Lecture 6 14.11.16

<u>Theorem 9</u>: (solution theory for (ESP))
Let $\mathring{\mathrm{rot}}\,\mathring{R}$, $\mathring{\nabla}\mathring{H}^1$ be closed. Then (ESP) is uniquely solvable in $\mathring{R}\cap\varepsilon^{-1}D$ iff $F\in\mathring{\mathrm{rot}}\,\mathring{R}$, $g\in L^2$, $D\in\mathcal{D}_\varepsilon$. The unique solution is given by

$$E = E_F + E_f + D \in (\mathring{R}\cap\varepsilon^{-1}\mathrm{rot}\,R)\oplus(\varepsilon^{-1}D\cap\mathring{\nabla}\mathring{H}^1)\oplus\mathcal{D}_\varepsilon$$

where

$$E_F := \mathring{\mathrm{rot}}^{-1}F \in \mathring{R}\cap\varepsilon^{-1}\mathrm{rot}\,R = \mathring{R}\cap\varepsilon^{-1}D_0\cap\mathcal{D}_\varepsilon^{\perp_{L^2_\varepsilon}}$$
$$E_f := (-\mathrm{div}\,\varepsilon)^{-1}f \in \varepsilon^{-1}D\cap\mathring{\nabla}\mathring{H}^1 = \varepsilon^{-1}D\cap\mathring{R}_0\cap\mathcal{D}_\varepsilon^{\perp_{L^2_\varepsilon}}$$

and depends continuously on the data, i.e.

$$|E|^2_{L^2_\varepsilon} \leq c_m^2\,|F|^2_{L^2} + c_F^2\,|f|^2_{L^2} + |D|_{L^2_\varepsilon}$$
$$|E|^2_{\mathring{R}\cap\varepsilon^{-1}D} \leq (1+c_m^2)\,|F|^2_{L^2} + (1+c_F^2)\,|f|^2_{L^2} + |D|^2_{L^2_\varepsilon}.$$

Furthermore: $E_F = \pi_{\varepsilon^{-1}\mathrm{rot}}E$, $E_f = \pi_{\mathring{\nabla}}E_0$

<u>Remark</u>:

① E_F, E_f solve

$$\begin{aligned} \mathring{\mathrm{rot}}\,E_F &= F & \mathring{\mathrm{rot}}\,E_f &= 0 \\ -\mathrm{div}\,\varepsilon E_F &= 0 & -\mathrm{div}\,\varepsilon E_f &= f \\ \pi_D E_F &= 0 & \pi_D E_f &= 0 \end{aligned}$$

② Again:
$\mathring{\mathrm{rot}}\,\mathring{R}$ is closed $\Leftrightarrow$ $\mathrm{rot}\,R$ is closed

$\mathring{\nabla}\mathring{H}^1$ is closed $\Leftrightarrow$ div $\mathbb{D}$ is closed

Both we get e.g. from

$$\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow L^2_\varepsilon$$

$$\Rightarrow \begin{cases} |\mathcal{D}_\varepsilon| < \infty \\ \mathring{\mathrm{rot}}^{-1}, \mathrm{rot}^{-1}, \mathring{\nabla}^{-1}, \nabla^{-1}, \mathrm{div}^{-1}, \dots \\ \text{are compact.} \end{cases}$$

„Variational Formulation" for (ESP)

$$E_F = A_1^* H_F = \varepsilon^{-1} \mathrm{rot}\, H_F\,,$$
$$\Rightarrow \quad H_F = (A_1^*)^{-1} E_F$$
$$= (\varepsilon^{-1}\mathrm{rot})^{-1} E_F \in R \cap \mathring{\mathrm{rot}}\,\mathring{R}$$

can be found by:

Find $H_F \in R \cap \mathring{\mathrm{rot}}\,\mathring{R}$ such that

$\forall \phi \in R \cap \mathring{\mathrm{rot}}\,\mathring{R}$:

$$\langle \varepsilon^{-1}\mathrm{rot}\, H_F, \varepsilon^{-1}\mathrm{rot}\,\phi \rangle_{L^2_\varepsilon} = \langle F, \phi \rangle_{L^2}$$
$$\langle \mathrm{rot}\, H_F, \mathrm{rot}\,\phi \rangle_{L^2_{\varepsilon^{-1}}}$$

$\mathring{\mathrm{rot}}\,\mathring{R} = R_0^\perp$

$\Leftrightarrow$ Find $H_F \in R$ such that

$$\forall \phi \in R: \langle \mathrm{rot}\, H_F, \mathrm{rot}\,\phi \rangle_{L^2_{\varepsilon^{-1}}} = \langle F, \phi \rangle_{L^2}$$
$$\forall \psi \in R_0: \quad \langle H_F, \psi \rangle_{L^2} = 0$$

Now assume: $\mathcal{N} = \{0\}$

$\Leftrightarrow$ Ω is simply connected.

$R_0 = \nabla H^1 \oplus \mathcal{N}$

$\Leftrightarrow$ Find $(H_F, u_F) \in R \times H^1_\perp$ such that

$\forall \phi \in R:$

$$\langle \operatorname{rot} H_F, \operatorname{rot} \phi \rangle_{L^2_{\varepsilon^{-1}}} + \langle \phi, \nabla u_F \rangle_{L^2} = \langle F, \phi \rangle_{L^2}$$

$\forall \psi \in H^1: \langle H_F, \nabla \psi \rangle_{L^2} = 0$

One can also put $H^1_\perp$.

So: $E_f \rightsquigarrow$ rot-rot-problem

$E_f = A_0 u_f = \mathring{\nabla} u_f$, $u_f = A_0^{-1} E_f = \mathring{\nabla}^{-1} E_f \in \mathring{H}^1$ can be found by

Find $u_f \in \mathring{H}^1$ such that
$\forall \phi \in \mathring{H}^1 \; \langle \nabla u_f, \nabla \phi \rangle_{L^2_\varepsilon} = \langle f, \phi \rangle_{L^2}$

So: $u_f \rightsquigarrow -\Delta_0$-problem

"*Variational formulation*" *for* (MSP)

$H_G = A_1 E_G = \operatorname{rot} E_G$,
$\Rightarrow E_G = (A_1)^{-1} H_G$
$= \operatorname{rot}^{-1} H_G \in \mathring{R} \cap \operatorname{rot} R$

can be found by
Find $(E_G, v_G) \in \mathring{R} \times \mathring{H}^1$ such that

$$\forall \phi \in \mathring{R}: \langle \operatorname{rot} E_G, \operatorname{rot} \phi \rangle_{L^2_{\mu^{-1}}} + \langle \phi, \nabla v_G \rangle_{L^2} = \langle G, \phi \rangle_{L^2}$$

$\forall \psi \in \mathring{H}^1: \langle E_G, \nabla \psi \rangle_{L^2} = 0$

$$H_G = A_2^* u_g = -\nabla u_g \,,$$
$$\Rightarrow \quad u_g = (A_2^*)^{-1} H_g$$
$$= (-\nabla)^{-1} H_g \in H^1$$

can be found by

Find $u_g \in H^1_\perp$ such that

$$\forall \phi \in H^1_\perp : \ \langle \nabla u_g, \nabla \phi \rangle_{L^2_\mu} = \langle g, \phi \rangle_{L^2}$$

So far we can handle:

$A_i x = f$	$A_i^* A_i x = f$	$A_i^* A_i x = f$
$A_{i-1}^* x = g$	$A_{i-1}^* x = g$	$A_{i-1} A_{i-1}^* x = g$
$\pi_i x = h$	$\pi_i x = h$	$\pi_i x = h$
kernel: K_i	kernel: K_i	kernel: K_i

$\uparrow$

since:

$$0 = \langle A_i^* A_i x, x \rangle = \| A_i x \|^2$$
$$\Rightarrow \quad A_i x = 0$$

now: <u>case ②</u>

solution theory for: $(M - \lambda) x = f$, $\lambda \in \mathbb{C}$?

<u>$\lambda = 0$:</u> ✓ $\rightsquigarrow$ case ③ (static case)

<u>$\lambda \in \mathbb{C} \setminus \mathbb{R}$:</u> ✓ $\rightsquigarrow$ M is selfadjoint

$$\Rightarrow \sigma(M) \subset \mathbb{R}$$
$$\Rightarrow \mathbb{C} \setminus \mathbb{R} \subset \rho(M)$$

so we are left with: $\boxed{\lambda \in \mathbb{R} \setminus \{0\}}$

recall:

$$* \; A = \begin{pmatrix} 0 & -\mathring{\mathrm{rot}}^* \\ \mathring{\mathrm{rot}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{rot} \\ \mathring{\mathrm{rot}} & 0 \end{pmatrix}$$

$$* \; \Lambda = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$$

$$* \; \mathcal{M} = i \Lambda^{-1} A = i \begin{pmatrix} 0 & -\varepsilon^{-1}\mathrm{rot} \\ \mu^{-1}\mathring{\mathrm{rot}} & 0 \end{pmatrix}$$

$\mathcal{M} : D(\mathcal{M}) = \mathring{R} \times R \subset L^2_\Lambda = L^2_\varepsilon \times L^2_\mu \longrightarrow L^2_\Lambda$
is selfadjoint for all Ω (open) $\subset \mathbb{R}^3$

$\mathfrak{M} : D(\mathfrak{M}) = D(\mathcal{M}) \cap \overline{R(\mathcal{M})} \subset \overline{R(\mathcal{M})} \longrightarrow \overline{R(\mathcal{M})}$

<u>assume</u>: $\boxed{D(\mathfrak{M}) \hookrightarrow\hookrightarrow R(\mathcal{M}) \,/\, L^2_\Lambda \,/\, L^2}$ (*)

($\rightsquigarrow$ critical point for Fredholm-alternative)

$D(\mathfrak{M}) = (\mathring{R} \cap \varepsilon^{-1}\overline{\mathrm{rot}\, R}) \times (R \cap \mu^{-1}\overline{\mathring{\mathrm{rot}}\,\mathring{R}}) \hookrightarrow\hookrightarrow L^2_\Lambda$

$\Leftrightarrow \mathring{R} \cap \varepsilon^{-1}\overline{\mathrm{rot}\, R} \hookrightarrow\hookrightarrow L^2_\varepsilon \wedge R \cap \mu^{-1}\overline{\mathring{\mathrm{rot}}\,\mathring{R}} \hookrightarrow\hookrightarrow L^2_\mu$

$$\Uparrow$$

$$\boxed{\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow\hookrightarrow L^2_\varepsilon \wedge R \cap \mu^{-1}\mathring{D} \hookrightarrow\hookrightarrow L^2_\mu}$$

(Weck's selection theorem; true for e.g. Ω bounded, weak Lipschitz)

$\Rightarrow R(\mathfrak{M}) = R(\mathcal{M})$ is closed

$\wedge \; \forall x \in D(\mathfrak{M}) : |x| \leq c_m |\mathcal{M}x|$

$\wedge \; \mathfrak{M}^{-1} : R(\mathcal{M}) \longrightarrow D(\mathfrak{M})$

is continuous.

$\wedge$ $\boxed{\begin{array}{l} \mathcal{M}^{-1}: R(\mathcal{M}) \to R(\mathcal{M}) / L^2_\Lambda \\ \text{is compact} \end{array}}$

Now: $\mathcal{M}_\lambda := \mathcal{M} - \lambda, \ \lambda \in \mathbb{R} \setminus \{0\}$

$\rightsquigarrow$ $\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) = D(\mathcal{M}) \subset L^2_\Lambda \to L^2_\Lambda$ is linear, densely defined, closed and self-adjoint.

$\ulcorner$ $\mathcal{M}_\lambda$ is closed, since:

$$x_n \to x \text{ in } L^2_\Lambda, \ \mathcal{M}_\lambda x_n \to y \text{ in } L^2_\Lambda$$
$$\Rightarrow x_n \to x \text{ in } L^2_\Lambda$$
$$\wedge \ \mathcal{M} x_n - \lambda x_n \to y \text{ in } L^2_\Lambda$$
$$\Rightarrow x_n \to x \text{ in } L^2_\Lambda$$
$$\wedge \ \mathcal{M} x_n \to y + \lambda x \text{ in } L^2_\Lambda$$

$\mathcal{M}$ is closed

$$\Rightarrow x \in D(\mathcal{M}) = D(\mathcal{M}_\lambda)$$
$$\wedge \ \mathcal{M} x = y + \lambda x \Leftrightarrow \mathcal{M}_\lambda x = y.$$

$\mathcal{M}_\lambda$ is selfadjoint, since:

$$(\mathcal{M} - \lambda)^* = \mathcal{M}^* - \bar{\lambda} \underset{\uparrow}{=} \mathcal{M} - \lambda$$

$\mathcal{M}$ is selfadjoint $\lrcorner$

$\rightsquigarrow$ $\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) = D(\mathcal{M}_\lambda) \cap \overline{R(\mathcal{M}_\lambda)}$
$= D(\mathcal{M}) \cap \overline{R(\mathcal{M}_\lambda)} \subset \overline{R(\mathcal{M}_\lambda)} \to \overline{R(\mathcal{M}_\lambda)}$

Let's solve $(\mathcal{M} - \lambda) x = f \in L^2_\Lambda, \ x \in D(\mathcal{M})$:
Suppose x solves the system, then:

$$L^2 = N(\mathcal{M}) \oplus \overline{R(\mathcal{M})}$$

We decompose f, x:

$$f = f_n + f_r \in N(M) \oplus \overline{R(M)}$$
$$x = x_n + x_r \in N(M) \oplus D(M) \quad (x \in D(M))$$

Then:

$$(M-\lambda)x = f$$
$$\Leftrightarrow (M-\lambda)(x_n + x_r) = f_n + f_r$$
$$\Leftrightarrow -\lambda x_n + (M-\lambda)x_r = f_n + f_r$$
$$\Leftrightarrow \underbrace{(M-\lambda)x_r - f_r}_{\in \overline{R(M)}} = \underbrace{f_n + \lambda x_n}_{\in N(M)}$$

$$N(M) = \overline{R(M)}^{\perp} \longrightarrow \Downarrow$$
$$M_\lambda x_r = f_r \in \overline{R(M)},\ x_r \in D(M)$$
$$-\lambda x_n = f_n$$

So, for solving we take: $f \in L^2_\Delta$

<u>1. step:</u> decompose

$$f = f_n + f_r \in N(M) \oplus \overline{R(M)}$$
$$\rightsquigarrow \text{define: } x_n := -\lambda^{-1} f_n \in N(M)$$

<u>2. step:</u> solve (?)

find $x_r \in D(M)$: $M_\lambda x_r = f_r \in \overline{R(M)}$

We end up with a reduced problem:

$$\boxed{\begin{array}{c}\text{Find } x \in D(M) \text{ such that} \\ M_\lambda x = f \in \overline{R(M)}\end{array}}$$

$$\rightsquigarrow M : \mathring{R} \times R \subset L^2_\varepsilon \times L^2_\mu \to L^2_\varepsilon \times L^2_\mu$$

$$\rightsquigarrow \quad \mathcal{M} : (\mathring{R} \times R) \cap (\varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}} \mathring{R}})$$
$$\subset \varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}}\mathring{R}} \to \varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}}\mathring{R}}$$

$$\overline{R(\mathcal{M})} = \varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}}\mathring{R}}$$
$$= (\varepsilon^{-1} D_0 \cap \mathcal{D}_\varepsilon^{\perp_{L^2_\varepsilon}}) \times (\mu^{-1}\mathring{D}_0 \cap \mathcal{N}_\mu^{\perp_{L^2_\mu}})$$

$$N(A_0^*) = K_1 \oplus \overline{R(A_1^*)}$$
$$N(A_2) = K_2 \oplus \overline{R(A_1)}$$

$$(*) \Rightarrow \quad \mathcal{M}_\lambda x = (\mathcal{M} - \lambda)x = f \in \overline{R(\mathcal{M})} = R(\mathcal{M})$$
$$\wedge \ \mathcal{M}^{-1} : R(\mathcal{M}) \to R(\mathcal{M}) \text{ is compact}$$
$$(\text{since: } \mathcal{M}^{-1} : R(\mathcal{M}) \to D(\mathcal{M}) \hookrightarrow R(\mathcal{M}))$$

So

$$\mathcal{M}_\lambda x = (\mathcal{M} - \lambda)x = f \in R(\mathcal{M})$$
$$\Updownarrow \mathcal{M}^{-1}$$
$$\underbrace{\mathcal{M}^{-1}\mathcal{M}x}_{=x} - \lambda \mathcal{M}^{-1}x = \mathcal{M}^{-1}f$$

$\ulcorner$ $\mathcal{M}^{-1}\mathcal{M}x = x$, since $\mathcal{M}$ is injective &

$$\mathcal{M}(\underbrace{\mathcal{M}^{-1}\mathcal{M}x - x}_{\in D(\mathcal{M})}) = 0$$

$\uparrow$ $\mathcal{M}^{-1}$ is right inverse $\lrcorner$

Now: $\quad x - \lambda\mathcal{M}^{-1}x = \mathcal{M}^{-1}f$

$$\Leftrightarrow \quad (1/\lambda\, I - \mathcal{M}^{-1})x = 1/\lambda\, \mathcal{M}^{-1}f \quad (*_2)$$

Apply <u>Fredholm - alternative</u>:

$$\Rightarrow \quad \sigma(\mathcal{M}^{-1}) \subset \sigma_p(\mathcal{M}^{-1}) \subset J := [-|\mathcal{M}^{-1}|, |\mathcal{M}^{-1}|]$$

$\wedge \quad \forall\, \omega = 1/\lambda \in \sigma_p(\mathcal{M}^{-1}):$

$$|\mathcal{N}(\mathcal{M}^{-1} - \omega)| < \infty$$

$\wedge \quad \sigma_p(\mathcal{M}^{-1})$ has no accumulation point in $\mathcal{J} \setminus \{0\}$

Furthermore:

$$(\mathcal{M}^{-1} - 1/\lambda)\, x = 0 \,, \quad x \in D(\mathcal{M})$$

$$\Leftrightarrow \quad (\mathcal{M} - \lambda)\, x = 0 \,, \quad x \in D(\mathcal{M})$$

so: $\boxed{\begin{array}{l} x \text{ is eigenfunction of } (\mathcal{M}^{-1}, 1/\lambda) \\ \Leftrightarrow x \text{ is eigenfunction of } (\mathcal{M}, \lambda) \end{array}}$

$$\Rightarrow \quad |\mathcal{N}(\mathcal{M} - \lambda)| < \infty \quad \wedge \quad \sigma(\mathcal{M}) \subset \underbrace{\sigma_p(\mathcal{M})}_{= \, \sigma_p(\mathcal{M}^{-1})^{-1}}$$

We therefore can solve $(*_2)$, iff

$$\mathcal{M}^{-1} f \perp \mathcal{N}(\mathcal{M}^{-1} - 1/\lambda) = \mathcal{N}(\mathcal{M} - \lambda)$$

$$\Leftrightarrow \quad \langle \mathcal{M}^{-1} f, y \rangle = 0 \quad \forall y \in \mathcal{N}(\mathcal{M} - \lambda)$$

$$\langle \mathcal{M}^{-1} f, y \rangle = 1/\lambda \, \langle \mathcal{M}^{-1} f, \mathcal{M} y \rangle = 1/\lambda \, \langle f, y \rangle$$

$(\, y \in \mathcal{N}(\mathcal{M} - \lambda) \Leftrightarrow \mathcal{M} y = \lambda y \,)$

$$\Leftrightarrow \quad \boxed{f \perp \mathcal{N}(\mathcal{M} - \lambda)}$$