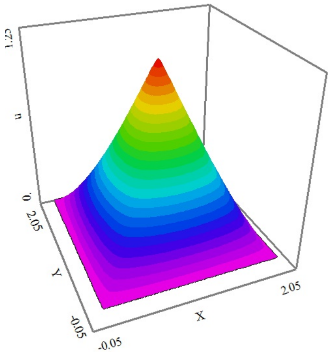


(a) $p = 5$



(b) $p = 10$

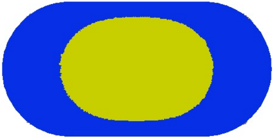
Figure: The eigenfunctions $\hat{u}$ corresponding to the optimal sets for different values of p.

Example

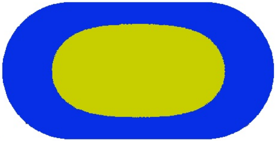
Let Ω be a stadium which is defined as the set of points at distance less than 1 from the line segment joining points $(-1,0)$ and $(0,1)$. We know that $|\Omega| = \pi + 4$ and we set $A = 4$. The initial guess for our algorithm is chosen as follows

$$\varrho_0 = \begin{cases} 2 & D_0, \\ 1 & D_0^c, \end{cases}$$

where $D_0 = \{(x_1, x_2) \in \mathbb{R}^2 : \ -1 < x_1 < 1, \ -1 < x_2 < 1\}$.

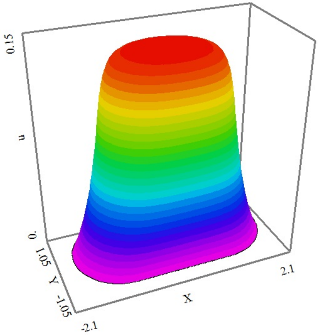


(a) $p = 1.1,\ \hat{\lambda} = 1.02$

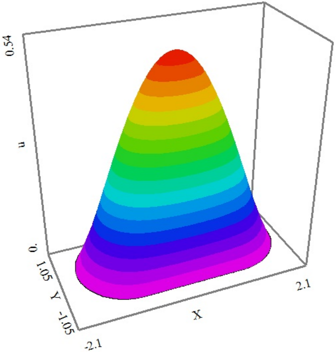


(b) $p = 2,\ \hat{\lambda} = 1.63$

Figure: The minimizer sets corresponding to different values of p are in yellow and $\hat{\lambda}$ is the associated optimal eigenvalue.

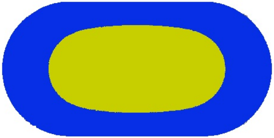


(a) $p = 1.1$

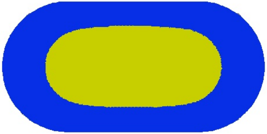


(b) $p = 2$

Figure: The eigenfunctions $\hat{u}$ corresponding to the optimal sets for different values of p.

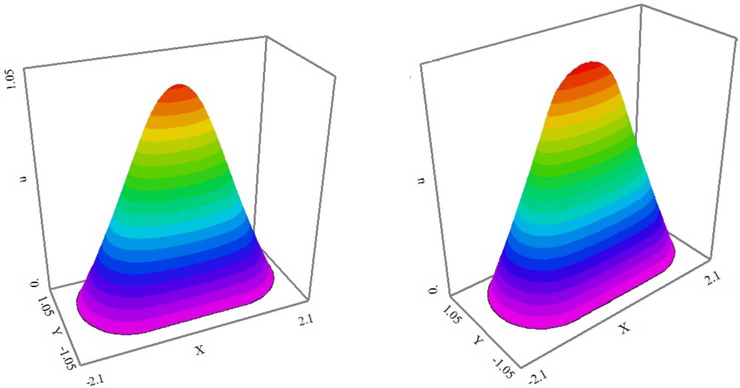


(a) $p = 5,\ \hat{\lambda} = 3.31$



(b) $p = 10,\ \hat{\lambda} = 5.41$

Figure: The minimizer sets corresponding to different values of p are in yellow and $\hat{\lambda}$ is the associated optimal eigenvalue.



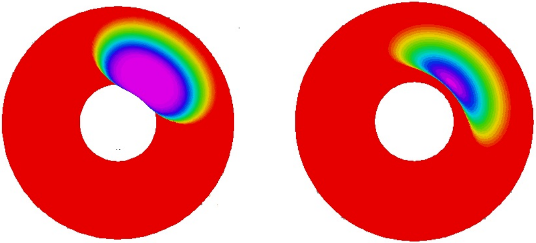
(a) $p = 5$ (b) $p = 10$

Figure: The eigenfunctions $\hat{u}$ corresponding to the optimal sets for different values of p.

Example

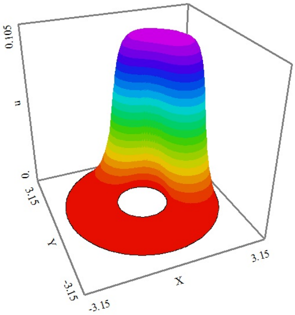
Let $O = (0,0)$ and Ω be the annulus $\mathcal{B}(O,3) \setminus \mathcal{B}(O,1)$. Note that $|\Omega| = 8\pi$ and we set $A = 1.62\pi$. Consider $q_1 = (2,0)$ and $q_2 = (0,2)$, two points in Ω. Algorithm 1 is started with the following density function

$$\varrho_0 = \begin{cases} 2 & x \in \mathcal{B}(q_1, 0.9) \cup \mathcal{B}(q_2, 0.9), \\ 1 & \text{otherwise.} \end{cases}$$

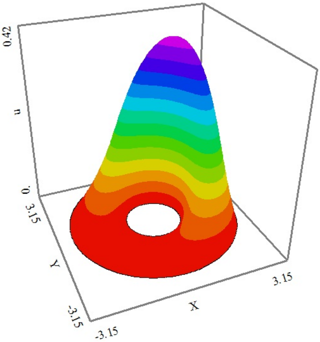


(a) $p = 1.1$, $\hat{\lambda} = 0.90$ (b) $p = 2$, $\hat{\lambda} = 1.40$

Figure: The compliment of the minimizer sets, $\hat{D}^c$, corresponding to different values of p are in red and $\hat{\lambda}$ is the associated optimal eigenvalue.

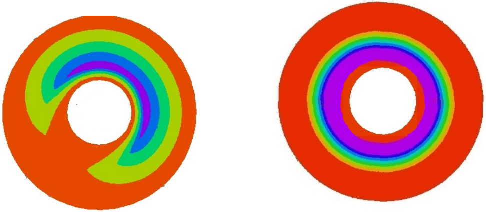


(a) $p = 1.1$



(b) $p = 2$

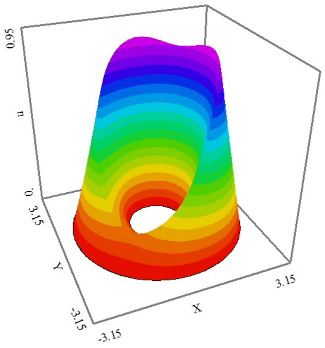
Figure: The eigenfunctions $\hat{u}$ corresponding to the optimal sets for different values of p.



(a) $p = 10, \ \hat{\lambda} = 5.20$

(b) $p = 15, \ \hat{\lambda} = 8.17$

Figure: The minimizer sets, $\hat{D}^c$, corresponding to different values of p are in red and $\hat{\lambda}$ is the associated optimal eigenvalue.



(a) $p = 10$ (b) $p = 15$

Figure: The eigenfunctions $\hat{u}$ corresponding to the optimal sets for different values of p.

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Thanks for your attention