lecture 4 31.10.16

<u>Recall:</u>

$A: D(A) \subset H_1 \to H_2$ linear, densely defined, closed

$A^*: D(A^*) \subset H_2 \to H_1$ linear, densely defined, closed

$\Big\downarrow$ switch to inj. operators

$$H_1 = N(A) \oplus_{H_1} \overline{R(A^*)}$$
$$H_2 = N(A^*) \oplus_{H_2} \overline{R(A)}$$
$$\hookrightarrow \; N(A)^{\perp} = \overline{R(A^*)}$$

$$\mathcal{A}: D(\mathcal{A}) := D(A) \cap \overline{R(A^*)} \subset \overline{R(A^*)} \to \overline{R(A)}$$
$$\mathcal{A}^*: D(\mathcal{A}^*) := D(A^*) \cap \overline{R(A)} \subset \overline{R(A)} \to \overline{R(A^*)}$$

$\Rightarrow$ (A, A^*) & $(\mathcal{A}, \mathcal{A}^*)$ are dual pairs
[i.e. $A^{**} = A$ resp. $\mathcal{A}^{**} = \mathcal{A}$]

$$D(A) = N(A) \oplus_{H_1} D(\mathcal{A}) \Rightarrow R(A) = R(\mathcal{A})$$
$$D(A^*) = N(A^*) \oplus_{H_2} D(\mathcal{A}^*) \Rightarrow R(A^*) = R(\mathcal{A}^*)$$

cf. exercise 1, Problem 6 we have

$$D(\mathcal{A}) \hookrightarrow H_1 \Leftrightarrow D(\mathcal{A}^*) \hookrightarrow H_2$$

$\Downarrow$

$R(A) = R(\mathcal{A})$ is closed

$\Leftrightarrow \exists\, c_A > 0 \;\; \forall x \in D(\mathcal{A}):$

$$|x|_{H_1} \leq c_A \, |Ax|_{H_2},$$

$R(A^*) = R(\mathcal{A}^*)$ is closed

$\Leftrightarrow \exists\, c_{A^*} > 0 \quad \forall y \in D(\mathcal{A}^*):$

$$|y|_{H_2} \le c_{A^*} |A^* y|_{H_1}$$

<u>We even have</u>: $c_A = c_{A^*}$,

if we define:

$$1/c_A := \inf_{0 \ne x \in D(\mathcal{A})} \frac{|Ax|_{H_2}}{|x|_{H_1}}, \quad 1/c_{A^*} := \inf_{0 \ne y \in D(\mathcal{A}^*)} \frac{|A^* y|_{H_1}}{|y|_{H_2}}$$

<u>sequences</u>:

Let $A_0 : D(A_0) \subset H_0 \to H_1$, $A_1 : D(A_1) \subset H_1 \to H_2$, $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_0^*, \mathcal{A}_1^*$ as above and assume

$\boxed{A_1 A_0 = 0}$ sequence (complex) property

(indeed here should be a „$\subset$", since 0 has a bigger domain of defin.)

$\Leftrightarrow R(A_0) \subset N(A_1)$

$\overset{?}{\Leftrightarrow} R(A_1^*) \subset N(A_0^*)$

(We can also put closure bars, since closed operators have closed kernels)

<u>Theorem 5</u>: (HH - decompositions)

* $H_0 = N(A_0) \oplus_{H_0} \overline{R(A_0^*)}$

* $H_1 = \underset{\cup}{N(A_1)} \oplus_{H_1} \underset{\cap}{\overline{R(A_1^*)}}$
  $= \overline{R(A_0)} \oplus_{H_1} N(A_0^*)$ $\Rightarrow N(A_1) = \overline{R(A_0)} \oplus_{H_1} K_1$

$$\Rightarrow \quad H_1 = \overline{R(A_0)} \oplus_{H_1} K_1 \oplus_{H_1} \overline{R(A_1^*)}$$

If additionally $A_2 : D(A_2) \subset H_2 \to H_3$ linear, densely defined, closed with $A_2 A_1 = 0$, we have:

$$* \quad H_2 = \underset{\cup}{N(A_2)} \oplus_{H_2} \overline{R(A_2^*)} \qquad \Rightarrow \quad N(A_2) = \overline{R(A_1)} \oplus_{H_2} K_2$$
$$= \overline{R(A_1)} \oplus_{H_2} \overset{\cap}{N(A_1^*)}$$

$$\Rightarrow \quad H_2 = \overline{R(A_1)} \oplus_{H_2} K_2 \oplus_{H_2} \overline{R(A_2^*)}$$

$$* \quad H_3 = \overline{R(A_2)} \oplus_{H_3} N(A_2^*)$$

The spaces $K_1 := N(A_0^*) \cap N(A_1)$ and $K_2 := N(A_1^*) \cap N(A_2)$ are called „<u>cohomology groups</u>"

<u>Proof:</u>

Exercise 2, Problem 2

$\boxtimes$

<u>Primal & dual sequences:</u>

$$D(A_0) \xrightarrow{A_0} D(A_1) \xrightarrow{A_1} D(A_2) \xrightarrow{A_2} H_3 \quad \text{(primal)}$$
$$H_0 \xleftarrow{A_0^*} D(A_0^*) \xleftarrow{A_1^*} D(A_1^*) \xleftarrow{A_2^*} D(A_2^*) \quad \text{(dual)}$$

<u>Definition:</u>

We call a sequence

* <u>complex</u> $:\Leftrightarrow R(A_i) \subset N(A_{i+1})$
* <u>closed</u> $:\Leftrightarrow R(A_i)$ is closed
* <u>exact</u> $:\Leftrightarrow K_i = \{0\}$

  (then: $\overline{R(A_i)} = N(A_{i+1})$)

Remark:

We have

* primal sequence is a complex
  $\Leftrightarrow$ dual sequence is a complex.
* primal complex is closed
  $\Leftrightarrow$ dual complex is closed.
  („closed range theorem")
* primal complex is exact
  $\Leftrightarrow$ dual complex is exact.
  $\left(N(A_2) = \overline{R(A_1)} \oplus_{H_2} K_2 ,\ N(A_1^*) = K_2 \oplus \overline{R(A_2^*)} \right)$

If (A, A^*) is a dual pair, A linear, densely defined, closed, then

* A^*A, AA^* are selfadjoint.
* $\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} : D(A) \times D(A^*) \subset H_1 \times H_2 \to H_1 \times H_2$
  is selfadjoint.
* $\begin{pmatrix} 0 & -A^* \\ A & 0 \end{pmatrix}$ is skew-selfadjoint.

  $\rightsquigarrow i \begin{pmatrix} 0 & -A^* \\ A & 0 \end{pmatrix}$ is selfadjoint.

Assume: $R(A_0)$ & $R(A_1)$ are closed

$\Rightarrow H_1 = R(A_0) \oplus_{H_1} K_1 \oplus_{H_1} R(A_1^*)$

$\quad \hookrightarrow D(A_1) = R(A_0) \oplus_{H_1} K_1 \oplus_{H_1} D(\mathcal{A}_1)$

$$\Rightarrow D_1 := D(A_1) \cap D(A_0^*)$$
$$= D(\mathcal{A}_0^*) \oplus_{H_1} K_1 \oplus_{H_1} D(\mathcal{A}_1)$$

<u>Lemma 6</u>:

Let $R(A_0)$ & $R(A_1)$ be closed. Then:

$$\forall x \in D_1 \cap K_1^\perp : |x|_{H_1}^2 \leq c_{A_0}^2 |A_0^* x|_{H_0}^2 + c_{A_1}^2 |A_1 x|_{H_2}^2$$

<u>Proof</u>:

Let $y \in D_1 \cap K_1^\perp$

$$\Rightarrow y = y_0 + y_1 \in D(\mathcal{A}_0^*) \oplus_{H_1} D(\mathcal{A}_1)$$

$$\Rightarrow |y|_{H_1}^2 \leq |y_0|_{H_1}^2 + |y_1|_{H_1}^2$$

$$\leq c_{A_0}^2 |A_0^* y_0|_{H_0}^2 + c_{A_1}^2 |A_1 y_1|_{H_2}^2$$

$D(\mathcal{A}_0^*) \subset N(A_1)$
$D(\mathcal{A}_1) \subset N(A_0^*)$

$$= c_{A_0}^2 |A_0^* y|_{H_0}^2 + c_{A_1}^2 |A_1 y|_{H_2}^2$$

$\boxed{\varepsilon}$

Apply „toolbox" to the Maxwell case:

Let $\Omega \subset \mathbb{R}^3$ open; $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times3}$;
$\varepsilon, \mu \in L^\infty(\Omega)$, symmetric and uniformly positive definite.

$$\hat{A}_0 := \hat{\nabla} : \mathring{C}^\infty \subset L^2 \to L^2_\varepsilon, \; \hat{A}_0 u = \nabla u$$
$$\hat{A}_1 := \mu^{-1} \hat{\mathrm{rot}} : \mathring{C}^\infty \subset L^2_\varepsilon \to L^2_\mu, \; \hat{A}_1 u = \mu^{-1} \mathrm{rot}\, u$$
$$\hat{A}_2 := \hat{\mathrm{div}}\, \mu : \mu^{-1} \mathring{C}^\infty \subset L^2_\mu \to L^2, \; \hat{A}_2 u = \mathrm{div}(\mu u)$$

$\rightsquigarrow$ all operators are linear, densely defined; but are they closable?

e.g. $\hat{A}_1$ closable:

$\forall (x_n)_n \subset D(A_1) = \overset{\circ}{C}^\infty$ with
$x_n \to 0$ & $\hat{A}_1 x_n = \mu^{-1} \operatorname{rot} x_n \to y$

$\Rightarrow \forall \phi \in \overset{\circ}{C}^\infty$:

$$\langle y, \phi \rangle_{L^2_\mu} = \langle y, \mu \phi \rangle_{L^2}$$
$$= \lim_{n\to\infty} \langle \mu^{-1} \operatorname{rot} x_n, \mu\phi \rangle_{L^2}$$
$$= \lim_{n\to\infty} \langle \operatorname{rot} x_n, \phi \rangle_{L^2}$$
$$= \lim_{n\to\infty} \langle x_n, \operatorname{rot} \phi \rangle_{L^2} = 0$$

$\Rightarrow y = 0$
$\Rightarrow \hat{A}_1$ is closable.

$\boxed{\text{q}}$

$\rightsquigarrow$ closed operators: $D(\overline{\hat{A}_i}) = \overline{D(\hat{A}_i)}^{G(\hat{A}_i)}$

$$A_0 := \overline{\hat{A}_0} = \overset{\circ}{\nabla} : \overset{\circ}{H}{}^1 \subset L^2 \to L^2_\varepsilon$$
$$A_1 := \overline{\hat{A}_1} = \mu^{-1} \overset{\circ}{\operatorname{rot}} : \overset{\circ}{R} \subset L^2_\varepsilon \to L^2_\mu$$
$$A_2 := \overline{\hat{A}_2} = \overset{\circ}{\operatorname{div}}_\mu : \mu^{-1} \overset{\circ}{D} \subset L^2_\mu \to L^2$$

(of course: linear, densely defined
and now closed)

$\rightsquigarrow$ compute the adjoints:

* $u \in D(\overset{\circ}{\nabla}{}^*) \wedge \overset{\circ}{\nabla}{}^* u = f$

$\Leftrightarrow u \in L^2_\varepsilon \wedge \exists f \in L^2 \ \forall \varphi \in D(\overset{\circ}{\nabla}) = \overset{\circ}{H}{}^1$:

$$\langle u, \nabla\varphi \rangle_{L^2_\varepsilon} = \langle f, \varphi \rangle_{L^2}$$
$$\|$$
$$\langle \varepsilon u, \nabla\varphi \rangle_{L^2}$$

$$\Leftrightarrow \varepsilon u \in D \wedge f = -\operatorname{div} \varepsilon u$$
$$\Leftrightarrow u \in \varepsilon^{-1} D \wedge f = -\operatorname{div} \varepsilon u$$

$$\rightsquigarrow \boxed{A_0^* = -\operatorname{div} \varepsilon : \varepsilon^{-1} D \subset L^2_\varepsilon \to L^2}$$

$*$ $u \in D(\mu^{-1} \mathring{\operatorname{rot}}) \wedge \mu^{-1} \mathring{\operatorname{rot}}^* u = f$

$$\Leftrightarrow u \in L^2_\mu \wedge \exists f \in L^2_\varepsilon \; \forall \varphi \in D(\mu^{-1} \mathring{\operatorname{rot}}) = \mathring{R} :$$
$$\underbrace{\langle u, \mu^{-1} \operatorname{rot} \varphi \rangle_{L^2_\mu}}_{= \langle u, \operatorname{rot} \varphi \rangle_{L^2}} = \underbrace{\langle f, \varphi \rangle_{L^2_\varepsilon}}_{= \langle \varepsilon f, \varphi \rangle_{L^2}}$$
$$\Leftrightarrow u \in R \wedge \operatorname{rot} u = \varepsilon f$$
$$\Leftrightarrow u \in R \wedge f = \varepsilon^{-1} \operatorname{rot} u$$

$$\rightsquigarrow \boxed{A_1^* = \varepsilon^{-1} \operatorname{rot} : R \subset L^2_\mu \to L^2_\varepsilon}$$

$*$ $u \in D(\mathring{\operatorname{div}} \mu^*) \wedge \mathring{\operatorname{div}} \mu^* u = f$

$$\Leftrightarrow u \in L^2 \wedge \exists f \in L^2_\mu \; \forall \varphi \in D(\mathring{\operatorname{div}} \mu) = \mu^{-1} \mathring{D} :$$
$$\langle u, \operatorname{div} \underbrace{\mu \varphi}_{\in \mathring{D}} \rangle_{L^2} = \underbrace{\langle f, \varphi \rangle_{L^2_\mu}}_{= \langle f, \mu \varphi \rangle_{L^2}}$$
$$\Leftrightarrow u \in H^1 \wedge f = -\nabla u$$

$$\rightsquigarrow \boxed{A_2^* = -\nabla : H^1 \subset L^2 \to L^2_\mu}$$

Now apply Theorem 5:

$*$ $L^2 = N(\mathring{\nabla}) \oplus_{L^2} \overline{R(-\operatorname{div} \varepsilon)}$
$$= \{0\} \oplus_{L^2} \overline{R(\operatorname{div})} = \overline{R(\operatorname{div})}$$
$$= \overline{\operatorname{div} D}$$

* $$L^2_\varepsilon = N(\mu^{-1}\mathring{\mathrm{rot}}) \oplus_{L^2_\varepsilon} \overline{R(\varepsilon^{-1}\mathrm{rot})}$$
$$= \mathring{R}_0 \oplus_{L^2_\varepsilon} \varepsilon^{-1}\overline{\mathrm{rot}\, R}$$
$$= \overline{R(\mathring{\nabla})} \oplus_{L^2_\varepsilon} N(-\mathrm{div}\,\varepsilon)$$
$$= \overline{\mathring{\nabla}\mathring{H}^1} \oplus_{L^2_\varepsilon} \varepsilon^{-1} D_0$$
$$= \underbrace{\overline{R(\mathring{\nabla})} \oplus_{L^2_\varepsilon} \left[N(-\mathrm{div}\,\varepsilon) \cap N(\mu^{-1}\mathring{\mathrm{rot}})\right]}_{= N(\mu^{-1}\mathring{\mathrm{rot}})} \oplus_{L^2_\varepsilon} \overline{R(\varepsilon^{-1}\mathrm{rot})}$$
$$\left[N(-\mathrm{div}\,\varepsilon) \cap N(\mu^{-1}\mathring{\mathrm{rot}})\right] \oplus_{L^2_\varepsilon} \overline{R(\varepsilon^{-1}\mathrm{rot})} = N(-\mathrm{div}\,\varepsilon)$$
$$= \underbrace{\overline{\mathring{\nabla}\mathring{H}^1} \oplus_{L^2_\varepsilon} (\mathring{R}_0 \cap \varepsilon^{-1} D_0)}_{= \mathring{R}_0} \oplus_{L^2_\varepsilon} \varepsilon^{-1}\overline{\mathrm{rot}\, R}$$
$$(\mathring{R}_0 \cap \varepsilon^{-1} D_0) \oplus_{L^2_\varepsilon} \varepsilon^{-1}\overline{\mathrm{rot}\, R} = \varepsilon^{-1} D_0$$

* $$L^2_\mu = N(\mathrm{div}\,\mu) \oplus_{L^2_\mu} \overline{R(\nabla)}$$
$$= \mu^{-1}\mathring{D}_0 \oplus_{L^2_\mu} \overline{\nabla H^1}$$
$$= \overline{R(\mu^{-1}\mathring{\mathrm{rot}})} \oplus_{L^2_\mu} N(\varepsilon^{-1}\mathrm{rot})$$
$$= \mu^{-1}\overline{\mathring{\mathrm{rot}}\,\mathring{R}} \oplus_{L^2_\mu} R_0$$
$$= \underbrace{\overline{R(\mu^{-1}\mathring{\mathrm{rot}})} \oplus_{L^2_\mu} \left[N(\mathrm{div}\,\mu) \cap N(\varepsilon^{-1}\mathrm{rot})\right]}_{= N(\mathrm{div}\,\mu)} \oplus_{L^2_\mu} \overline{R(\nabla)}$$
$$\left[N(\mathrm{div}\,\mu) \cap N(\varepsilon^{-1}\mathrm{rot})\right] \oplus_{L^2_\mu} \overline{R(\nabla)} = N(\varepsilon^{-1}\mathrm{rot})$$
$$= \underbrace{\mu^{-1}\overline{\mathring{\mathrm{rot}}\,\mathring{R}} \oplus_{L^2_\mu} (\mu^{-1}\mathring{D}_0 \cap R_0)}_{= \mu^{-1}\mathring{D}_0} \oplus_{L^2_\mu} \overline{\nabla H^1}$$
$$(\mu^{-1}\mathring{D}_0 \cap R_0) \oplus_{L^2_\mu} \overline{\nabla H^1} = R_0$$

* $$L^2 = \overline{R(\mathring{\mathrm{div}}\,\mu)} \oplus_{L^2} N(\nabla)$$
$$= \overline{\mathring{\mathrm{div}}\,\mathring{D}} \oplus_{L^2} \mathbb{R}$$

We call:

* $\mathcal{H}_D := \mathring{R}_0 \cap \varepsilon^{-1} D_0$ "Dirichlet Fields"
* $\mathcal{H}_N := R_0 \cap \mu^{-1}\mathring{D}_0$ "Neumann Fields"

$A_0^* A_0 = \operatorname{div} \varepsilon \mathring{\nabla}$, $A_1^* A_1 = \varepsilon^{-1} \operatorname{rot} \mu^{-1} \mathring{\operatorname{rot}}$ and
$A_2^* A_2 = -\nabla \mathring{\operatorname{div}} \mu$ are selfadjoint.

skew - selfadjoint are for example:

$$\begin{pmatrix} 0 & -A_0^* \\ A_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{div} \varepsilon \\ \mathring{\nabla} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -A_1^* \\ A_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon^{-1} \operatorname{rot} \\ \mu^{-1} \mathring{\operatorname{rot}} & 0 \end{pmatrix}$$

$\hookrightarrow$ Maxwell - operator

$$\begin{pmatrix} 0 & -A_2^* \\ A_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nabla \\ \mathring{\operatorname{div}} \mu & 0 \end{pmatrix}$$

lecture 4½ 02.11.16

We have seen:

$N(A_0) = \{0\} \quad \Rightarrow \quad \overline{R(A_0)} = L^2 = \overline{\operatorname{div} D}$

$N(A_2^*) = \mathbb{R} \quad \Rightarrow \quad \overline{R(A_2)} = \mathbb{R}^\perp = L_0^2 = \overline{\mathring{\operatorname{div}} \mathring{D}}$

$\rightsquigarrow$ reduced operators (injective)

$\mathcal{A}_0 : \mathring{H}^1 \subset L^2 \to \overline{\mathring{\nabla} \mathring{H}^1}^{L_\varepsilon^2}$

$\mathcal{A}_1 : \mathring{R} \cap \varepsilon^{-1} \overline{\operatorname{rot} R} \subset \varepsilon^{-1} \overline{\operatorname{rot} R} \to \mu^{-1} \overline{\mathring{\operatorname{rot}} \mathring{R}}$

$\mathcal{A}_2 : \mu^{-1} \mathring{D} \cap \overline{\nabla H^1} \subset \overline{\nabla H^1} \to \overline{\mathring{\operatorname{div}} \mathring{D}} = \mathbb{R}^\perp$

$\mathcal{A}_0^* : \varepsilon^{-1} D \cap \overline{\mathring{\nabla} \mathring{H}^1} \subset \overline{\mathring{\nabla} \mathring{H}^1} \to L^2$

$\mathcal{A}_1^* : R \cap \mu^{-1} \overline{\mathring{\operatorname{rot}} \mathring{R}} \subset \mu^{-1} \overline{\mathring{\operatorname{rot}} \mathring{R}} \to \varepsilon^{-1} \overline{\operatorname{rot} R}$

$\mathcal{A}_2^* : H^1 \cap L_0^2 \subset L_0^2 \to \nabla H^1 \quad (L_0^2 = L_1^2)$

$D(A_0^*) \cap D(A_1) \overset{c}{\hookrightarrow} H_2 \Leftrightarrow D(\mathcal{A}_0) \overset{c}{\hookrightarrow} H_0 \wedge D(\mathcal{A}_1) \overset{c}{\hookrightarrow} H_1 \wedge$
K_1 is finite dimensional

$\rightsquigarrow$ Furthermore:

* $D(\mathcal{A}_0) \hookrightarrow\hookrightarrow H_0 \iff D(\mathcal{A}_0^*) \hookrightarrow\hookrightarrow H_1$
* $D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_1 \iff D(\mathcal{A}_1^*) \hookrightarrow\hookrightarrow H_2$
* $D(\mathcal{A}_0^*) \cap D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_2$

$\Rightarrow$ $R(\mathcal{A}_0)$ & $R(\mathcal{A}_1)$ are closed

$\iff$ Poincaré-type estimates

<u>In more detail:</u>

$D(\mathcal{A}_0) \hookrightarrow\hookrightarrow H_0$	$\mathring{H}^1 \hookrightarrow\hookrightarrow L^2$ Rellich's sel. th. (need: Ω bd.)
$\Updownarrow$	$\Updownarrow$
$D(\mathcal{A}_0^*) \hookrightarrow\hookrightarrow H_1$	$\underbrace{\varepsilon^{-1} D \cap \overline{\nabla \mathring{H}^1}}_{= \varepsilon^{-1} D \cap \mathring{R}_0 \cap \mathcal{H}_D^{\perp}} \hookrightarrow\hookrightarrow L^2_\varepsilon$
$D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_1$	$\underbrace{\mathring{R} \cap \varepsilon^{-1} \overline{\operatorname{rot} R}}_{= \mathring{R} \cap \varepsilon^{-1} D_0 \cap \mathcal{H}_D^{\perp_{L^2_\varepsilon}}} \hookrightarrow\hookrightarrow L^2_\varepsilon$ (?)
$\Updownarrow$	$\Updownarrow$
$D(\mathcal{A}_1^*) \hookrightarrow\hookrightarrow H_2$	$\underbrace{R \cap \mu^{-1} \overline{\mathring{\operatorname{rot}} \mathring{R}}}_{= R \cap \mu^{-1} \mathring{D}_0 \cap \mathcal{H}_N^{\perp_{L^2_\mu}}} \hookrightarrow\hookrightarrow L^2_\mu$
$D(\mathcal{A}_2) \hookrightarrow\hookrightarrow H_2$	$\underbrace{\mu^{-1} \mathring{D} \cap \overline{\nabla H^1}}_{= \mu^{-1} \mathring{D} \cap R \cap \mathcal{H}_N^{\perp}} \hookrightarrow\hookrightarrow L^2_\mu$
$\Updownarrow$	$\Updownarrow$
$D(\mathcal{A}_2^*) \hookrightarrow\hookrightarrow H_3$	$H^1 \cap L^2_\perp = H^1_\perp \hookrightarrow\hookrightarrow L^2$ Rellich's sel. theorem (need: Ω bd, segment property) [Ω weak Lipschitz:

	$H^1 \hookrightarrow\hookrightarrow L^2$]
$D(A_0^*) \cap D(A_1) \hookrightarrow\hookrightarrow H_1$ $\Updownarrow$ $D(\mathcal{A}_0) \hookrightarrow\hookrightarrow H_0$ $\wedge\ D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_1$ $\wedge\ K_1 = N(A_1) \cap N(A_0^*)$ is finite dimens.	$\varepsilon^{-1} D \cap \mathring{R} \hookrightarrow\hookrightarrow L^2_\varepsilon$ Weck's sel. theorem (need: Ω bounded, weak Lipschitz) $\Updownarrow$ $\mathring{H}^1 \hookrightarrow\hookrightarrow L^2$ (already: ✓) $\wedge\ \mathring{R} \cap \varepsilon^{-1} \operatorname{rot} R \hookrightarrow\hookrightarrow L^2_\varepsilon$ $\wedge\ \mathring{R}_0 \cap \varepsilon^{-1} D_0 = \mathcal{H}_D$ is finite dimensional.
$D(A_1^*) \cap D(A_2) \hookrightarrow\hookrightarrow H_2$ $\Updownarrow$ $D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_1$ $\wedge\ D(\mathcal{A}_2^*) \hookrightarrow\hookrightarrow H_3$ $\wedge\ K_2 = N(A_2) \cap N(A_1^*)$ is finite dimens.	$R \cap \mu^{-1} \mathring{D} \hookrightarrow\hookrightarrow L^2_\mu$ Weck's selection theorem (need: Ω bounded, weak Lipschitz) $\Updownarrow$ $\mathring{R} \cap \varepsilon^{-1} \operatorname{rot} R \hookrightarrow\hookrightarrow L^2_\varepsilon$ (?) $\wedge\ H^1_\perp \hookrightarrow\hookrightarrow L^2$ $\wedge\ R_0 \cap \mu^{-1} \mathring{D}_0 = \mathcal{H}_N$ is finite dimensional.

Poincaré-type estimates:

$R(A_i)$ closed $\Leftrightarrow \forall x \in D(\mathcal{A}_i): |x|_{H_i} \le c_i |\mathcal{A} x|_{H_{i+1}}$

$\Updownarrow$

$R(A_i^*)$ closed $\Leftrightarrow \forall x \in D(\mathcal{A}_i^*): |x|_{H_{i+1}} \le c_i |\mathcal{A}^*_i x|_{H_i}$

$\rightsquigarrow$ apply to the Maxwell case:

* $R(A_0) = R(\mathring{\nabla}) = \mathring{\nabla}\mathring{H}^1$ closed (follows e.g. from Rellich's selection theorem)

  $\Leftrightarrow \forall x \in \mathring{H}^1 : |x|_{L^2} \le c_F |\nabla x|_{L^2_\varepsilon}$

  $\Leftrightarrow \forall u \in \varepsilon^{-1} D \cap \mathring{\nabla}\mathring{H}^1 : |u|_{L^2_\varepsilon} \le c_F |\operatorname{div} \varepsilon u|_{L^2}$

* $R(A_1) = \mu^{-1}\mathring{\operatorname{rot}}\mathring{R}$ is closed $\Leftrightarrow \mathring{\operatorname{rot}}\mathring{R}$ is closed (follows e.g. from Weck's selection theorem)

  $\Leftrightarrow \forall x \in \mathring{R} \cap \varepsilon^{-1}\operatorname{rot} R : |x|_{L^2_\varepsilon} \le c_m |\mu^{-1}\operatorname{rot} x|_{L^2_\mu}$
  $= c_m |\operatorname{rot} x|_{L^2_{\mu^{-1}}}$

  $\Leftrightarrow \forall u \in R \cap \mu^{-1}\mathring{\operatorname{rot}}\mathring{R} : |u|_{L^2_\mu} \le c_m |\varepsilon^{-1}\operatorname{rot} u|_{L^2_\varepsilon}$
  $= c_m |\operatorname{rot} u|_{L^2_{\varepsilon^{-1}}}$

* $R(A_2) = L^2_\perp$ is closed $\Leftrightarrow R(A_2^*)$ is closed $\Leftrightarrow \nabla H^1$ is closed (follows e.g. from Rellich's selection theorem: $H^1 \hookrightarrow L^2$)

  $\Leftrightarrow \forall x \in \mu^{-1}\mathring{D} \cap \nabla H^1 : |x|_{L^2_\mu} \le c_p |\operatorname{div} \mu x|_{L^2}$

  $\Leftrightarrow \forall u \in H^1_\perp : |u|_{L^2} \le c_p |\nabla u|_{L^2_\mu}$

Moreover:

* $N(A_1) = K_1 \oplus_{L^2_\varepsilon} R(A_0)$

  $\Rightarrow R(A_0) = N(A_1) \cap K_1^{\perp_{L^2_\varepsilon}}$

  $\Rightarrow \mathring{\nabla}\mathring{H}^1 = \mathring{R}_0 \cap \mathcal{H}_D^{\perp_{L^2_\varepsilon}}$

* $N(A_0^*) = K_1 \oplus_{L^2_\varepsilon} R(A_1^*)$

$$\Rightarrow \; R(A_1^*) = N(A_0^*) \cap K_1^{\perp_{L^2_\varepsilon}}$$

$$\Rightarrow \; \varepsilon^{-1} \operatorname{rot} R = \varepsilon^{-1} D_0 \cap \mathcal{H}_D^{\perp_{L^2_\varepsilon}}$$

$$* \quad N(A_2) = K_2 \oplus_{L^2_\mu} R(A_1)$$

$$\Rightarrow \; R(A_1) = N(A_2) \cap K_2^{\perp_{L^2_\mu}}$$

$$\Rightarrow \; \mu^{-1} \mathring{\operatorname{rot}} \mathring{R} = \mu^{-1} \mathring{D}_0 \cap \mathcal{H}_N^{\perp_{L^2_\mu}}$$

$$* \quad N(A_1^*) = K_2 \oplus_{L^2_\mu} R(A_2^*)$$

$$\Rightarrow \; R(A_2^*) = N(A_1^*) \cap K_2^{\perp_{L^2_\mu}}$$

$$\Rightarrow \; \nabla H^1 = R_0 \cap \mathcal{H}_N^{\perp_{L^2_\mu}}$$

<u>Lemma 7</u>:

If Ω bounded and convex, we have

$$c_m \le c_p \le \frac{\operatorname{diam} \Omega}{\pi}.$$