Lecture 7 28.11.16

Reminder:

* $A := \begin{pmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{pmatrix} : \mathring{R} \times R \subset L^2_\Lambda \to L^2_\Lambda$

  is linear, densely defined and closed,

* $L^2_\Lambda := L^2_\varepsilon \times L^2_\mu$, $\operatorname{rot} = \mathring{\operatorname{rot}}^*$

* $M := i\Lambda^{-1}A : \mathring{R} \times R \subset L^2_\Lambda \to L^2_\Lambda$ is linear, densely defined, closed and selfadjoint

* $\mathcal{M} : D(\mathcal{M}) = D(M) \cap \overline{R(M)} \subset \overline{R(M)} \to \overline{R(M)}$,

  $R(M) = \varepsilon^{-1}\operatorname{rot} R \times \mu^{-1}\mathring{\operatorname{rot}}\mathring{R}$

We are still interested in solving:

$$(M - \lambda)x = f$$

$\hookrightarrow$ $\lambda = 0$: (static case) ✓

$\hookrightarrow$ $\lambda \in \mathbb{C}\setminus\mathbb{R}$: ✓, since $\mathbb{C}\setminus\mathbb{R} \subset \rho(M)$.

<u>Left with</u>: $\boxed{\lambda \in \mathbb{R}\setminus\{0\}}$

Starting with $f \in L^2_\Lambda$ we are looking for $x \in D(M)$ with $(M-\lambda)x = f$.

<u>idea</u>: decompose x and f

* $L^2_\Lambda = N(M) \oplus \overline{R(M)}$

  $\rightsquigarrow$ $f = f_0 + f_R \in N(M) \oplus \overline{R(M)}$

* $D(M) = N(M) \oplus D(\mathcal{M})$

  $\rightsquigarrow$ $x = x_0 + x_R \in N(M) \oplus D(\mathcal{M})$

Then we get:

$$(\mathcal{U}-\lambda)x = f$$

$$\Leftrightarrow (\mathcal{U}-\lambda)x_0 + (\mathcal{U}-\lambda)x_R = f_0 + f_R$$

$$\Leftrightarrow -\lambda x_0 - f_0 = f_R - (\mathcal{U}-\lambda)x_R$$

$\overline{R(\mathcal{U})} = N(\mathcal{U})^{\perp}$

$$\Leftrightarrow \begin{cases} -\lambda x_0 = f_0 \in N(\mathcal{U}), & x_0 \in N(\mathcal{U}) \\ (\mathcal{U}-\lambda)x_R = f_R \in \overline{R(\mathcal{U})}, & x_R \in D(\mathcal{U}) \end{cases}$$

Since $x_0 := -\lambda^{-1} f_0 \in N(\mathcal{U})$ is a solution to the first problem, we just have to deal with the second one:

$$(\mathcal{U}-\lambda)x_R = f_R \in \overline{R(\mathcal{U})}, \quad x_R \in D(\mathcal{U})$$

<u>crucial assumption</u>: $\boxed{D(\mathcal{U}) \hookrightarrow L^2}$

$\Rightarrow$ $R(\mathcal{U})$ is closed, Poincaré-estimates and $\mathcal{U}^{-1}: R(\mathcal{U}) \to D(\mathcal{U})$ is continuous

<u>first idea</u>: apply $\mathcal{U}^{-1}$

$$(\mathcal{U}-\lambda)x_R = f_R$$
$$\Leftrightarrow (1/\lambda \, \mathrm{id} - \mathcal{U}^{-1})x_R = 1/\lambda \, \mathcal{U}^{-1} f_R$$

Now $\mathcal{U}^{-1}: R(\mathcal{U}) \to R(\mathcal{U})$ is compact and we can use Fredholms alternative.

<u>alternative idea</u>: use toolbox

Look at: $\mathcal{U}_\lambda : D(\mathcal{U}_\lambda) = D(\mathcal{U}) \subset H \to H$,
$\mathcal{U}_\lambda := \mathcal{U} - \lambda$

still we assume: $\boxed{D(\mathcal{M}) \hookrightarrow H}$

and again we have:

$$\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) \subset R(M_\lambda) \to R(M_\lambda)$$
$$\text{with } D(\mathcal{M}_\lambda) = D(M) \cap R(M_\lambda)$$

<u>But</u>: Is $D(\mathcal{M}_\lambda) \hookrightarrow H$?

$\rightsquigarrow$ Yes it is, but it has to be proven!

To avoid this „gap", we instead start with another operator:

$$M_\lambda : D(M_\lambda) = D(\mathcal{M}) \subset R(M) \to R(\mathcal{M})$$
$$\Rightarrow \mathcal{M}_\lambda : D(\mathcal{M}_\lambda) \subset R(M_\lambda) \to R(\mathcal{M}_\lambda)$$
$$\text{with } D(\mathcal{M}_\lambda) = D(\mathcal{M}) \cap R(M_\lambda)$$

Of course we have to check if M_λ is of the „<u>right kind</u>":

* M_λ is linear: clear!
* M_λ is densely defined:

  Take $f \in R(M) \subset H$. Since $D(M)$ is dense in H there exists $(f_n)_{n\in\mathbb{N}} \subset D(M)$, $f_n \to f$ in H. By $D(M) = N(M) \oplus D(\mathcal{M})$ we get for $n \in \mathbb{N}$:

  $$f_n = f_n^1 + f_n^2 \in N(M) \oplus D(\mathcal{M})$$

  Then clearly $f_n^1 \to 0$ in H and $f_n^2 \to f$ in H.

* $\mathcal{M}_\lambda$ is closed:

Take $(x_n)_n \subset D(\mathcal{M})$, $x_n \to x$ in $R(\mathcal{M})$ and $\mathcal{M}_\lambda x_n \to f$ in $R(\mathcal{M})$

$\Rightarrow \mathcal{M} x_n = \mathcal{M}_\lambda x_n + \lambda x_n \to f + \lambda x$

Since $\mathcal{M}$ is closed we have $x \in D(\mathcal{M})$ and $\mathcal{M} x = f + \lambda x$

$\Rightarrow x \in D(\mathcal{M})$ and $\mathcal{M}_\lambda x = f$.

Since $D(\mathcal{M}_\lambda) = D(\mathcal{M}) \cap R(\mathcal{M}_\lambda) \subset D(\mathcal{M})$ we have:

$$\boxed{D(\mathcal{M}_\lambda) \hookrightarrow R(\mathcal{M})}$$

$\Rightarrow R(\mathcal{M}_\lambda)$ is closed and with

$R(\mathcal{M}) = N(\mathcal{M}_\lambda) \oplus R(\mathcal{M}_\lambda)$

$\hookrightarrow D(\mathcal{M}) = D(\mathcal{M}_\lambda)$

$= N(\mathcal{M}_\lambda) \oplus [D(\mathcal{M}) \cap R(\mathcal{M}_\lambda)]$

we have:

$\forall x \in D(\mathcal{M}_\lambda) = D(\mathcal{M}) \cap R(\mathcal{M}_\lambda)$:

$$|x|_H \leq c\, |\mathcal{M}_\lambda x|_H$$

Furthermore:

$\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) \subset R(\mathcal{M}_\lambda) \cap R(\mathcal{M}) \to R(\mathcal{M}_\lambda) \cap R(\mathcal{M})$

$\wedge\ \mathcal{M}_\lambda^{-1} : R(\mathcal{M}_\lambda) \cap R(\mathcal{M}) \to D(\mathcal{M}_\lambda)$ is continuous.

$\wedge\ \mathcal{M}_\lambda^{-1} : R(\mathcal{M}_\lambda) \cap R(\mathcal{M}) \to R(\mathcal{M}_\lambda) \cap R(\mathcal{M})$ is compact.

Thus we just proved:

Theorem: (Fredholms alternative)

Let $f \in R(\mathcal{M})$. Then $\mathcal{M}_\lambda x = f$ is solvable in $D(\mathfrak{M})$, iff $f \perp N(\mathcal{M}_\lambda)$. Choosing e.g. $x \perp N(\mathcal{M}_\lambda)$ makes x unique.

Lemma:

$\sigma_p(\mathcal{M}) \setminus \{0\} = \sigma(\mathcal{M}) \setminus \{0\} = \sigma(\mathfrak{M}) = \sigma_p(\mathfrak{M}) \subset \mathbb{R} \setminus \{0\}$ is discrete and $\sigma(\mathfrak{M})$ can only accumulate at ∞. All $N(\mathcal{M}_\lambda)$ are finite dimensional.

Proof:

$0 \in \rho(\mathfrak{M})$, since $\mathfrak{M}^{-1}$ is continuous. Then ($0 \in \sigma_p(\mathcal{M})$ or $0 \in \rho(\mathfrak{M})$) we have $0 \notin \sigma_p(\mathcal{M})$. Furthermore:

$$
\begin{aligned}
0 \neq \lambda \notin \sigma_p(\mathcal{M}) &\Rightarrow N(\mathcal{M}_\lambda) = 0 \\
&\Rightarrow R(\mathcal{M}_\lambda) = R(\mathcal{M}) \\
\mathcal{M}_\lambda^{-1} \text{ is contin.} &\Rightarrow \mathfrak{M}_\lambda = \mathcal{M}_\lambda \\
&\Rightarrow \lambda \in \rho(\mathcal{M})
\end{aligned}
$$

If $|N(\mathcal{M}_\lambda)| = \infty$ or there is an accum. point of $\sigma(\mathfrak{M})$ in $\mathbb{R}$ we find $(x_n)_{n \in \mathbb{N}}$ orthonormalsequence in $N(\mathcal{M}_{\lambda_n})$ for $(\lambda_n)_{n \in \mathbb{N}}$, $\lambda_n \to \lambda$.

⌈ clear for $|N(\mathcal{M}_\lambda)| = \infty$. If $\sigma(\mathfrak{M})$ has an accumulation point λ, we can choose a sequence $(\lambda_n)_{n \in \mathbb{N}}$, $\lambda_n \neq \lambda_m$ for $n \neq m$ with $\lambda_n \to \lambda$, and a corresponding sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in N(\mathcal{M}_{\lambda_n})$. Then:

$$\lambda_n < x_n, x_m >$$
$$= < \mathcal{M} x_n, x_m >$$
$$= < x_n, \mathcal{M} x_m > = \lambda_m < x_n, x_m >$$
$$\Rightarrow (\lambda_n - \lambda_m) < x_n, x_m > = 0$$
$$\Rightarrow < x_n, x_m > = 0$$

$\Rightarrow$ $(x_n)_n$ is orthogonalsequence. ⌋

As an orthonormalsequence $x_n \to 0$ in H resp. $R(\mathcal{M})$. Now $x_n \in D(\mathcal{M})$, $|x_n|_H = 1$ and therefore:

$$|\mathcal{M} x_n| = |\mathcal{M}_\lambda x_n + \lambda_n x_n|$$
$$\leq |\lambda_n| |x_n| \leq |\lambda|,$$

which means $(x_n)_{n\in\mathbb{N}}$ is bounded in $D(\mathcal{M})$.
$D(\mathcal{M}) \hookrightarrow\hookrightarrow H \Rightarrow \exists$ subsequence $(x_{\pi(n)})_{n\in\mathbb{N}}$ with $x_{\pi(n)} \to x = 0$ in H. ↯

$\boxed{\xi}$

Weck's selection theorem or the Maxwell compactness property:

Theorem:

$\Omega \subset \mathbb{R}^3$ bounded and strong lipschitz, $\varepsilon : \Omega \to \mathbb{R}^{3\times3}$ uniformly positive definite, symmetric and with L^∞-coefficients.

Then:

$$\mathring{R}(\Omega) \cap \varepsilon^{-1} D(\Omega) \hookrightarrow L^2(\Omega)$$
$$R(\Omega) \cap \varepsilon^{-1} \mathring{D}(\Omega) \hookrightarrow L^2(\Omega)$$

<u>Ideas for a proof</u>:

'50/'60: $\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow H^1 \hookrightarrow L^2$

need: Ω, ε smooth

'74: Weck for strong Lipschitz (in $\mathbb{R}^n$ or manifolds)

'84: Weber for strong Lipschitz in $\mathbb{R}^3$ with potentials

We will follow the idea of Weber.

<u>Lemma n</u>:

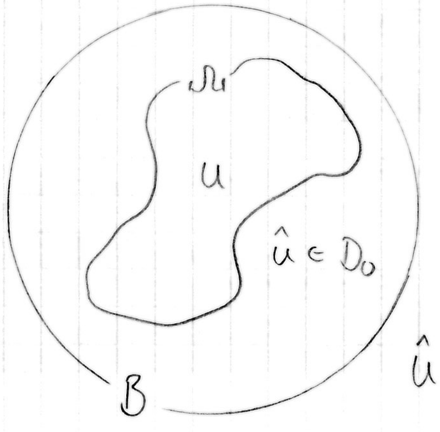
$\Omega \subset \mathbb{R}^3$ bd., str. Lipschitz and topological trivial, i.e. simply connected and $\partial\Omega$ is connected ($\partial\Omega$ connected is enough). Then $D_0 = \operatorname{rot} H^1$ with continuous potential, i.e. $\forall\, u \in D_0\ \exists\, \phi \in H^1$:

$$\operatorname{rot} \phi = u \quad \text{and} \quad |\phi|_{H^1} \le c\,|u|_{L^2}$$

<u>Proof</u>:

„$\supset$": $\operatorname{rot} H^1 \subset \operatorname{rot} R \subset D_0$

„$\subset$": let $u \in D_0(\Omega)$



<u>wish</u>: Extend u by $\hat{u} \in D_0(\mathbb{R}^3)$

<u>idea</u>: Extend u by $\hat{u} \in \mathring{D}_0(B)$

Assume we have constructed $\hat{u} \in \mathring{D}_0(B)$.
Then for all $\varphi \in H^1(B)$ we have

$$
\begin{aligned}
0 &= \langle \hat{u}, \nabla\varphi \rangle_{L^2(B)} \\
&= \langle u, \nabla\varphi \rangle_{L^2(\Omega_i)} + \langle \hat{u}, \nabla\varphi \rangle_{L^2(B\setminus\overline{\Omega_i})}
\end{aligned}
$$

So we make the ansatz:

$\hat{u} = \nabla v$, $v \in H^1_\perp(B\setminus\overline{\Omega_i})$:

$\forall \psi \in H^1_\perp(B\setminus\overline{\Omega_i})$:

$$\langle \nabla v, \nabla\psi \rangle_{L^2(B\setminus\overline{\Omega_i})} = -\langle u, \nabla(\mathcal{E}\psi)\rangle_{L^2(\Omega_i)}$$

where $\mathcal{E}: H^1(B\setminus\overline{\Omega_i}) \to H^1(B)$ is an extension operator (Calderon / Stein, ...)

⌜ need: $B\setminus\overline{\Omega_i}$ strong Lipschitz, connected. But this is given, since Ω_i is topological trivial (resp. $\partial\Omega_i$ is connected) and therefore $B\setminus\overline{\Omega}$ a domain. ⌟

$\Rightarrow$ By Riesz we get: $\exists!\, v \in H^1_\perp(B\setminus\overline{\Omega_i})$.

Then we define:

$$\hat{u} := \begin{cases} u &, \text{ in } \Omega_i \\ \nabla v &, \text{ in } B\setminus\overline{\Omega_i} \\ 0 &, \text{ in } \mathbb{R}^3\setminus\overline{B} \end{cases}$$

Now pick $\varphi \in \mathring{C}^\infty(\mathbb{R}^3)$ and define

$\psi := \varphi - |\Omega_i|^{-1}\langle \varphi, 1\rangle_{L^2(B\setminus\overline{\Omega_i})} \in H^1_\perp(B\setminus\overline{\Omega_i})$

Then $\nabla\psi = \nabla\varphi$ and we have:

$$\langle \hat{u}, \nabla\varphi \rangle_{L^2(\mathbb{R}^3)}$$
$$= \langle \nabla v, \nabla\varphi \rangle_{L^2(B\setminus\overline{\Omega_1})} + \langle u, \nabla\varphi \rangle_{L^2(\Omega_1)}$$
$$= \langle \nabla v, \nabla\psi \rangle_{L^2(B\setminus\overline{\Omega_1})} + \langle u, \nabla\psi \rangle_{L^2(\Omega_1)}$$
$$= \langle u, \nabla(\psi - \mathcal{E}\psi) \rangle_{L^2(\Omega_1)}$$
$$= - \langle \operatorname{div} u, \psi - \mathcal{E}\psi \rangle_{L^2(\Omega_1)} = 0$$

⌜ beachte:

$\psi - \mathcal{E}\psi \in H^1(B)$ und

$\psi - \mathcal{E}\psi = 0$ in $B\setminus\overline{\Omega_1}$

$\Rightarrow \psi - \mathcal{E}\psi \in \mathring{H}^1(\Omega_1)$ ⌟

$\Rightarrow$ $\hat{u} \in D_0(\mathbb{R}^3)$ and furthermore $\hat{u} \in D_0(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ since $\operatorname{supp} \hat{u} \subset \overline{B}$

Now we define $\Phi := \xi/r^2 \times \mathcal{F}(\hat{u})$ with $\xi(x) = x$, $r(x) = |x|$ and

$$\mathcal{F}^{\pm 1}(v)(x) = \int_{\mathbb{R}^3} e^{\mp ixy} v(y)\, dy$$

Now: $\mathcal{F}(\hat{u}) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, since

$$|\mathcal{F}(\hat{u})|(x) \leq c\, |\hat{u}|_{L^1(\mathbb{R}^3)}$$

$\Rightarrow$ $\Phi \in L^2(\mathbb{R}^3\setminus\overline{B}_1) \cap L^2(B_1) = L^2(\mathbb{R}^3)$, since:

$$|\Phi|_{L^2(B_1)} \le c \int_0^1 r^{-2} r^2 dr \le c.$$

$$\Rightarrow r\Phi \in L^2(\mathbb{R}^3) \Leftrightarrow \mathcal{F}^{-1}(\Phi) \in H^1(\mathbb{R}^3).$$

We define: $\psi := -i\mathcal{F}^{-1}(\Phi)$

Now remember that:

$$\partial_n(\mathcal{F}^{\pm 1}(v)) = \mp i \mathcal{F}^{\pm 1}(\xi_n v)$$

$$\xi_n(\mathcal{F}^{\pm 1}(v)) = \mp i \mathcal{F}^{\pm 1}(\partial_n u)$$

$$\Rightarrow \quad \mathcal{F}^{\pm 1}(\operatorname{rot} v) = \pm i \xi \times \mathcal{F}^{\pm 1}(v),$$

$$\mathcal{F}^{\pm 1}(\operatorname{div} v) = \pm i \xi \cdot \mathcal{F}^{\pm 1}(v),$$

$$\mathcal{F}^{\pm 1}(\xi \times v) = \pm i \operatorname{rot} \mathcal{F}^{\pm 1}(v),$$

$$\mathcal{F}^{\pm 1}(\xi \cdot v) = \pm i \operatorname{div} \mathcal{F}^{\pm 1}(v)$$

and therefore

$*$ $\operatorname{div} \mathcal{F}^{-1}(\Phi) = i\mathcal{F}^{-1}(\xi \cdot \Phi) \underset{\uparrow}{=} 0$

$\xi \cdot \Phi = 0$

$*$
$$\begin{aligned}
\operatorname{rot} \psi &= -i \operatorname{rot} \mathcal{F}^{-1}(\Phi) \\
&= \mathcal{F}^{-1}(\xi \times \Phi) \\
&= \mathcal{F}^{-1}(\xi \times (\xi/r^2 \times \mathcal{F}(\hat{u}))) \\
&= \mathcal{F}^{-1}(\mathcal{F}(\hat{u}) - (\xi \cdot \mathcal{F}(\hat{u}))\xi/r^2) \\
&= \hat{u} - i\mathcal{F}^{-1}(\mathcal{F}(\operatorname{div} \hat{u})\xi/r^2) \\
&= \hat{u}
\end{aligned}$$

$$\Rightarrow \psi \in H^1(\mathbb{R}^3) \cap D_0(\mathbb{R}^3), \quad \operatorname{rot} \psi = \hat{u}$$

in Ω_1: $\psi \in H^1(\Omega_1) \wedge \operatorname{rot} \psi = \hat{u} = u$

Finally:

$$T : D_0(\Omega_1) \to H^1(\mathbb{R}^3) \cap D_0(\mathbb{R}^3)$$
$$u \mapsto -i\,\mathcal{F}^{-1}\left(\xi/r^2 \times \mathcal{F}(\hat{u})\right)$$

suffices rot $Tu = u$ in Ω_1 and

$$|Tu|_{H^1(\mathbb{R}^3)}$$
$$\leq |Tu|_{L^2(\mathbb{R}^3)} + |\nabla Tu|_{L^2(\mathbb{R}^3)}$$
$$\leq |\xi/r^2 \times \mathcal{F}(\hat{u})|_{L^2(\mathbb{R}^3)} + |\nabla \mathcal{F}^{-1}(\xi/r^2 \times \mathcal{F}(\hat{u}))|_{L^2(\mathbb{R}_3)}$$
$$\leq |\xi/r^2 \times \mathcal{F}(\hat{u})|_{L^2(\mathbb{R}^3)} + |\xi/r \times \mathcal{F}(\hat{u})|_{L^2(\mathbb{R}^3)}$$
$$\leq |\xi/r^2 \times \mathcal{F}(\hat{u})|_{L^2(B_1)} + |\hat{u}|_{L^2(\mathbb{R}^3)}$$
$$\leq c\,|\mathcal{F}(\hat{u})|_{L^\infty(\mathbb{R}_3)} + |\hat{u}|_{L^2(\mathbb{R}^3)}$$
$$\leq c\,|\hat{u}|_{L^1(\mathbb{R}^3)} + |\hat{u}|_{L^2(\mathbb{R}^3)}$$
$$\leq c\,|\hat{u}|_{L^2(B)}$$
$$\leq c\left(|u|_{L^2(\Omega_1)} + |\nabla v|_{L^2(B \setminus \overline{\Omega_1})}\right)$$
$$\leq c\,|u|_{L^2(\Omega_1)}$$

$\ulcorner$ $$\langle \nabla v, \nabla \varphi \rangle_{L^2(B \setminus \overline{\Omega_1})} = -\langle u, \nabla(\mathcal{E}\varphi)\rangle_{L^2(\Omega_1)}$$

$$\Rightarrow |\nabla v|^2_{L^2(B\setminus\overline{\Omega})}$$

$$\leq |u|_{L^2(\Omega)} |\nabla(\mathcal{E}v)|_{L^2(\Omega)}$$

$$< |u|_{L^2(\Omega)} |\mathcal{E}v|_{H^1(B)}$$

continuous extension

$$\leq |u|_{L^2(\Omega)} |v|_{H^1(B\setminus\overline{\Omega})}$$

$v \in H^1_\perp(B\setminus\overline{\Omega})$

$$\leq |u|_{L^2(\Omega)} |\nabla v|_{L^2(B\setminus\overline{\Omega})} \quad \lrcorner$$

$\boxed{4}$

Lecture 8 28.11.16

<u>Lemma n+1</u>:

$\Omega \subset \mathbb{R}^3$ bd., str. Lipschitz and topological trivial (simply connected is enough). Then $\mathring{D}_0 = \operatorname{rot} \mathring{H}^1$ with continuous potential, i.e. $\forall u \in \mathring{D}_0 \; \exists \phi \in \mathring{H}^1$:

$\operatorname{rot} \phi = u \;\land\; |\phi|_{H_1} \leq c\,|u|_{L^2}$

<u>Proof</u>:

„$\supset$": $\operatorname{rot} \mathring{H}^1 \subset \operatorname{rot} \mathring{R} \subset \mathring{D}_0$

„$\subset$": Take $u \in \mathring{D}_0(\Omega)$. We define

$$\hat{u} := \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^3\setminus\overline{\Omega} \end{cases} \Rightarrow \hat{u} \in D_0(\mathbb{R}^3)$$

(cf. exercise 3)

As in the proof of lemma n we get $\psi \in H^1(\mathbb{R}^3)$, $\operatorname{rot}\psi = \hat{u}$ in $\mathbb{R}^3$,

$|\psi|_{H^1(\mathbb{R}^3)} \leq c\,|u|_{L^2(\Omega)}$

In $B\setminus\overline{\Omega}$ we have: $\operatorname{rot}\psi = 0$

Since $B \setminus \overline{\Omega_i}$ is simply connected, we have:

$$\psi = \nabla v, \quad v \in H^1_1(B \setminus \overline{\Omega_i})$$
$$\Rightarrow v \in H^2(B \setminus \overline{\Omega_i})$$

$\psi \in H^1(\mathbb{R}^3)$

Now we extend v to $\hat{v} \in H^2(\mathbb{R}^3)$ (again with Calderon / Stein ...)

⌜ now: $B \setminus \overline{\Omega_i}$ is simply connected since Ω_i is simply connected. ⌟

We define: $\phi := \psi - \nabla \hat{v} \in H^1(\mathbb{R}^3)$

$$\Rightarrow \operatorname{rot} \phi = \operatorname{rot} \psi = \hat{u} \text{ in } \mathbb{R}^3$$

and in $B \setminus \overline{\Omega_i}$:

$$\phi = \psi - \nabla \hat{v} = \psi - \nabla v = 0$$

$$\Rightarrow \phi \in H^1(B), \ \phi = 0 \text{ in } B \setminus \overline{\Omega_i}$$
$$\Rightarrow \phi \in \mathring{H}^1(\Omega_i)$$

Finally

$$S: \mathring{D}_0(\Omega_i) \to H^1(\mathbb{R}^3) \cap \mathring{H}^1(\Omega_i)$$
$$u \mapsto \psi - \nabla \hat{v}$$

suffices $\operatorname{rot} Su = u$ in Ω_i and

$$|Su|_{H^1(\mathbb{R}^3)}$$
$$\leq c \left(|\psi|_{H^1(\mathbb{R}^3)} + |\hat{v}|_{H^2(\mathbb{R}^3)} \right)$$
$$\leq c \left(|\psi|_{H^1(\mathbb{R}^3)} + |v|_{H^2(B \setminus \overline{\Omega_i})} \right)$$

$$\leq c\left(|\psi|_{H^1(\mathbb{R}^3)} + |\nabla v|_{H^1(B\setminus\overline{\Omega})}\right)$$

$v \in H^1_\perp(B\setminus\overline{\Omega})$

$$\leq c\,|\psi|_{H^1(\mathbb{R}^3)} \leq c\cdot |u|_{L^2(\Omega)}.$$

$\boxed{\varepsilon}$

Corollary:

$\Omega \subset \mathbb{R}^3$ bd., strong lipschitz and topological trivial. Then:

$$\mathcal{D}_\varepsilon = \mathring{R}_0 \cap \varepsilon^{-1} D_0 = \{0\}$$
$$\mathcal{N}_\mu = R_0 \cap \mu^{-1}\mathring{D}_0 = \{0\}$$

Furthermore:

$$\varepsilon^{-1} D_0 = \varepsilon^{-1}\operatorname{rot} R \oplus_{L^2_\varepsilon} \mathcal{D}_\varepsilon = \varepsilon^{-1}\operatorname{rot} H^1$$
$$\mu^{-1}\mathring{D}_0 = \mu^{-1}\operatorname{rot}\mathring{R} \oplus_{L^2_\mu} \mathcal{N}_\mu = \mu^{-1}\operatorname{rot}\mathring{H}^1$$

and

$$L^2_\varepsilon = \nabla\mathring{H}^1 \oplus_{L^2_\varepsilon} \varepsilon^{-1}\operatorname{rot} H^1$$
$$= \mathring{R}_0 \oplus_{L^2_\varepsilon} \varepsilon^{-1} D_0$$
$$L^2_\mu = \nabla H^1 \oplus_{L^2_\mu} \mu^{-1}\operatorname{rot}\mathring{R}$$
$$= R_0 \oplus_{L^2_\mu} \mu^{-1}\mathring{D}_0$$

Remark:

$\Omega \subset \mathbb{R}^3$ topological trivial. Then $\mathcal{D}_\varepsilon = \mathcal{N}_\mu = \{0\}$ can be shown by elementary calculations.

Lemma:

$\Omega \subset \mathbb{R}^3$ bd., str. Lipschitz and topological trivial. Then we have:

$$\mathring{R}(\Omega) \cap \varepsilon^{-1} D(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$
$$R(\Omega) \cap \varepsilon^{-1} \mathring{D}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$

Again with $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}$ uniformly positive definite, symmetric and with L^∞-coefficients.

Proof:

Observe that:

$$L^2_\varepsilon = \nabla \mathring{H}^1 \oplus_{L^2_\varepsilon} \varepsilon^{-1} D_0$$
$$\Rightarrow \mathring{R} \cap \varepsilon^{-1} D = (\nabla \mathring{H}^1 \cap \varepsilon^{-1} D) \oplus_{L^2_\varepsilon} (\mathring{R} \cap \varepsilon^{-1} D_0)$$

⌜ $\nabla \mathring{H}^1$ is closed by Rellich's selection theorem. ⌟

Let $(u_n)_n \subset \mathring{R} \cap \varepsilon^{-1} D$ bd. in $R \cap \varepsilon^{-1} D$

$$\Rightarrow u_n = u_{\nabla,n} + u_{0,n} \in (\nabla \mathring{H}^1 \cap \varepsilon^{-1} D) \oplus_{L^2_\varepsilon} (\mathring{R} \cap \varepsilon^{-1} D_0),$$
$$|u_{\nabla,n}|^2_{L^2_\varepsilon} + |u_{0,n}|^2_{L^2_\varepsilon} = |u_n|^2_{L^2_\varepsilon},$$
$$\operatorname{rot} u_n = \operatorname{rot} u_{0,n} \wedge \operatorname{div} u_n = \operatorname{div} u_{\nabla,n}$$

Therefore $(u_{0,n})_n \subset \mathring{R} \cap \varepsilon^{-1} D$ is bounded in $R \cap \varepsilon^{-1} D$ and also $(u_{\nabla,n})_n \subset \nabla \mathring{H}^1 \cap \varepsilon^{-1} D$ is bounded in $R \cap \varepsilon^{-1} D$. Then $u_{\nabla,n} = \nabla v_n$ for $v_n \in \mathring{H}^1$ and we have:

$$|v_n|_{L^2_\varepsilon} \le c\, |\nabla v_n|_{L^2_\varepsilon}$$
$$= c\, |u_{\nabla,n}|_{L^2_\varepsilon} \le c < \infty$$

Therefore $(v_n)_n$ is bounded in H^1 and by Rellich's selection theorem we can choose a subsequence $(v_{\pi(n)})_n$ converging in L^2. Then by

$$u_{\nabla,n,m} := u_{\nabla,n} - u_{\nabla,m}$$
$$u_{0,n,m} := u_{0,n} - u_{0,m}$$

and

$$|u_{\nabla,n,m}|^2_{L^2_\varepsilon} = \langle u_{\nabla,n,m} \,,\, \nabla v_{n,m} \rangle_{L^2_\varepsilon}$$
$$= - \langle \operatorname{div} \varepsilon u_{\nabla,n,m} \,,\, v_{n,m} \rangle_{L^2}$$
$$= - \langle \operatorname{div} \varepsilon u_{n,m} \,,\, v_{n,m} \rangle_{L^2}$$
$$\le c\, |v_{n,m}|_{L^2}$$

we get that $(u_{\nabla,\pi(n)})_n$ is a cauchy-sequence in L^2 and therefore converging. Additionally we get from Lemma n:

$$\varepsilon u_{0,n} = \operatorname{rot} \phi_n,\ \phi_n \in H^1 \text{ with}$$
$$|\phi_n|_{H_1} \le c\, |u_{0,n}|_{L^2_\varepsilon} < c$$

By Rellich's selection theorem we again get a converging subsequence

$(\phi_{\pi_2(n)})_n$ of $(\phi_{\pi(n)})_n$. Then:

$$\begin{aligned} |u_{0,\pi_2(n,m)}|^2_{L^2_\varepsilon} &= \langle u_{0,\pi_2(n,m)}, \varepsilon^{-1} \operatorname{rot} \phi_{\pi_2(n,m)} \rangle_{L^2_\varepsilon} \\ &= \langle \operatorname{rot} u_{0,\pi_2(n,m)}, \phi_{\pi_2(n,m)} \rangle_{L^2} \\ &= \langle \operatorname{rot} u_{\pi_2(n,m)}, \phi_{\pi_2(n,m)} \rangle_{L^2} \\ &\leq c\, |\phi_{\pi_2(n,m)}| \end{aligned}$$

and therefore $(u_{0,\pi_2(n)})_n$ is a cauchy-sequence in L^2. In total $(u_{\pi_2(n)})_n$ is a cauchysequence in L^2 and therefore converging in L^2.

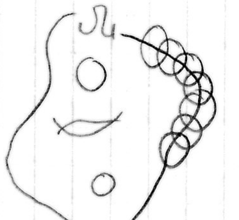
$\square$

Remark:

The second assertion follows analogously with Lemma $n+1$.

Proof of the theorem:

Take $(E_n)_n \subset \mathring{R} \cap \varepsilon^{-1} D$ bounded in $R \cap \varepsilon^{-1} D$.



Now $\forall x \in \overline{\Omega}\ \exists\, U_x := B_{r_x}(x)$ open with $x \in U_x$ and $\Omega_x := U_x \cap \Omega$ is topological trivial.

$$\Rightarrow \overline{\Omega} \subset \bigcup_{x \in \overline{\Omega}} \Omega_x \underset{\uparrow}{\Rightarrow} \Omega \subset \bigcup_{i=1}^{k} \Omega_{x_i}$$

$\overline{\Omega}$ is compact

Now let $\varphi_i \in \mathring{C}^\infty(U_{x_i})$ with

$$\sum_{i=1}^{k} \varphi_i = 1 \text{ in } \overline{\Omega}. \quad \Rightarrow \quad E_n = \sum_{i=1}^{k} \varphi_i E_n$$

Then $(\varphi_i E_n)_n \subset \overset{\circ}{R}(\Omega_{x_i}) \cap \varepsilon^{-1} D(\Omega_{x_i})$ is bounded in $L^2(\Omega_{x_i})$. Furthermore

$$\operatorname{rot}(\varphi_i E_n) = \varphi_i \operatorname{rot} E_n - \nabla \varphi_i \times E_n$$
$$\operatorname{div}(\varphi_i \varepsilon E_n) = \varphi_i \operatorname{div} \varepsilon E_n + \nabla \varphi_i \cdot \varepsilon E_n$$

$\Rightarrow$ $(\varphi_i E_n)_n$ is bounded in $R(\Omega_{x_i}) \cap \varepsilon^{-1} D(\Omega_{x_i})$

$\ulcorner$ since: $\operatorname{supp}(\varphi_i E_n) \subset \operatorname{supp}(\varphi_i) \subset U_{x_i}$ $\lrcorner$

From Lemma above we can choose a subsequence $(\varphi_1 E_{\pi_1(n)})_n$ of $(\varphi_1 E_n)_n$ converging in $L^2(\Omega_{x_1})$. Then $(\varphi_2 E_{\pi_1(n)})_n$ is also bounded in $R(\Omega_{x_2}) \cap \varepsilon^{-1} D(\Omega_{x_2})$ such that we can extract a subsequence $(\varphi_2 E_{\pi_2(n)})_n$ converging in $L^2(\Omega_{x_2})$. Continuing this procedure until $i = k$ we end up with a subsequence $(E_{\pi_k(n)})_n$ such that $(\varphi_i E_{\pi_k(n)})_n$ is converging in $L^2(\Omega_{x_i})$ for all $i \in \{1, \dots, k\}$.

$\Rightarrow$ $(E_{\pi_k(n)})_n$ is converging in $L^2(\Omega)$.

The assertion for $R \cap \varepsilon^{-1} \overset{\circ}{D}$ follows analogously.