Maxwell - School, Linz 2016

Lecture 1, 10.10.16

Ω (open, connected) $\subset \mathbb{R}^3$;
$I \subset \mathbb{R}$ (time interval)

$E : I \times \Omega \to \mathbb{C}^3$ electric field
$D : I \times \Omega \to \mathbb{C}^3$ electric displacement field
$B : I \times \Omega \to \mathbb{C}^3$ magnetic field
$H : I \times \Omega \to \mathbb{C}^3$ magnetizing field
$\varepsilon : I \times \Omega \to \mathbb{R}^{3\times 3}$ permittivity
$\mu : I \times \Omega \to \mathbb{R}^{3\times 3}$ permeability
$\rho : I \times \Omega \to \mathbb{C}$ charge density
$P : I \times \Omega \to \mathbb{C}$ polarization
$\mathcal{M} : I \times \Omega \to \mathbb{C}^3$ magnetization
$\sigma : I \times \Omega \to \mathbb{R}^{3\times 3}$ conductivity
$J : I \times \Omega \to \mathbb{C}^3$ current density

$\partial_t D - \operatorname{rot} H = -J$ in Ω (Ampere's law)
$\partial_t B + \operatorname{rot} E = 0$ in Ω (Faraday's law)
$\operatorname{div} D = \rho$ in Ω
$\operatorname{div} B = 0$ in Ω (Gauss law)

$D = \varepsilon E + P$
$B = \mu H + \mathcal{M}$
$J = \sigma E - F$

Assumptions:
$P = \mathcal{M} = \sigma = 0$

⇝ end up with:

$$\partial_t \varepsilon E - \operatorname{rot} H = F$$
$$\partial_t \mu H + \operatorname{rot} E = G \quad (\text{usually } G = 0)$$
$$\operatorname{div} \varepsilon E = \rho$$
$$\operatorname{div} \mu H = \tau \quad (\text{usually } \tau = 0)$$

(E, H) fields of interest

⇝ initial conditions:

$$E(0) = E_0 \;\&\; H(0) = H_0$$

⇝ boundary conditions:

later on ...

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some identities:

$$\varphi, \psi \in C^\infty(\Omega, \mathbb{C}) \; ; \; \phi \in C^\infty(\Omega, \mathbb{C}^3)$$

$$\rightsquigarrow \nabla \varphi = \begin{bmatrix} \partial_1 \varphi \\ \partial_2 \varphi \\ \partial_3 \varphi \end{bmatrix}, \quad \operatorname{rot} \phi = \begin{bmatrix} \partial_2 \phi_3 - \partial_3 \phi_2 \\ \partial_3 \phi_1 - \partial_1 \phi_3 \\ \partial_1 \phi_2 - \partial_2 \phi_1 \end{bmatrix}$$

$$\operatorname{div} \phi = \partial_1 \phi_1 + \partial_2 \phi_2 + \partial_3 \phi_3$$

$$\rightsquigarrow \operatorname{rot} \nabla \varphi = 0 \quad (\text{Schwarz lemma})$$
$$\operatorname{div} \operatorname{rot} \phi = 0$$
$$\Delta \varphi = \operatorname{div} \nabla \varphi = \partial_1^2 \varphi + \partial_2^2 \varphi + \partial_3^2 \varphi$$

$$\rightsquigarrow \operatorname{rot} \operatorname{rot} \phi - \nabla \operatorname{div} \phi = - \Delta \phi = - \begin{bmatrix} \Delta \phi_1 \\ \Delta \phi_2 \\ \Delta \phi_3 \end{bmatrix}$$

$$\rightsquigarrow \nabla (\varphi \psi) = \psi \nabla \varphi + \varphi \nabla \psi$$
$$\operatorname{rot} (\varphi \phi) = \varphi \operatorname{rot} \phi + \nabla \varphi \times \phi$$
$$\operatorname{div} (\varphi \phi) = \varphi \operatorname{div} \phi + \nabla \varphi \cdot \phi$$

$\Rightarrow$ commutators are related to algebraic operations.

Suppose ε, μ are time independent:

$$\begin{aligned}\partial_t^2 \varepsilon E &= \partial_t(\partial_t \varepsilon E)\\ &= \partial_t(\operatorname{rot} H) + \partial_t F\\ &= \operatorname{rot}(\partial_t H) + \partial_t F\\ &= \operatorname{rot}(-\mu^{-1}\operatorname{rot} E + \mu^{-1} G) + \partial_t F\end{aligned}$$

$$\Rightarrow \partial_t^2 \varepsilon E + \operatorname{rot}\mu^{-1}\operatorname{rot} E = \tilde{\tilde{F}}$$
with $\tilde{\tilde{F}} := \operatorname{rot}\mu^{-1} G + \partial_t F$

$\varepsilon = \mu = id$: (wave equation)

$$\begin{aligned}\Rightarrow \partial_t^2 E + \operatorname{rot}\operatorname{rot} E &= \tilde{\tilde{F}}\\ \Rightarrow \partial_t^2 E - \Delta E &= \tilde{\tilde{F}} + \nabla \operatorname{div} E\\ &= \tilde{\tilde{F}} + \nabla \rho\end{aligned}$$

$\rightsquigarrow$ similar with H: $\partial_t^2 H - \Delta H = \tilde{G}$

Applying div to equations we get:

$$\partial_t \operatorname{div} \varepsilon E = \operatorname{div} F$$
$$\partial_t \operatorname{div} \mu H = \operatorname{div} G$$

$$\Rightarrow \rho = \operatorname{div} \varepsilon E = \operatorname{div} \varepsilon E_0 + \int_0^t \operatorname{div} F$$

$$\tau = \operatorname{div} \mu H = \operatorname{div} \mu H_0 + \int_0^t \operatorname{div} G$$

$\Rightarrow$ second two equations are already included.

Abstract framework, spectral - theory

$$\Big(\partial_t \underbrace{\begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}}_{=: \Lambda} + \underbrace{\begin{bmatrix} 0 & -\mathrm{rot} \\ \mathrm{rot} & 0 \end{bmatrix}}_{=: A} \Big) \underbrace{\begin{bmatrix} E \\ H \end{bmatrix}}_{=: x} = \underbrace{\begin{bmatrix} F \\ G \end{bmatrix}}_{=: \tilde{F}}$$

$$(\partial_t \Lambda + A)\, x = \hat{F}$$

$$\Leftrightarrow (\partial_t + \Lambda^{-1} A)\, x = \Lambda^{-1} \hat{F} =: f$$

$$\Rightarrow \partial_t^2 x = \partial_t (f - \Lambda^{-1} A x)$$

$$= - \Lambda^{-1} A\, \partial_t x + \partial_t f$$

$$= \Lambda^{-1} A\, \Lambda^{-1} A + \partial_t f + \Lambda^{-1} A F$$

$$\Rightarrow (\partial_t^2 - \Lambda^{-1} A\, \Lambda^{-1} A)\, x = \partial_t f + \Lambda^{-1} A F$$

(generalized wave equation)

$\phi, \psi \in C^\infty(\Omega, \mathbb{C}^3)$

$\Rightarrow \operatorname{div}(\phi \times \psi)$

$$= \operatorname{div} \begin{bmatrix} \phi_2 \psi_3 - \phi_3 \psi_2 \\ \phi_3 \psi_1 - \phi_1 \psi_3 \\ \phi_1 \psi_2 - \phi_2 \psi_1 \end{bmatrix}$$

$$= (\partial_1 \phi_2) \psi_3 + \phi_2 (\partial_1 \psi_3) - (\partial_1 \phi_3) \psi_2$$

$$- \phi_3 (\partial_1 \psi_2) + (\partial_2 \phi_3) \psi_1 + \phi_3 (\partial_2 \psi_1)$$

$$- (\partial_2 \phi_1) \psi_3 - \phi_1 (\partial_2 \psi_3) + \dots$$

$$= \psi \cdot \mathrm{rot}\, \phi - \phi \cdot \mathrm{rot}\, \psi$$

$\Rightarrow \forall \phi, \psi \in C^\infty(\overline{\Omega}, \mathbb{C}^3) := C^\infty(\mathbb{R}^3, \mathbb{C}^3)|_\Omega$

$$\int_\Omega \operatorname{rot}\phi \cdot \psi - \phi \cdot \operatorname{rot}\psi \, dx$$

$$= \int_\Omega \operatorname{div}(\phi \times \psi) \, dx$$

$$= \int_{\partial\Omega} n \cdot (\phi \times \psi) \, dS$$

$$= \int_{\partial\Omega} (n \times \phi) \cdot \psi \, dS = \int_{\partial\Omega} (n \times \phi)(n \times \psi \times n) \, dS$$

since $\psi = (\psi \cdot n) \cdot n$

Notation:

$$\langle \phi, \psi \rangle_\Omega := \int_\Omega \phi \cdot \overline{\psi} \, dx \quad (L^2 \text{ inner product})$$

$$\langle \phi, \psi \rangle_\Gamma := \int_\Gamma \phi \cdot \overline{\psi} \, dS \quad (\text{on boundary})$$

$$\rightsquigarrow \langle \operatorname{rot}\phi, \psi \rangle_\Omega - \langle \phi, \operatorname{rot}\psi \rangle_\Omega = \langle n \times \phi, \psi \rangle_\Gamma$$

$$\operatorname{div}(\varphi\phi) = \varphi \operatorname{div}\phi + \nabla\varphi \cdot \phi$$

$$\rightsquigarrow \langle \nabla\varphi, \phi \rangle_\Omega + \langle \varphi, \operatorname{div}\phi \rangle_\Omega = \langle \varphi, n \cdot \phi \rangle_\Gamma$$

$\phi, \psi \in C^\infty(\overline{\Omega}, \mathbb{C}^3)$, $\psi \in \mathring{C}^\infty(\Omega, \mathbb{C}^3)$

$$< \begin{bmatrix} E \\ H \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix} >_D := < E, \phi >_D + < H, \psi >_D$$

$$< A \begin{bmatrix} E \\ H \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix} >_\Omega$$

$$= < -\operatorname{rot} H, \phi >_\Omega + < \operatorname{rot} E, \psi >_\Omega$$

$$= < H, -\operatorname{rot} \phi >_\Omega + < E, \operatorname{rot} \psi >_\Omega$$

$$= - < \begin{bmatrix} E \\ H \end{bmatrix}, A \begin{bmatrix} \phi \\ \psi \end{bmatrix} >$$

$\Rightarrow$ A is skew-symmetric operator

$\Rightarrow$ $\Lambda^{-1} A$ is skew-symmetric (with respect to <u>weighted</u> inner product)

$\Rightarrow$ $\boxed{i \Lambda^{-1} A}$ is symmetric and a good candidate for a „self-adjoint"-operator.

rewrite:

$$\Rightarrow \left(\partial_t - i \underbrace{i \Lambda^{-1} A}_{=: M} \right) x = f$$

Maxwell-operator is self-adjoint.

spectral theorem

$$\Rightarrow x(t) = e^{itM} x_0 + \int_0^t e^{-i(t-s)M} f(s) \, ds$$

boundary conditions:

$\varphi, \phi \in C^\infty(\overline{\Omega})$:

$$\langle \operatorname{rot}\phi, \nabla\varphi \rangle_{\Omega}$$

$$\overset{①}{=} -\langle \underbrace{\operatorname{div}\operatorname{rot}\phi}_{=0}, \varphi \rangle_{\Omega} + \int_\Gamma \overline{\varphi}\, n\cdot \operatorname{rot}\phi \, dS$$

$$\overset{②}{=} \langle \phi, \underbrace{\operatorname{rot}\nabla\varphi}_{=0} \rangle + \int_\Gamma (n\times\phi)\cdot\nabla\overline{\varphi}\, dS$$

$$\Rightarrow \int_\Gamma \overline{\varphi}\,(n\cdot\operatorname{rot}\phi)\, dS = \int_\Gamma \nabla\overline{\varphi}\cdot(n\times\phi)\, dS$$

$$= \int_\Gamma -(n\times\nabla\overline{\varphi})\cdot\phi\, dS$$

Therefore:

$*$ $\varphi = 0$ at Γ $\Rightarrow$ $\int_\Gamma (n\times\nabla\overline{\varphi})\cdot\phi\, dS = 0$

since $C^\infty(\overline{\Omega})_{|\Gamma}$ dense in $L^2(\Gamma)$ $\curvearrowright$ $\Rightarrow$ $n\times\nabla\varphi = 0$ at Γ

$*$ $n\times\phi = 0$ at Γ $\Rightarrow$ $n\cdot\operatorname{rot}\phi = 0$ at Γ

$\leadsto$ so: apply $n\times E_{|\Gamma} = 0$

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time-harmonic ansatz:

$$E(t,x) = e^{i\omega t}\,\tilde{E}(x), \quad H(t,x) = e^{i\omega t}\,\hat{H}(x)$$

$\Rightarrow \quad (\partial_t - i\mathcal{U})x = f \Leftrightarrow i(\omega - \mathcal{U})\tilde{x} = \hat{f}$

$\Leftrightarrow \quad (\mathcal{U} - \omega)\tilde{x} = -i\tilde{f}$

↳ apply Fredholm alternative

$-i\varepsilon^{-1}\operatorname{rot} H - \omega E = F \quad \text{in } \Omega_1$

$i\mu^{-1}\operatorname{rot} E - \omega H = G \quad \text{in } \Omega_1$

with homogeneous boundary condition:

$n \times E_{|\Gamma} = 0$

① $\underline{\omega \neq 0}$:

$\varrho = \operatorname{div} \varepsilon E = 1/\omega \operatorname{div} \varepsilon F$

$\tau = \operatorname{div} \mu H = 1/\omega \operatorname{div} \mu G$

$\rightsquigarrow \varrho, \tau$ already given by initial data

② $\underline{\omega = 0}$: (static case)

$\operatorname{rot} E = F$

$\operatorname{rot} H = G \qquad n \times E = 0 \text{ on } \Gamma$

now we have to add:

$\operatorname{div} \varepsilon E = \varrho$

$\operatorname{div} \mu H = \tau \qquad n \cdot \mu H = 0 \text{ on } \Gamma$

↗ after proj.: $n\mu G = 0$

observe:

$n \times E = 0 \Rightarrow n \cdot \operatorname{rot} E = 0$

$\Rightarrow n(\omega\mu H + \mu G) = 0 \text{ on } \Gamma$

kernels: $\mathcal{H}_D$ (for E), $\mathcal{H}_N$ (for H)

- uniqueness by: $E \perp \mathcal{H}_D$ or $\pi_{\mathcal{H}_D} E = K$
  for K given.

Lecture 2, 10.10.16

∇, rot, div ; $\operatorname{rot} \nabla = 0$, $\operatorname{div} \operatorname{rot} = 0$

$\Rightarrow R(\nabla) \subset N(\operatorname{rot})$, $R(\operatorname{rot}) \subset N(\operatorname{div})$

$$\mathbb{R} \xrightarrow{\iota_{\mathbb{R}}} C^\infty(\bar{\Omega}) \xrightarrow{\nabla} C^\infty(\bar{\Omega}) \xrightarrow{\operatorname{rot}} C^\infty(\bar{\Omega}) \xrightarrow{\operatorname{div}} C^\infty(\bar{\Omega}) \xrightarrow{0}$$

sequence / complex

Sobolev spaces:

- $L^2\underbrace{(\Omega)}_{\text{skipped if clear}}$ either scalar/vector/tensor-valued
- $u \in L^2(\Omega)$, $\partial_m u \in L^2(\Omega)$
  $\Leftrightarrow \exists v_m \in L^2 \ \forall \varphi \in \mathring{C}^\infty : \langle u, \partial_m \varphi \rangle_\Omega = - \langle v_m, \varphi \rangle_\Omega$
  (if v_m exists, it is uniquely defined)
  $\leadsto$ set $\partial_m u := v_m$
- $H^1 := \{ u \in L^2 : \partial_m u \in L^2 \ \forall m = 1,2,3 \}$
  $H^l := \{ u \in L^2 : \forall |\alpha| \le l \ \partial^\alpha u \in L^2 \}$
- $R := \{ E \in L^2 : \operatorname{rot} E \in L^2 \}$

  $\exists F \in L^2 \ \forall \Phi \in C^\infty : \langle E, \operatorname{rot} \Phi \rangle_\Omega = \langle F, \Phi \rangle_\Omega$
- $D := \{ E \in L^2 : \operatorname{div} E \in L^2 \}$

  $\exists F \in L^2 \ \forall \phi \in C^\infty : \langle E, \nabla \phi \rangle_\Omega = - \langle f, \phi \rangle_\Omega$

$$\mathbb{R} \xrightarrow{\iota_{\mathbb{R}}} H^1 \xrightarrow{\nabla} R \xrightarrow{\operatorname{rot}} D \xrightarrow{\operatorname{div}} L^2 \xrightarrow{0}$$

$$\langle u, v \rangle_{L^2} := \langle u, v \rangle_{\Omega},$$

$$\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2}$$

$$\langle u, v \rangle_{H^l} := \sum_{|\alpha| \leq l} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2}$$

$$\langle u, v \rangle_{R} := \langle u, v \rangle_{L^2} + \langle \operatorname{rot} u, \operatorname{rot} v \rangle_{L^2}$$

$$\langle u, v \rangle_{D} := \langle u, v \rangle_{L^2} + \langle \operatorname{div} u, \operatorname{div} v \rangle_{L^2}$$

$\rightsquigarrow$ spaces are complete.

$(u_n)_n$ cauchy-sequence in R

$\Rightarrow$ $(u_n)_n$, $(\operatorname{rot} u_n)_n$ are cauchy-sequences in L^2

L^2 complete $\Rightarrow$ $u_n \to u \in L^2$

$\operatorname{rot} u_n \to v \in L^2$

$\Rightarrow$ $\forall \phi \in \overset{\circ}{C}{}^\infty$:

$$\begin{aligned} &\langle u, \operatorname{rot} \phi \rangle_{L^2} \\ &= \lim_{n \to \infty} \langle u_n, \operatorname{rot} \phi \rangle_{L^2} \\ &= \lim_{n \to \infty} \langle \operatorname{rot} u_n, \phi \rangle_{L^2} \\ &= \langle v, \phi \rangle_{L^2} \end{aligned}$$

$\Rightarrow$ $u \in R$, $\operatorname{rot} u = v$

$\Rightarrow$ R complete.

$\boxed{\lightning}$

$*$ $\overset{\circ}{H}{}^1 := \overline{\overset{\circ}{C}{}^\infty}^{H^1} \ni u : \langle \nabla u, \phi \rangle_{L^2} = - \langle u, \operatorname{div} \phi \rangle_{L^2}$ for all $\phi \in D$.

$*$ $\overset{\circ}{R} := \overline{\overset{\circ}{C}{}^\infty}^{R} \ni u : \langle \operatorname{rot} u, \phi \rangle_{L^2} = \langle u, \operatorname{rot} \phi \rangle_{L^2}$ for all $\phi \in R$

* $\mathring{D} := \overline{\mathring{C}^{\infty}}^{D} \ni u : \langle \operatorname{div} u, \phi \rangle_{L^2} = -\langle u, \nabla \phi \rangle_{L^2}$
for all $\phi \in H^1$

<u>Proof</u>:

$u \in \mathring{C}^{\infty}, \phi \in C^{\infty}(\bar{\Omega}) : \langle \nabla u, \phi \rangle_{L^2} = -\langle u, \operatorname{div} \phi \rangle_{L^2}$

now:

$u \in \mathring{H}^1, \phi \in D$

$\Rightarrow \exists (u_n)_n \subset \mathring{C}^{\infty}, u_n \to u$ in H^1

$\Rightarrow \langle \nabla u, \phi \rangle_{L^2}$

$= \lim\limits_{n \to \infty} \langle \nabla u_n, \phi \rangle_{L^2}$

by defin. of weak ∇ $\rightsquigarrow$ $= \lim\limits_{n \to \infty} -\langle u_n, \operatorname{div} \phi \rangle_{L^2}$

$= -\langle u, \operatorname{div} \phi \rangle_{L^2}$ $\square$

($u \in H^1, \phi \in \mathring{D}$ also possible since you only need $\mathring{C}^{\infty}$ dense in the space)

<u>so</u>:

$u \in \mathring{H}^1 \rightsquigarrow$ „ $u_{|\Gamma} = 0$ "

$u \in \mathring{H}^1 \cap C^{\infty}(\bar{\Omega})$

$\Rightarrow \langle u, n \cdot \phi \rangle_{L^2(\Gamma)} = 0 \quad \forall \phi \in C^{\infty}(\bar{\Omega})$

$n \cdot C^{\infty}(\bar{\Omega})_{|\Gamma}$ dense in $L^2(\Gamma)$

$u \in \mathring{R} \rightsquigarrow$ „ $n \times u_{|\Gamma} = 0$ "

$u \in \mathring{D} \rightsquigarrow$ „ $n \cdot u_{|\Gamma} = 0$ "

Remarks:

- * so far: no smoothness needed
- * also holds in unbounded case
- * can say $(u - u_0) \in \mathring{R}$ to „speak" about traces.

$$\mathring{H}^1 \xrightarrow{\nabla} \mathring{R} \xrightarrow{\mathrm{rot}} \mathring{D} \xrightarrow{\mathrm{div}} L^2 \xrightarrow{0}$$

<u>Lemma:</u>

(i) $\overline{\nabla H^1} \subset R_0$ (i') $\overline{\nabla \mathring{H}^1} \subset \mathring{R}_0$

(ii) $\overline{\mathrm{rot}\, R} \subset D_0$ (ii') $\overline{\mathrm{rot}\, \mathring{R}} \subset \mathring{D}_0$

<u>Proof of (ii):</u>

Enough to show: $\mathrm{rot}\, R \subset D_0$

(because D_0 is closed - being a kernel)

$u \in R$, $\varphi \in \mathring{C}^\infty$:

$$\langle \mathrm{rot}\, u, \nabla \varphi \rangle_{L^2} = \langle u, \mathrm{rot}\, \nabla \varphi \rangle = 0$$

$$\Rightarrow \mathrm{rot}\, u \in D \wedge \mathrm{div}\,\mathrm{rot}\, u = 0$$

<u>Proof of (ii'):</u>

$u \in \mathring{R} \Leftrightarrow \exists (u_n)_n \subset \mathring{C}^\infty,\ u_n \to u$ in R

$\Rightarrow$ $u_n \to u$ in L^2

$\mathrm{rot}\, u_n \to \mathrm{rot}\, u$ in L^2

$\Rightarrow$ $\mathrm{rot}\, u_n \to \mathrm{rot}\, u$ in D

$\Rightarrow$ $\mathrm{rot}\, u \in \mathring{D}_0$

$\boxed{\lightning}$

$(\mathring{H}^1 = \mathbb{R}$ (or $\mathbb{C}$), $\mathring{H}^1_0 = \{0\})$

Lemma:

$\varphi \in C^\infty(\overline{\Omega}, \mathbb{C})$

(i) $u \in \overset{(\circ)}{H}{}^1 \Rightarrow \varphi u \in \overset{(\circ)}{H}{}^1$,

$$\nabla(\varphi u) = u \nabla \varphi + \varphi \nabla u$$

(ii) $u \in \overset{(\circ)}{R} \Rightarrow \varphi u \in \overset{(\circ)}{R}$,

$$\operatorname{rot}(\varphi u) = \varphi \operatorname{rot} u + \nabla \varphi \times u$$

(iii) $u \in \overset{(\circ)}{D} \Rightarrow \varphi u \in \overset{(\circ)}{D}$,

$$\operatorname{div}(\varphi u) = \varphi \operatorname{div} u + \nabla \varphi \cdot u$$

(property of first order operators)

Proof:

* $u \in R, \varphi \in C^\infty(\overline{\Omega})$:

$\Rightarrow \forall \psi \in \mathring{C}^\infty(\Omega)$:

$$\begin{aligned} &\langle \varphi u, \operatorname{rot} \psi \rangle_{L^2} \\ &= \langle u, \varphi \operatorname{rot} \psi \rangle_{L^2} \\ &= \langle u, \operatorname{rot}(\varphi \psi) \rangle_{L^2} - \langle u, \nabla \varphi \times \psi \rangle_{L^2} \\ &= \langle \operatorname{rot} u, \varphi \psi \rangle_{L^2} + \langle \nabla \varphi \times u, \psi \rangle_{L^2} \\ &= \langle \varphi \operatorname{rot} u, \psi \rangle_{L^2} + \langle \nabla \varphi \times u, \psi \rangle_{L^2} \end{aligned}$$

* $u \in \mathring{R} \Rightarrow \exists (u_n)_n \subset \mathring{C}^\infty, \; u_n \to u$ in R

$\Rightarrow \varphi u_n \to \varphi u$ in L^2

$$\begin{aligned} \operatorname{rot}(\varphi u_n) = \varphi &\operatorname{rot} u_n + \nabla \varphi \times u_n \\ &\Big\downarrow \text{ in } L^2 \\ \varphi &\operatorname{rot} u + \nabla \varphi \times u \end{aligned}$$

$\Rightarrow \varphi u_n \to \varphi u$ in $R \Rightarrow \varphi u \in \mathring{R}$

Definition:

A sequence is called closed, iff all ranges are closed.

$$\left.\begin{array}{l}\nabla : \mathring{C}^\infty \subset L^2 \to L^2 \\ \mathrm{rot} : \mathring{C}^\infty \subset L^2 \to L^2 \\ \mathrm{div} : \mathring{C}^\infty \subset L^2 \to L^2\end{array}\right\}$$ linear operators, densly defined, poss. unbounded

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Functional Analysis Toolbox

$A : D(A) \subset H_1 \to H_2$ linear

$D(A)$: domain of definition

H_1, H_2: Hilbert spaces

$G(A) := \{ (x, Ax) \in H_1 \times H_2 : x \in D(A) \}$

"graph of A"

- A is densly defined $:\Leftrightarrow \overline{D(A)}^{H_1} = H_1$
- A is closed $:\Leftrightarrow G(A)$ closed

  $\Leftrightarrow \forall (x_n)_n \subset D(A),\ x_n \to x$ in H_1, $Ax_n \to y$ in H_2

  $\Rightarrow x \in D(A) \wedge Ax = y$

  (i.e. ∇, rot, div as above are not closed)
- A is closable $:\Leftrightarrow \exists B : \overline{G(A)} = G(B)$, $B = \bar{A}$

  $\Leftrightarrow \forall (x_n)_n \subset D(A),\ x_n \to 0$ in H_1, $Ax_n \to y$ in $H_2 \Rightarrow y = 0$.

Definition:

$A: D(A) \subset H_1 \to H_2$ densly defined, linear

$\Rightarrow \exists A^*: D(A^*) \subset H_2 \to H_1$ „adjoint"

$y \in D(A^*) \wedge A^* y = f \in H_1$

$\Leftrightarrow y \in H_2 \wedge \exists f \in H_1 \; \forall \varphi \in D(A):$

$\langle A\varphi, y \rangle_{H_2} = \langle \varphi, f \rangle_{H_1}$, $A^* y := f$

$\rightsquigarrow A^*$ is unique, since

$\langle \varphi, f_1 - f_2 \rangle = 0 \quad \forall \varphi \in D(A)$

$\Rightarrow f_1 - f_2 = 0$ ($D(A)$ is dense in H_1)

$\bar{A}: D(\bar{A}) \subset H_1 \to H_2$, $D(\bar{A}) = \overline{D(A)}^{G(A)}$

$\|.\|_{G(A)} = (|.|^2 + |A.|^2)^{1/2}$ graph norm

$\rightsquigarrow A$ densly defined, closed: $A = \bar{A} = A^{**}$

then: (A, A^*) is called dual pair

General assumptions:

A linear, densly defined, closed

($\Rightarrow A^*$ closed , $A^* = \bar{A}^* = \overline{A^*}$)

Lecture 3, 10.10.16

projection theorem

$\Rightarrow H_2 = \overline{R(A)} \oplus N(A^*)$, $H_1 = \overline{R(A^*)} \oplus N(A)$

$A^*: D(A^*) \subset H_2 \to H_1$ densly defined, closed and (A, A^*) dual pair.

reduced operators: (project kernels away)

$$\mathcal{A} : \underbrace{D(A) \cap \overline{R(A^*)}}_{=: D(\mathcal{A})} \subset \overline{R(A^*)} \to R(A)$$

$$\mathcal{A}^* : \underbrace{D(A^*) \cap \overline{R(A)}}_{=: D(\mathcal{A}^*)} \subset \overline{R(A)} \to R(A^*)$$

- $(\mathcal{A}, \mathcal{A}^*)$ dual pair
- $\mathcal{A}, \mathcal{A}^*$ injective (possibly unbounded)

  $\Rightarrow \exists\, \mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$

  $(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$

If the ranges $R(A), R(A^*)$ are closed
$\Rightarrow$ inverses are continuous

closed range theorem:

$$R(A) \text{ closed} \Leftrightarrow R(A^*) \text{ closed}$$

<u>Lemma 1</u>:

A densly defined, closed linear operator. Then the following assertions are equivalent:

(i) $\forall x \in D(\mathcal{A}) : |x|_{H_1} \leq c_A |Ax|_{H_2}$

(i') $\forall y \in D(\mathcal{A}^*) : |y|_{H_2} \leq c_A |A^* y|_{H_1}$

(ii) $R(A) = R(\mathcal{A})$ closed

(ii') $R(A^*) = R(\mathcal{A}^*)$ closed

(iii) $\mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$ bounded

(iii*) $(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$ bounded

Then:

$$|\mathcal{A}^{-1}|_{R(A),\,R(A^*)} \leq c_A$$

$$|\mathcal{A}^{-1}|_{R(A),\,D(A)} \leq (1 + c_A^2)^{1/2}$$

$$|(\mathcal{A}^*)^{-1}|_{R(A^*),\,R(A)} \leq c_{A^*}$$

$$|(\mathcal{A}^*)^{-1}|_{R(A^*),\,D(A^*)} \leq \left(1 + c_{A^*}^2\right)^{1/2}$$

<u>idea of the proof</u>:

$$D(A) = \overline{R(A^*)} \cap D(A) \oplus N(A)$$

$$\Rightarrow R(A) = R(\mathcal{A})$$

(same for the adjoint: $R(A^*) = R(\mathcal{A}^*)$)

(ii) $\Leftrightarrow$ (ii') by closed range theorem.

Show:

(i) $\Rightarrow$ (ii') $\Rightarrow$ (iii') $\Rightarrow$ (i')

$\boxed{4}$

$$1/c_A := \inf_{\substack{x \in D(\mathcal{A}) \\ x \neq 0}} \frac{|Ax|}{|x|} \quad ; \quad 1/c_{A^*} := \inf_{\substack{y \in D(\mathcal{A}^*) \\ y \neq 0}} \frac{|A^* y|}{|y|}$$

$$|\mathcal{A}^{-1}|_{R(A),\,R(A^*)} = \sup_{\substack{y \in R(A) \\ y \neq 0}} \frac{|\mathcal{A}^{-1} y|}{|y|}$$

$$x := \mathcal{A}^{-1} y \in D(\mathcal{A}) \curvearrowright = \sup_{\substack{x \in D(\mathcal{A}) \\ x \neq 0}} \frac{|x|}{|Ax|}$$

$$= \Bigg(\inf_{\substack{x \in D(\mathcal{A}) \\ x \neq 0}} \frac{|Ax|}{|x|} \Bigg)^{-1} = c_A$$

Lemma 2:

$$C_A = C_{A^*}$$

Proof:

Let $y \in D(A^*) = D(A^*) \cap R(A)$.

$\Rightarrow \quad y = Ax \, , \quad x \in D(A)$

$$\begin{aligned} \Rightarrow \quad |y|^2 &= \langle y, Ax \rangle \\ &= \langle A^* y, x \rangle \\ &\leq |A^* y| \, C_A \, |Ax| \\ &= C_A \, |A^* y| \, |y| \end{aligned}$$

$\Rightarrow \quad C_{A^*} \leq C_A$

The other way around : $C_A \leq C_{A^*}$

$\square$

Remark:

Even the whole spectrum coincides.

$\leadsto$ **Need**: e.g. $R(A)$ closed
(for solution theory)

Lemma 3:

If $D(A) \hookrightarrow\hookrightarrow H_1$, then Lemma 1 holds.

Proof:

Like standard Poincare proof with cauchy sequence.

$\square$

$$\Rightarrow \quad \mathcal{A}^{-1} : R(A) \to D(\mathcal{A}) \to R(A^*) / H_1$$

is compact.

(solution operator is compact
$\Rightarrow$ discrete spectrum, Fredholm)

$$\left.\begin{array}{l} A_1 : D(A_1) \subset H_1 \to H_2 \\ A_2 : D(A_2) \subset H_2 \to H_3 \end{array}\right. \quad \text{densly defined, closed}$$

$$\begin{aligned} A_2 A_1 = 0 \quad &\Leftrightarrow \quad R(A_1) \subset \mathcal{N}(A_2) \\ &\Leftrightarrow \quad A_1^* A_2^* = 0 \\ &\Leftrightarrow \quad R(A_2^*) \subset \mathcal{N}(A_1^*) \end{aligned}$$

<u>Helmholtz decompositions</u>:

$$\begin{array}{rlcccc} H_1 = & \overline{R(A_1^*)} \oplus \mathcal{N}(A_1) & & & & \\ H_2 = & \overline{R(A_1)} \oplus \mathcal{N}(A_1^*) & = & \overline{R(A_1)} & \oplus & \mathcal{N}(A_1^*) \\ & & & \cap & & \cup \\ = & \overline{R(A_2^*)} \oplus \mathcal{N}(A_2) & = & \mathcal{N}(A_2) & \oplus & \overline{R(A_2^*)} \\ H_3 = & \overline{R(A_2)} \oplus \mathcal{N}(A_2^*) & & & & \end{array}$$

$$\Rightarrow \quad \mathcal{N}(A_2) = \overline{R(A_1)} \oplus [\, \mathcal{N}(A_1^*) \cap \mathcal{N}(A_2) \,]$$

$\rightsquigarrow$ $K_2 = \mathcal{N}(A_2) \cap \mathcal{N}(A_1^*)$ cohom. group
(„harmonic fields" in Maxwell)

$$\Rightarrow \quad H_2 = \underbrace{\overline{R(A_1)} \oplus K_2}_{\mathcal{N}(A_2)} \oplus \overline{R(A_2^*)}$$

$$\underbrace{K_2 \oplus \overline{R(A_2^*)}}_{\mathcal{N}(A_1^*)}$$

$$\begin{aligned} \Rightarrow \quad D(A_2) &= \overline{R(A_1)} \oplus K_2 \oplus [\, D(A_2) \cap \overline{R(A_2^*)} \,] \\ &= \mathcal{N}(A_2) \oplus D(\mathcal{A}_2) \end{aligned}$$

$$\Rightarrow \; D(A_2) \cap D(A_1^*) = D(\mathcal{A}_2) \oplus K_2 \oplus D(\mathcal{A}_1^*)$$

<u>*Lemma $2\frac{1}{2}$:*</u>

$$D(\mathcal{A}) \hookrightarrow\hookrightarrow H_1 \iff D(\mathcal{A}^*) \hookrightarrow\hookrightarrow H_2$$

<u>*Lemma 4:*</u>

$$D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_1 \wedge D(\mathcal{A}_2) \hookrightarrow\hookrightarrow H_2 \wedge K_2 \hookrightarrow\hookrightarrow H_2$$
$$\iff D_2 := D(A_2) \cap D(A_1^*) \hookrightarrow\hookrightarrow H_2$$

<u>*Proof:*</u>

$$D(\mathcal{A}_1^*) = D(A_1^*) \cap \overline{R(A_1)}$$
$$\subset D(A_1^*) \cap N(A_2) \subset D_2$$

$$D(\mathcal{A}_2) = D(A_2) \cap \overline{R(A_2^*)}$$
$$\subset D(A_2) \cap N(A_1^*) \subset D_2$$

$$K_2 \subset D_2$$

⇝ For other direction:
refined Helmholtz equation

$\square$

---

sequence:

$$\begin{tikzcd} D(A_1) \arrow[r, "A_1"] & D(A_2) \arrow[r, "A_2"] & H_3 \\ H_1 & D(A_1^*) \arrow[l, "A_1^*"'] & D(A_2^*) \arrow[l, "A_2^*"'] \end{tikzcd}$$

$$A_2 A_1 = 0 \;\Updownarrow\; A_1^* A_2^* = 0$$

(one can also use $\mathcal{A} \leftrightarrow A$)

⇝ sequence is <u>exact</u>
$$:\iff K_2 = N(A_2) \ominus \overline{R(A_1)} = \{0\}$$

A densly defined, linear, closed

$\Rightarrow$ A^*A self-adjoint

$$D(A^*A) = \{ x \in D(A) : Ax \in D(A^*) \}$$

$$\Rightarrow \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} : D(A) \times D(A^*) \subset H_1 \times H_2 \to H_1 \times H_2$$

is self-adjoint operator.

Maxwell-operator:

$\begin{bmatrix} 0 & -A^* \\ A & 0 \end{bmatrix}$ is skew-self-adjoint

$\rightsquigarrow$ $\mathcal{M} := i \begin{bmatrix} 0 & -A^* \\ A & 0 \end{bmatrix}$ is self-adjoint

$\Rightarrow$ $\mathcal{M}^{-1}$ exists

Assume: ε, μ Material-parameters

$\rightsquigarrow$ apply Functional-Analysis-Toolbox:

$$\hat{A}_0 := \nabla : \mathring{C}^\infty \subset L^2 \to L^2$$
$$\hat{A}_1 := \operatorname{rot} : \mathring{C}^\infty \subset L^2 \to L^2$$
$$\hat{A}_2 := \operatorname{div} : \mathring{C}^\infty \subset L^2 \to L^2$$

linear & densly defined

closable, since:

$(E_n) \subset \mathring{C}^\infty$, $E_n \to 0$ in L^2

$\operatorname{rot} E_n \to H$ in L^2

$$\Rightarrow \quad 0 = \lim_{n \to \infty} \langle E_n, \operatorname{rot} \varphi \rangle_{L^2}$$
$$= \lim_{n \to \infty} \langle \operatorname{rot} E_n, \varphi \rangle_{L^2}$$

$$= \langle H, \varphi \rangle_{L^2}$$

$$\Rightarrow \quad H = 0.$$

closures:

$A_0 := \mathring{\nabla} : \mathring{H}^1 \subset L^2 \to L^2_\varepsilon$

$A_1 := \mu^{-1} \operatorname{rot} : \mathring{R} \subset L^2_\varepsilon \to L^2_\mu$

$A_2 := \operatorname{div} \mu : \mu^{-1} \mathring{D} \subset L^2_\mu \to L^2$

densly defined, closed

adjoints:

$* \; E \in D(A_0^*) \wedge A_0^* E = f \in L^2$

$\Leftrightarrow \; E \in L^2 \wedge \exists f \in L^2 \; \forall \varphi \in D(A_0) = \mathring{H}^1 :$

$\langle \varepsilon E, \nabla \varphi \rangle_{L^2} = \langle f, \varphi \rangle_{L^2}$

$\Rightarrow \; E \in \varepsilon^{-1} D \wedge f = -\operatorname{div} \varepsilon E$

$\Rightarrow \; A_0^* = -\operatorname{div} \varepsilon : \varepsilon^{-1} D \subset L^2_\varepsilon \to L^2$

$* \; E \in D(A_1^*) \wedge A_1^* E = f \in L^2$

$\Leftrightarrow \; E \in L^2 \wedge \exists f \in L^2 \; \forall \varphi \in D(A_1) :$

$\langle E, \operatorname{rot} \varphi \rangle_{L^2} = \langle \varepsilon f, \varphi \rangle_{L^2}$

$\Rightarrow \; E \in R \wedge \operatorname{rot} E = \varepsilon f$

$\Rightarrow \; A_1^* = \varepsilon^{-1} \operatorname{rot} : R \subset L^2_\mu \to L^2_\varepsilon$

$* \; E \in D(A_2^*) \wedge A_2^* E = f \in L^2$

$\Leftrightarrow \; E \in L^2 \wedge \exists f \in L^2 \; \forall \varphi \in D(A_2) :$

$\langle E, \operatorname{div} \mu \varphi \rangle_{L^2} = \langle \mu f, \varphi \rangle_{L^2}$

$\Rightarrow \; E \in H^1 \wedge -\nabla E = f$

$\Rightarrow \; A_2^* = -\nabla : H^1 \subset L^2 \to L^2_\mu$

reduced operators:

$$L^2 = \overline{\operatorname{div} D} \oplus \underbrace{N(\mathring{\nabla})}_{= \{ c \in \mathbb{R} : c = 0 \text{ auf } \Gamma \} = \{0\}}$$

$$L^2 = \overline{\operatorname{div} \mathring{D}} \oplus \underbrace{N(\nabla)}_{= \mathbb{R}}$$

$$\mathcal{A}^0 = \mathring{\nabla} : \mathring{H}^1 \cap \underbrace{\overline{\operatorname{div} D}}_{= L^2} \subset \underbrace{\overline{\operatorname{div} D}}_{L^2} \longrightarrow \overline{\nabla \mathring{H}^1}$$

$$\mathcal{A}^1 = \mu^{-1} \operatorname{rot} : \mathring{R} \cap \overline{\varepsilon^{-1} \operatorname{rot} R} \subset \overline{\varepsilon^{-1} \operatorname{rot} R} \longrightarrow \overline{\mu^{-1} \operatorname{rot} \mathring{R}}$$

$$\mathcal{A}^2 = \operatorname{div} \mu : \mu^{-1} \mathring{D} \cap \overline{\nabla H^1} \subset \overline{\nabla H^1} \longrightarrow \mathbb{R}^\perp$$

$$\mathcal{A}_0^* = -\operatorname{div} \varepsilon : \varepsilon^{-1} D \cap \overline{\nabla \mathring{H}^1} \subset \overline{\nabla \mathring{H}^1} \longrightarrow L^2$$

$$\mathcal{A}_1^* = \varepsilon^{-1} \operatorname{rot} : R \cap \overline{\mu^{-1} \operatorname{rot} \mathring{R}} \subset \overline{\mu^{-1} \operatorname{rot} \mathring{R}} \longrightarrow \overline{\varepsilon^{-1} \operatorname{rot} R}$$

$$\mathcal{A}_2^* = -\nabla : H^1 \cap \mathbb{R}^\perp \subset \mathbb{R}^\perp \longrightarrow \nabla H^1$$

Toolbox delivers: Poincaré estimates

$\forall u \in \mathring{H}^1 : \ |u|_{L^2} \leq c_{\mathring{\nabla}} |\nabla u|_{L^2_\varepsilon} \quad (c_{\mathring{\nabla}} = c_F)$

$\forall E \in \mathring{R} \cap \varepsilon^{-1} \operatorname{rot} R : \ |E|_{L^2_\varepsilon} \leq c_{\mathring{\operatorname{rot}}} |\operatorname{rot} E|_{L^2_{\mu^{-1}}}$

$\forall E \in \mu^{-1} \mathring{D} \cap \nabla H^1 : \ |E|_{L^2_\mu} \leq c_{\mathring{\operatorname{div}}} |\operatorname{div} E|_{L^2}$

$\forall E \in \varepsilon^{-1} D \cap \nabla \mathring{H}^1 : \ |E|_{L^2_\varepsilon} \leq c_\nabla |\operatorname{div} E|_{L^2}$

$\forall E \in R \cap \mu^{-1} \operatorname{rot} \mathring{R} : \ |E|_{L^2_\mu} \leq c_\mu |\operatorname{rot} E|_{L^2_{\varepsilon^{-1}}}$

$\forall u \in H^1 \cap \mathbb{R}^\perp : \ |u|_{L^2} \leq c_P |\nabla u|_{L^2_\mu}$

Rellich: $H^1 \hookrightarrow\hookrightarrow L^2$

if we assume:

∇H^1 closed , $\operatorname{rot} \mathring{R}$ closed , $\nabla \mathring{H}^1$ closed

$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow$

$\operatorname{div} \mathring{D}$ closed $\quad \operatorname{rot} R$ closed $\quad \operatorname{div} D$ closed

Weck's selection theorem:

$\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow\hookrightarrow L^2$

lecture 4 31.10.16

<u>Recall</u>:

$A : D(A) \subset H_1 \to H_2$ linear, densely defined, closed

$A^* : D(A^*) \subset H_2 \to H_1$ linear, densely defined, closed

$$\Big\downarrow \text{ switch to inj. operators}$$

$H_1 = N(A) \oplus_{H_1} \overline{R(A^*)}$

$H_2 = N(A^*) \oplus_{H_2} \overline{R(A)}$

$\quad \hookrightarrow N(A)^\perp = \overline{R(A^*)}$

$\mathcal{A} : D(\mathcal{A}) := D(A) \cap \overline{R(A^*)} \subset \overline{R(A^*)} \to \overline{R(A)}$

$\mathcal{A}^* : D(\mathcal{A}^*) := D(A^*) \cap \overline{R(A)} \subset \overline{R(A)} \to \overline{R(A^*)}$

$\Rightarrow (A, A^*)$ & $(\mathcal{A}, \mathcal{A}^*)$ are dual pairs
[i.e. $A^{**} = A$ resp. $\mathcal{A}^{**} = \mathcal{A}$]

$D(A) = N(A) \oplus_{H_1} D(\mathcal{A}) \Rightarrow R(A) = R(\mathcal{A})$

$D(A^*) = N(A^*) \oplus_{H_2} D(\mathcal{A}^*) \Rightarrow R(A^*) = R(\mathcal{A}^*)$

cf. exercise 1, Problem 6 we have

$$D(\mathcal{A}) \hookrightarrow H_1 \Leftrightarrow D(\mathcal{A}^*) \hookrightarrow H_2$$

$$\Updownarrow$$

$R(A) = R(\mathcal{A})$ is closed

$\Leftrightarrow \exists\, c_A > 0 \;\forall x \in D(\mathcal{A}) :$

$$|x|_{H_1} \leq c_A |Ax|_{H_2},$$

$R(A^*) = R(\mathcal{A}^*)$ is closed

$\Leftrightarrow \exists\, c_{A^*} > 0 \ \ \forall\, y \in D(\mathcal{A}^*)$:

$$|y|_{H_2} \leq c_{A^*} \, |A^* y|_{H_1}$$

<u>We even have:</u> $c_A = c_{A^*}$,

if we define:

$$1/c_A := \inf_{0 \neq x \in D(\mathcal{A})} \frac{|Ax|_{H_2}}{|x|_{H_1}}, \quad 1/c_{A^*} := \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|A^* y|_{H_1}}{|y|_{H_2}}$$

<u>sequences:</u>

Let $A_0 : D(A_0) \subset H_0 \to H_1$, $A_1 : D(A_1) \subset H_1 \to H_2$, $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_0^*, \mathcal{A}_1^*$ as above and assume

$$\boxed{A_1 A_0 \underset{\uparrow}{=} 0}$$ sequence (complex) property

(indeed here should be a „$\subset$", since 0 has a bigger domain of defin.)

$\Leftrightarrow R(A_0) \subset N(A_1)$

$\overset{?}{\Leftrightarrow} R(A_1^*) \subset N(A_0^*)$

(We can also put closure bars, since closed operators have closed kernels)

<u>Theorem 5:</u> (HH-decompositions)

$$* \ H_0 = N(A_0) \oplus_{H_0} \overline{R(A_0^*)}$$

$$* \ H_1 = \underset{\cup}{N(A_1)} \oplus_{H_1} \underset{\cap}{\overline{R(A_1^*)}}$$
$$= \overline{R(A_0)} \oplus_{H_1} N(A_0^*) \quad \Rightarrow \quad N(A_1) = \overline{R(A_0)} \oplus_{H_1} K_1$$

$$\Rightarrow \quad H_1 = \overline{R(A_0)} \oplus_{H_1} K_1 \oplus_{H_1} \overline{R(A_1^*)}$$

If additionally $A_2 : D(A_2) \subset H_2 \to H_3$ linear, densely defined, closed with $A_2 A_1 = 0$, we have:

$$* \quad \begin{aligned} H_2 &= \underset{\cup}{N(A_2)} \oplus_{H_2} \underset{\cap}{\overline{R(A_2^*)}} \\ &= \overline{R(A_1)} \oplus_{H_2} N(A_1^*) \end{aligned} \quad \Rightarrow \quad N(A_2) = \overline{R(A_1)} \oplus_{H_2} K_2$$

$$\Rightarrow \quad H_2 = \overline{R(A_1)} \oplus_{H_2} K_2 \oplus_{H_2} \overline{R(A_2^*)}$$

$$* \quad H_3 = \overline{R(A_2)} \oplus_{H_3} N(A_2^*)$$

The spaces $K_1 := N(A_0^*) \cap N(A_1)$ and $K_2 := N(A_1^*) \cap N(A_2)$ are called „cohomology groups"

Proof:

Exercise 2, Problem 2

$\boxed{\lightning}$

Primal & dual sequences:

$$\begin{tikzcd} D(A_0) \arrow[r, "A_0"] & D(A_1) \arrow[r, "A_1"] & D(A_2) \arrow[r, "A_2"] & H_3 & \text{(primal)} \\ H_0 & D(A_0^*) \arrow[l, "A_0^*"'] & D(A_1^*) \arrow[l, "A_1^*"'] & D(A_2^*) \arrow[l, "A_2^*"'] & \text{(dual)} \end{tikzcd}$$

Definition:

We call a sequence

- complex $:\Leftrightarrow R(A_i) \subset N(A_{i+1})$
- closed $:\Leftrightarrow R(A_i)$ is closed
- exact $:\Leftrightarrow K_i = \{0\}$

(then: $\overline{R(A_i)} = N(A_{i+1})$)

Remark:

We have

- primal sequence is a complex
  $\Leftrightarrow$ dual sequence is a complex.
- primal complex is closed
  $\Leftrightarrow$ dual complex is closed.
  („closed range theorem")
- primal complex is exact
  $\Leftrightarrow$ dual complex is exact.
  $\left(N(A_2) = \overline{R(A_1)} \oplus_{H_2} K_2 , \; N(A_1^*) = K_2 \oplus \overline{R(A_2^*)} \right)$

If (A, A^*) is a dual pair, A linear, densely defined, closed, then

- A^*A, AA^* are selfadjoint.
- $\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} : D(A) \times D(A^*) \subset H_1 \times H_2 \to H_1 \times H_2$
  is selfadjoint.
- $\begin{pmatrix} 0 & -A^* \\ A & 0 \end{pmatrix}$ is skew-selfadjoint.

  $\rightsquigarrow i \begin{pmatrix} 0 & -A^* \\ A & 0 \end{pmatrix}$ is selfadjoint.

Assume: $R(A_0)$ & $R(A_1)$ are closed

$\Rightarrow H_1 = R(A_0) \oplus_{H_1} K_1 \oplus_{H_1} R(A_1^*)$

$\hookrightarrow D(A_1) = R(A_0) \oplus_{H_1} K_1 \oplus_{H_1} D(\mathcal{A}_1)$

$$\Rightarrow \quad D_1 := D(A_1) \cap D(A_0^*)$$
$$= D(\mathcal{A}_0^*) \oplus_{H_1} K_1 \oplus_{H_1} D(\mathcal{A}_1)$$

Lemma 6:

Let $R(A_0)$ & $R(A_1)$ be closed. Then:

$$\forall x \in D_1 \cap K_1^\perp : |x|^2_{H_1} \leq c^2_{A_0} |A_0^* x|^2_{H_0} + c^2_{A_1} |A_1 x|^2_{H_2}$$

Proof:

Let $y \in D_1 \cap K_1^\perp$

$$\Rightarrow \quad y = y_0 + y_1 \in D(\mathcal{A}_0^*) \oplus_{H_1} D(\mathcal{A}_1)$$

$$\Rightarrow \quad |y|^2_{H_1} \leq |y_0|^2_{H_1} + |y_1|^2_{H_1}$$

$$\leq c^2_{A_0} |A_0^* y_0|^2_{H_0} + c^2_{A_1} |A_1 y_1|^2_{H_2}$$

$D(\mathcal{A}_0^*) \subset N(A_1)$
$D(\mathcal{A}_1) \subset N(A_0^*)$

$$= c^2_{A_0} |A_0^* y|^2_{H_0} + c^2_{A_1} |A_1 y|^2_{H_2}$$

$\boxed{\xi}$

Apply „toolbox" to the Maxwell case:

Let $\Omega \subset \mathbb{R}^3$ open; $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$; $\varepsilon, \mu \in L^\infty(\Omega)$, symmetric and uniformly positive definite.

$$\hat{A}_0 := \hat{\nabla} : \mathring{C}^\infty \subset L^2 \to L^2_\varepsilon \;, \quad \hat{A}_0 u = \nabla u$$
$$\hat{A}_1 := \mu^{-1} \hat{\mathrm{rot}} : \mathring{C}^\infty \subset L^2_\varepsilon \to L^2_\mu \;, \quad \hat{A}_1 u = \mu^{-1} \mathrm{rot}\, u$$
$$\hat{A}_2 := \hat{\mathrm{div}}\, \mu : \mu^{-1} \mathring{C}^\infty \subset L^2_\mu \to L^2 \;, \quad \hat{A}_2 u = \mathrm{div}(\mu u)$$

$\rightsquigarrow$ all operators are linear, densely defined; but are they closable?

e.g. $\hat{A}_1$ closable:

$\forall (x_n)_n \subset D(A_1) = \overset{\circ}{C}^\infty$ with
$x_n \to 0$ & $\hat{A}_1 x_n = \mu^{-1} \operatorname{rot} x_n \to y$

$\Rightarrow \forall \phi \in \overset{\circ}{C}^\infty$:

$$\begin{aligned}\langle y, \phi \rangle_{L^2_\mu} &= \langle y, \mu \phi \rangle_{L^2} \\ &= \lim_{n \to \infty} \langle \mu^{-1} \operatorname{rot} x_n, \mu \phi \rangle_{L^2} \\ &= \lim_{n \to \infty} \langle \operatorname{rot} x_n, \phi \rangle_{L^2} \\ &= \lim_{n \to \infty} \langle x_n, \operatorname{rot} \phi \rangle_{L^2} = 0\end{aligned}$$

$\Rightarrow y = 0$
$\Rightarrow \hat{A}_1$ is closable.

$\square$

$\rightsquigarrow$ closed operators: $D(\overline{\hat{A}_i}) = \overline{D(\hat{A}_i)}^{G(\hat{A}_i)}$

$$\begin{aligned}A_0 &:= \overline{\hat{A}_0} = \overset{\circ}{\nabla} : \overset{\circ}{H}{}^1 \subset L^2 \to L^2_\varepsilon \\ A_1 &:= \overline{\hat{A}_1} = \mu^{-1} \overset{\circ}{\operatorname{rot}} : \overset{\circ}{R} \subset L^2_\varepsilon \to L^2_\mu \\ A_2 &:= \overline{\hat{A}_2} = \operatorname{div}_\mu : \mu^{-1} \overset{\circ}{D} \subset L^2_\mu \to L^2\end{aligned}$$

(of course: linear, densely defined
and now closed)

$\rightsquigarrow$ compute the adjoints:

$*$ $u \in D(\overset{\circ}{\nabla}{}^*) \wedge \overset{\circ}{\nabla}{}^* u = F$

$\Leftrightarrow u \in L^2_\varepsilon \wedge \exists F \in L^2 \; \forall \varphi \in D(\overset{\circ}{\nabla}) = \overset{\circ}{H}{}^1$:

$$\begin{aligned}&\langle u, \nabla \varphi \rangle_{L^2_\varepsilon} = \langle f, \varphi \rangle_{L^2} \\ &\quad \| \\ &\langle \varepsilon u, \nabla \varphi \rangle_{L^2}\end{aligned}$$

$$\Leftrightarrow \ \varepsilon u \in D \ \wedge \ f = -\operatorname{div}\varepsilon u$$

$$\Leftrightarrow \ u \in \varepsilon^{-1} D \ \wedge \ f = -\operatorname{div}\varepsilon u$$

$$\rightsquigarrow \ \boxed{A_0^* = -\operatorname{div}\varepsilon : \varepsilon^{-1} D \subset L^2_\varepsilon \to L^2}$$

$*$ $u \in D(\mu^{-1}\mathring{\operatorname{rot}}) \wedge \mu^{-1}\mathring{\operatorname{rot}}^* u = f$

$$\Leftrightarrow \ u \in L^2_\mu \wedge \exists f \in L^2_\varepsilon \ \forall \varphi \in D(\mu^{-1}\mathring{\operatorname{rot}}) = \mathring{R} :$$

$$\langle u, \mu^{-1}\operatorname{rot}\varphi\rangle_{L^2_\mu} = \langle f, \varphi\rangle_{L^2_\varepsilon}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\langle u, \operatorname{rot}\varphi\rangle_{L^2} \qquad \langle \varepsilon f, \varphi\rangle_{L^2}$$

$$\Leftrightarrow \ u \in R \ \wedge \ \operatorname{rot} u = \varepsilon f$$

$$\Leftrightarrow \ u \in R \ \wedge \ f = \varepsilon^{-1}\operatorname{rot} u$$

$$\rightsquigarrow \ \boxed{A_1^* = \varepsilon^{-1}\operatorname{rot} : R \subset L^2_\mu \to L^2_\varepsilon}$$

$*$ $u \in D(\mathring{\operatorname{div}}\mu^*) \wedge \mathring{\operatorname{div}}\mu^* u = f$

$$\Leftrightarrow \ u \in L^2 \wedge \exists f \in L^2_\mu \ \forall \varphi \in D(\mathring{\operatorname{div}}\mu) = \mu^{-1}\mathring{D} :$$

$$\langle u, \operatorname{div}\underbrace{\mu\varphi}_{\in \mathring{D}}\rangle_{L^2} = \langle f, \varphi\rangle_{L^2_\mu}$$

$$\|$$

$$\langle f, \mu\varphi\rangle_{L^2}$$

$$\Leftrightarrow \ u \in H^1 \ \wedge \ f = -\nabla u$$

$$\rightsquigarrow \ \boxed{A_2^* = -\nabla : H^1 \subset L^2 \to L^2_\mu}$$

Now apply Theorem 5:

$*$ $L^2 = N(\mathring{\nabla}) \oplus_{L^2} \overline{R(-\operatorname{div}\varepsilon)}$

$$= \{0\} \oplus_{L^2} \overline{R(\operatorname{div})} = \overline{R(\operatorname{div})}$$

$$= \overline{\operatorname{div} D}$$

$$
\begin{aligned}
* \quad L^2_\varepsilon &= N(\mu^{-1}\mathring{\mathrm{rot}}) \oplus_{L^2_\varepsilon} \overline{R(\varepsilon^{-1}\mathrm{rot})} \\
&= \mathring{R}_0 \oplus_{L^2_\varepsilon} \varepsilon^{-1}\overline{\mathrm{rot}\,R} \\
&= \overline{R(\mathring{\nabla})} \oplus_{L^2_\varepsilon} N(-\mathrm{div}\,\varepsilon) \\
&= \overline{\mathring{\nabla}\mathring{H}^1} \oplus_{L^2_\varepsilon} \varepsilon^{-1} D_0 \\
&= \rlap{$\underbrace{\phantom{\overline{R(\mathring{\nabla})} \oplus_{L^2_\varepsilon} [N(-\mathrm{div}\,\varepsilon) \cap N(\mu^{-1}\mathring{\mathrm{rot}})]}}_{= N(\mu^{-1}\mathring{\mathrm{rot}})}$}\overline{R(\mathring{\nabla})} \oplus_{L^2_\varepsilon} \underbrace{[N(-\mathrm{div}\,\varepsilon) \cap N(\mu^{-1}\mathring{\mathrm{rot}})] \oplus_{L^2_\varepsilon} \overline{R(\varepsilon^{-1}\mathrm{rot})}}_{= N(-\mathrm{div}\,\varepsilon)} \\
&= \rlap{$\underbrace{\phantom{\overline{\mathring{\nabla}\mathring{H}^1} \oplus_{L^2_\varepsilon} (\mathring{R}_0 \cap \varepsilon^{-1} D_0)}}_{= \mathring{R}_0}$}\overline{\mathring{\nabla}\mathring{H}^1} \oplus_{L^2_\varepsilon} \underbrace{(\mathring{R}_0 \cap \varepsilon^{-1} D_0) \oplus_{L^2_\varepsilon} \varepsilon^{-1}\overline{\mathrm{rot}\,R}}_{= \varepsilon^{-1} D_0}
\end{aligned}
$$

$$
\begin{aligned}
* \quad L^2_\mu &= N(\mathring{\mathrm{div}}\,\mu) \oplus_{L^2_\mu} \overline{R(\nabla)} \\
&= \mu^{-1}\mathring{D}_0 \oplus_{L^2_\mu} \overline{\nabla H^1} \\
&= \overline{R(\mu^{-1}\mathring{\mathrm{rot}})} \oplus_{L^2_\mu} N(\varepsilon^{-1}\mathrm{rot}) \\
&= \overline{\mu^{-1}\mathring{\mathrm{rot}}\,\mathring{R}} \oplus_{L^2_\mu} R_0 \\
&= \rlap{$\underbrace{\phantom{\overline{R(\mu^{-1}\mathring{\mathrm{rot}})} \oplus_{L^2_\mu} [N(\mathring{\mathrm{div}}\,\mu) \cap N(\varepsilon^{-1}\mathrm{rot})]}}_{= N(\mathring{\mathrm{div}}\,\mu)}$}\overline{R(\mu^{-1}\mathring{\mathrm{rot}})} \oplus_{L^2_\mu} \underbrace{[N(\mathring{\mathrm{div}}\,\mu) \cap N(\varepsilon^{-1}\mathrm{rot})] \oplus_{L^2_\mu} \overline{R(\nabla)}}_{= N(\varepsilon^{-1}\mathrm{rot})} \\
&= \rlap{$\underbrace{\phantom{\mu^{-1}\overline{\mathring{\mathrm{rot}}\,\mathring{R}} \oplus_{L^2_\mu} (\mu^{-1}\mathring{D}_0 \cap R_0)}}_{= \mu^{-1}\mathring{D}_0}$}\mu^{-1}\overline{\mathring{\mathrm{rot}}\,\mathring{R}} \oplus_{L^2_\mu} \underbrace{(\mu^{-1}\mathring{D}_0 \cap R_0) \oplus_{L^2_\mu} \overline{\nabla H^1}}_{= R_0}
\end{aligned}
$$

$$
\begin{aligned}
* \quad L^2 &= \overline{R(\mathring{\mathrm{div}}\,\mu)} \oplus_{L^2} N(\nabla) \\
&= \overline{\mathring{\mathrm{div}}\,\mathring{D}} \oplus_{L^2} \mathbb{R}
\end{aligned}
$$

We call:

- $* \quad \mathcal{H}_D := \mathring{R}_0 \cap \varepsilon^{-1} D_0$ „Dirichlet Fields"
- $* \quad \mathcal{H}_N := R_0 \cap \mu^{-1}\mathring{D}_0$ „Neumann Fields"

$A_0^* A_0 = \operatorname{div}\varepsilon\mathring{\nabla}$, $A_1^* A_1 = \varepsilon^{-1}\operatorname{rot}\mu^{-1}\mathring{\operatorname{rot}}$ and
$A_2^* A_2 = -\nabla\mathring{\operatorname{div}}\mu$ are selfadjoint.

skew - selfadjoint are for example:

$$\begin{pmatrix} 0 & -A_0^* \\ A_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{div}\varepsilon \\ \mathring{\nabla} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -A_1^* \\ A_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon^{-1}\operatorname{rot} \\ \mu^{-1}\mathring{\operatorname{rot}} & 0 \end{pmatrix}$$

↳ Maxwell - operator

$$\begin{pmatrix} 0 & -A_2^* \\ A_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nabla \\ \mathring{\operatorname{div}}\mu & 0 \end{pmatrix}$$

lecture 4½ 02.11.16

We have seen:

$N(A_0) = \{0\} \quad \Rightarrow \quad \overline{R(A_0)} = L^2 = \overline{\operatorname{div} D}$

$N(A_2^*) = \mathbb{R} \quad \Rightarrow \quad \overline{R(A_2)} = \mathbb{R}^\perp = L^2_0 = \overline{\mathring{\operatorname{div}}\mathring{D}}$

~> reduced operators (injective)

$\mathcal{A}_0 : \mathring{H}^1 \subset L^2 \to \overline{\mathring{\nabla}\mathring{H}^1}^{L^2_\varepsilon}$

$\mathcal{A}_1 : \mathring{R} \cap \varepsilon^{-1}\overline{\operatorname{rot} R} \subset \varepsilon^{-1}\overline{\operatorname{rot} R} \to \mu^{-1}\overline{\mathring{\operatorname{rot}}\mathring{R}}$

$\mathcal{A}_2 : \mu^{-1}\mathring{D} \cap \overline{\nabla H^1} \subset \overline{\nabla H^1} \to \overline{\mathring{\operatorname{div}}\mathring{D}} = \mathbb{R}^\perp$

$\mathcal{A}_0^* : \varepsilon^{-1} D \cap \overline{\mathring{\nabla}\mathring{H}^1} \subset \overline{\mathring{\nabla}\mathring{H}^1} \to L^2$

$\mathcal{A}_1^* : R \cap \mu^{-1}\overline{\mathring{\operatorname{rot}}\mathring{R}} \subset \mu^{-1}\overline{\mathring{\operatorname{rot}}\mathring{R}} \to \varepsilon^{-1}\overline{\operatorname{rot} R}$

$\mathcal{A}_2^* : H^1 \cap L^2_0 \subset L^2_0 \to \nabla H^1 \quad (L^2_0 = L^2_\perp)$

$D(A_0^*) \cap D(A_1) \overset{cc}{\hookrightarrow} H_2 \Leftrightarrow D(\mathcal{A}_0) \overset{cc}{\hookrightarrow} H_0 \wedge D(\mathcal{A}_1) \overset{cc}{\hookrightarrow} H_1 \wedge$
K_1 is finite dimensional

$\rightsquigarrow$ Furthermore:

- $* \; D(\mathcal{A}_0) \hookrightarrow\hookrightarrow H_0 \iff D(\mathcal{A}_0^*) \hookrightarrow\hookrightarrow H_1$
- $* \; D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_1 \iff D(\mathcal{A}_1^*) \hookrightarrow\hookrightarrow H_2$
- $* \; D(\mathcal{A}_0^*) \cap D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_2$

$\Rightarrow$ $R(\mathcal{A}_0)$ & $R(\mathcal{A}_1)$ are closed

$\Leftrightarrow$ Poincaré-type estimates

In more detail:

$D(\mathcal{A}_0) \hookrightarrow\hookrightarrow H_0$	$\mathring{H}^1 \hookrightarrow\hookrightarrow L^2$ Rellich's sel. th. (need: Ω bd.)
$\Updownarrow$	$\Updownarrow$
$D(\mathcal{A}_0^*) \hookrightarrow\hookrightarrow H_1$	$\underbrace{\varepsilon^{-1} D \cap \overline{\mathring{\nabla} \mathring{H}^1}}_{= \varepsilon^{-1} D \cap \mathring{R}_0 \cap \mathcal{H}_D^\perp} \hookrightarrow\hookrightarrow L^2_\varepsilon$
$D(\mathcal{A}_1) \hookrightarrow\hookrightarrow H_1$	$\underbrace{\mathring{R} \cap \varepsilon^{-1} \overline{\operatorname{rot} R}}_{= \mathring{R} \cap \varepsilon^{-1} D_0 \cap \mathcal{H}_D^{\perp_{L^2_\varepsilon}}} \hookrightarrow\hookrightarrow L^2_\varepsilon$ (?)
$\Updownarrow$	$\Updownarrow$
$D(\mathcal{A}_1^*) \hookrightarrow\hookrightarrow H_2$	$\underbrace{R \cap \mu^{-1} \overline{\mathring{\operatorname{rot}} \mathring{R}}}_{= R \cap \mu^{-1} \mathring{D}_0 \cap \mathcal{H}_N^{\perp_{L^2_\mu}}} \hookrightarrow\hookrightarrow L^2_\mu$
$D(\mathcal{A}_2) \hookrightarrow\hookrightarrow H_2$	$\underbrace{\mu^{-1} \mathring{D} \cap \overline{\nabla H^1}}_{= \mu^{-1} \mathring{D} \cap \mathring{R} \cap \mathcal{H}_N^\perp} \hookrightarrow\hookrightarrow L^2_\mu$
$\Updownarrow$	$\Updownarrow$
$D(\mathcal{A}_2^*) \hookrightarrow\hookrightarrow H_3$	$H^1 \cap L^2_\perp = H^1_\perp \hookrightarrow\hookrightarrow L^2$ Rellich's sel. theorem (need: Ω bd, segment property) [Ω weak Lipschitz:

	$H^1 \subset\subset L^2$]
$D(A_0^*) \cap D(A_1) \subset\subset H_1$ $\Updownarrow$ $D(\mathcal{A}_0) \subset\subset H_0$ $\wedge\ D(\mathcal{A}_1) \subset\subset H_1$ $\wedge\ K_1 = N(A_1) \cap N(A_0^*)$ is finite dimens.	$\varepsilon^{-1} D \cap \mathring{R} \subset\subset L^2_\varepsilon$ Weck's sel. theorem (need: Ω bounded, weak lipschitz) $\Updownarrow$ $\mathring{H}^1 \subset\subset L^2$ (already: ✓) $\wedge\ \mathring{R} \cap \varepsilon^{-1} \operatorname{rot} R \subset\subset L^2_\varepsilon$ $\wedge\ \mathring{R}_0 \cap \varepsilon^{-1} D_0 = \mathcal{H}_D$ is finite dimensional.
$D(A_1^*) \cap D(A_2) \subset\subset H_2$ $\Updownarrow$ $D(\mathcal{A}_1) \subset\subset H_1$ $\wedge\ D(\mathcal{A}_2^*) \subset\subset H_3$ $\wedge\ K_2 = N(A_2) \cap N(A_1^*)$ is finite dimens.	$R \cap \mu^{-1} \mathring{D} \subset\subset L^2_\mu$ Weck's selection theorem (need: Ω bounded, weak lipschitz) $\Updownarrow$ $\mathring{R} \cap \varepsilon^{-1} \operatorname{rot} R \subset\subset L^2_\varepsilon$ (?) $\wedge\ H^1_\perp \subset\subset L^2$ $\wedge\ R_0 \cap \mu^{-1} \mathring{D}_0 = \mathcal{H}_N$ is finite dimensional.

Poincaré-type estimates:

$R(A_i)$ closed $\iff \forall x \in D(\mathcal{A}_i): |x|_{H_i} \le c_i |\mathcal{A} x|_{H_{i+1}}$

$\Updownarrow$

$R(A_i^*)$ closed $\iff \forall x \in D(\mathcal{A}_i^*): |x|_{H_{i+1}} \le c_i |\mathcal{A}^* x|_{H_i}$

$\rightsquigarrow$ apply to the Maxwell case:

- $R(A_0) = R(\mathring{\nabla}) = \mathring{\nabla}\mathring{H}^1$ closed (follows e.g. from Rellich's selection theorem)

  $\Leftrightarrow \forall x \in \mathring{H}^1 : |x|_{L^2} \leq c_F |\nabla x|_{L^2_\varepsilon}$

  $\Leftrightarrow \forall u \in \varepsilon^{-1} D \cap \mathring{\nabla}\mathring{H}^1 : |u|_{L^2_\varepsilon} \leq c_F |\operatorname{div} \varepsilon u|_{L^2}$

- $R(A_1) = \mu^{-1} \mathring{\operatorname{rot}} \mathring{R}$ is closed $\Leftrightarrow \mathring{\operatorname{rot}} \mathring{R}$ is closed (follows e.g. from Weck's selection theorem)

  $\Leftrightarrow \forall x \in \mathring{R} \cap \varepsilon^{-1} \operatorname{rot} R : |x|_{L^2_\varepsilon} \leq c_m |\mu^{-1} \operatorname{rot} x|_{L^2_\mu}$
  $= c_m |\operatorname{rot} x|_{L^2_{\mu^{-1}}}$

  $\Leftrightarrow \forall u \in R \cap \mu^{-1} \mathring{\operatorname{rot}} \mathring{R} : |u|_{L^2_\mu} \leq c_m |\varepsilon^{-1} \operatorname{rot} u|_{L^2_\varepsilon}$
  $= c_m |\operatorname{rot} u|_{L^2_{\varepsilon^{-1}}}$

- $R(A_2) = L^2_\perp$ is closed $\Leftrightarrow R(A_2^*)$ is closed $\Leftrightarrow \nabla H^1$ is closed (follows e.g. from Rellich's selection theorem: $H^1 \hookrightarrow\hookrightarrow L^2$)

  $\Leftrightarrow \forall x \in \mu^{-1} \mathring{D} \cap \nabla H^1 : |x|_{L^2_\mu} \leq c_p |\operatorname{div} \mu x|_{L^2}$

  $\Leftrightarrow \forall u \in H^1_\perp : |u|_{L^2} \leq c_p |\nabla u|_{L^2_\mu}$

Moreover:

- $N(A_1) = K_1 \oplus_{L^2_\varepsilon} R(A_0)$

  $\Rightarrow R(A_0) = N(A_1) \cap K_1^{\perp_{L^2_\varepsilon}}$
  $\Rightarrow \mathring{\nabla}\mathring{H}^1 = \mathring{R}_0 \cap \mathcal{H}_D^{\perp_{L^2_\varepsilon}}$

- $N(A_0^*) = K_1 \oplus_{L^2_\varepsilon} R(A_1^*)$

$$\Rightarrow \quad R(A_1^*) = \mathcal{N}(A_0^*) \cap K_1^{\perp_{L^2_\varepsilon}}$$

$$\Rightarrow \quad \varepsilon^{-1} \operatorname{rot} R = \varepsilon^{-1} D_0 \cap \mathcal{H}_D^{\perp_{L^2_\varepsilon}}$$

$$* \quad \mathcal{N}(A_2) = K_2 \oplus_{L^2_\mu} R(A_1)$$

$$\Rightarrow \quad R(A_1) = \mathcal{N}(A_2) \cap K_2^{\perp_{L^2_\mu}}$$

$$\Rightarrow \quad \mu^{-1} \mathring{\operatorname{rot}} \mathring{R} = \mu^{-1} \mathring{D}_0 \cap \mathcal{H}_N^{\perp_{L^2_\mu}}$$

$$* \quad \mathcal{N}(A_1^*) = K_2 \oplus_{L^2_\mu} R(A_2^*)$$

$$\Rightarrow \quad R(A_2^*) = \mathcal{N}(A_1^*) \cap K_2^{\perp_{L^2_\mu}}$$

$$\Rightarrow \quad \nabla H^1 = R_0 \cap \mathcal{H}_N^{\perp_{L^2_\mu}}$$

Lemma 7:

If Ω bounded and convex, we have

$$c_m \leq c_p \leq \frac{\operatorname{diam} \Omega}{\pi}.$$

lecture 5 14.11.16

Ω (open) $\subset \mathbb{R}^3$, recall:

- $\Lambda := \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$

- $\mathcal{A} : \overset{\circ}{R} \times R \subset L^2 \times L^2 \to L^2 \times L^2$ defined by

$$\mathcal{A}\, x := \begin{pmatrix} 0 & -\overset{\circ}{\operatorname{rot}}^* \\ \overset{\circ}{\operatorname{rot}} & 0 \end{pmatrix} x = \begin{pmatrix} 0 & -\operatorname{rot} \\ \overset{\circ}{\operatorname{rot}} & 0 \end{pmatrix} x$$

is skew-selfadjoint.

- $\mathcal{M} : \overset{\circ}{R} \times R \subset L^2_\Lambda := L^2_\varepsilon \times L^2_\mu \to L^2_\Lambda$ defined by

$$\mathcal{M} x := i \Lambda^{-1} \mathcal{A}\, x = i \begin{pmatrix} 0 & -\varepsilon^{-1}\operatorname{rot} \\ \mu^{-1}\overset{\circ}{\operatorname{rot}} & 0 \end{pmatrix} x$$

is selfadjoint.

① <u>time dependent</u>:

$$\left(\partial_t - \underset{\nearrow}{i\mathcal{M}} \right) x = f \quad \& \quad \text{initial conditions.}$$

boundary conditions are already included.

↳ $\mathcal{M}$ selfadjoint $\Rightarrow$ solution theory by

(i) spectral theory

$$x(t) = e^{it\mathcal{M}} x_0 + \int_0^t e^{i(s-t)\mathcal{M}} f(s)\, ds$$

(ii) semi-group theory

(iii) Picard-theory.

② time - harmonic :

$$(\mathcal{M} - \omega)\, x = f \quad \rightsquigarrow \text{ Fredholm Alternative}$$

③ static case :

$$\mathcal{M} x = f \quad \& \quad \text{div - conditions}$$

$$\rightsquigarrow \text{ Toolbox}$$

For case ③:

Is $R(\mathcal{M})$ closed ? (central point)

for ② we need: $\boxed{D(\mathscr{M}) \hookrightarrow\hookrightarrow L^2_\Lambda}$

$\Rightarrow R(\mathcal{M}) = R(\mathscr{M})$ is closed.

$$\mathcal{M} : D(\mathcal{M}) \subset L^2_\Lambda \to L^2_\Lambda \,, \quad D(\mathcal{M}) = \mathring{R} \times R$$

$$\mathscr{M} : D(\mathscr{M}) = D(\mathcal{M}) \cap \overline{R(\mathcal{M})} \subset \overline{R(\mathcal{M})} \to \overline{R(\mathcal{M})}$$

with

$$D(\mathscr{M}) = (\mathring{R} \times R) \cap (\varepsilon^{-1} \overline{\operatorname{rot} R} \times \mu^{-1} \overline{\mathring{\operatorname{rot}} \mathring{R}})$$

$$= (\mathring{R} \cap \varepsilon^{-1} \overline{\operatorname{rot} R}) \times (R \cap \mu^{-1} \overline{\mathring{\operatorname{rot}} \mathring{R}})$$

$$\left(\text{clearly: } \overline{\varepsilon^{-1} \operatorname{rot} R} = \varepsilon^{-1} \overline{\operatorname{rot} R} \right)$$

so what we need to show:

$$D(\mathcal{A}_1) = \mathring{R} \cap \varepsilon^{-1} \overline{\operatorname{rot} R} \hookrightarrow\hookrightarrow L^2_\varepsilon$$

$$\Updownarrow \text{ cf. exercises}$$

$$D(\mathcal{A}_1^*) = R \cap \mu^{-1} \overline{\mathring{\operatorname{rot}} \mathring{R}} \hookrightarrow\hookrightarrow L^2_\mu$$

These indeed follow from :

$$\boxed{\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow L^2_\varepsilon \;\wedge\; R \cap \mu^{-1} \mathring{D} \hookrightarrow L^2_\mu}$$

(This we will show in the last lecture)

<u>The static problems:</u>

„$n \times E_{|_\Gamma} = 0$"

$$\begin{aligned}
&(\text{ESP}) && A_1 E = \mathring{\mathrm{rot}}\, E = F \in R(\mathring{\mathrm{rot}}) = R(A_1)\\
&\mu := 1 && A_0^* E = -\mathrm{div}\,\varepsilon E = f \in R(\mathrm{div}) = R(A_0^*)\\
& && \pi_1 E = \pi_D E = D \in \mathcal{D}_\varepsilon = K_1
\end{aligned}$$

$$\begin{aligned}
&(\text{MSP}) && A_1^* H = \mathrm{rot}\, H = G \in R(\mathrm{rot}) = R(A_1^*)\\
&\varepsilon := 1 && A_2 H = \mathring{\mathrm{div}}\, \mu H = g \in R(\mathrm{div}) = R(A_2)\\
& && \pi_2 H = \pi_N H = N \in \mathcal{N}_\mu = K_2
\end{aligned}$$

„$n \cdot H_{|_\Gamma} = 0$"

Remember:

$$\begin{aligned}
A_0 &= \mathring{\nabla} : \mathring{H}^1 \subset L^2 \to L^2_\varepsilon, \; u \mapsto \mathring{\nabla} u\\
A_1 &= \mu^{-1} \mathring{\mathrm{rot}} : \mathring{R} \subset L^2_\varepsilon \to L^2_\mu, \; E \mapsto \mu^{-1} \mathring{\mathrm{rot}}\, E\\
A_2 &= \mathrm{div}\,\mu : \mu^{-1} \mathring{D} \subset L^2_\mu \to L^2, \; H \mapsto \mathrm{div}\,\mu H\\
A_0^* &= -\mathrm{div}\,\varepsilon : \varepsilon^{-1} D \subset L^2_\varepsilon \to L^2, \; H \mapsto -\mathrm{div}\,\varepsilon H\\
A_1^* &= \varepsilon^{-1} \mathrm{rot} : R \subset L^2_\mu \to L^2_\varepsilon, \; E \mapsto \varepsilon^{-1} \mathrm{rot}\, E\\
A_2^* &= -\nabla : H^1 \subset L^2 \to L^2_\mu, \; u \mapsto -\nabla u
\end{aligned}$$

<u>Theorem 8:</u>

Let A_i, A_{i+1} be linear, densely defined, closed and $A_{i+1} A_i \subset 0$. Assume $R(A_i)$, $R(A_{i+1})$ are closed, then

$$A_{i+1} x = f \,, \quad A_i^* x = g \,, \quad \pi_{i+1} x = h$$

is uniquely solvable in $D_{i+1} := D(A_{i+1}) \cap D(A_i^*)$, iff $f \in R(A_{i+1})$, $g \in R(A_i^*)$, $h \in K_{i+1}$. The unique solution is given by

$$x = x_f + x_g + h \in D(\mathcal{A}_{i+1}) \oplus D(\mathcal{A}_i^*) \oplus K_{i+1} \,,$$

where

$$x_f = \mathcal{A}_{i+1}^{-1} f \in D(\mathcal{A}_{i+1}) = D(A_{i+1}) \cap R(A_{i+1}^*) \,,$$
$$x_g = (\mathcal{A}_i^*)^{-1} g \in D(\mathcal{A}_i^*) = D(A_i^*) \cap R(A_i) \,,$$

depends continuously on the data, i.e.

$$|x|^2_{H_{i+1}} \leq c_{i+1}^2 |x_f|^2_{H_{i+1}} + c_i^2 |x_g|^2_{H_{i-1}} + |h|^2_{H_i}$$

and

$$x_f = \pi_{A_{i+1}} x \,, \quad x_g = \pi_{A_i} x \,.$$

Remark:

① x_f, x_g solve

$$\begin{array}{ll} A_{i+1} x_f = f & A_{i+1} x_g = 0 \\ A_i^* x_f = 0 & A_i^* x_g = g \\ \pi_{i+1} x_f = 0 & \pi_{i+1} x_g = 0 \end{array}$$

② $D_{i+1} = D(A_{i+1}) \cap D(A_i^*) \hookrightarrow H_{i+1}$

$\Rightarrow$ $R(A_i)$, $R(A_{i+1})$ are closed.

& $K_{i+1} = N(A_{i+1}) \cap N(A_i^*)$ is finite dimensional

& ... Poincaré-type estimates,

$\mathcal{A}_{i+1}^{-1}$ is continuous/compact...

③ In (ESP): $i=0$; in (MSP): $i=1$.

Proof of Theorem 8:

"$\Rightarrow$": clear!

"$\Leftarrow$": $f \in R(A_{i+1}) = R(\mathcal{A}_{i+1})$ closed

$\overset{\text{toolbox}}{\Rightarrow}$ $\mathcal{A}_{i+1}^{-1} : R(A_{i+1}) \to D(\mathcal{A}_{i+1})$

is continuous.

We put:

$$x_f := \mathcal{A}_{i+1}^{-1} f \in D(\mathcal{A}_{i+1}) = D(A_{i+1}) \cap R(A_{i+1}^*)$$
$$\subset N(A_i^*) \cap K_{i+1}^{\perp}$$

⌜Helmholtz-decomposition:

$$H_{i+1} = R(A_i) \oplus K_{i+1} \oplus R(A_{i+1}^*)$$
$$\Rightarrow N(A_i^*) = R(A_{i+1}^*) \oplus K_{i+1}$$ ⌟

Then: $A_{i+1} x_f = f$, $x_f \in N(A_i^*) \cap K_{i+1}^{\perp}$

$g \in R(A_i^*) = R(\mathcal{A}_i^*)$ is closed

$\overset{\text{toolbox}}{\Rightarrow}$ $(\mathcal{A}_i^*)^{-1} : R(A_i^*) \to D(\mathcal{A}_i^*)$

is continuous.

We put:

$$x_g := (\mathcal{A}_i^*)^{-1} g \in D(\mathcal{A}_i^*) = D(A_i^*) \cap R(A_i)$$
$$\subset N(A_{i+1}) \cap K_{i+1}^{\perp}$$

Then: $A_i^* x_g = g$, $x_g \in N(A_{i+1}) \cap K_{i+1}^{\perp}$

Furthermore: $x := x_f + x_g + h$

$$\Rightarrow \quad |x|^2 = |x_f|^2 + |x_g|^2 + |h|^2$$

$$\leq c_{i+1}^2 |f|^2 + c_i^2 |g|^2 + |h|^2$$

$$\uparrow$$

$$\forall y \in D(\mathcal{A}_{i+1}) : \quad |y| \leq c_{i+1} |A_{i+1} y|$$

$$\forall z \in D(\mathcal{A}_i^*) : \quad |z| \leq c_i |A_i^* z|$$

$\square$

Numerical application:

„<u>Variational formulations</u>" to find x_g, x_f?

(interested in second order formulations; good for coercivity, symmetry, ...)

$$x_f := \mathcal{A}_{i+1}^{-1} f \in D(\mathcal{A}_{i+1}) = D(A_{i+1}) \cap R(A_{i+1}^*)$$

(first order potential for f)

$$y_f := (\mathcal{A}_{i+1}^*)^{-1} x_f = (\mathcal{A}_{i+1}^*)^{-1} \mathcal{A}_{i+1}^{-1} f$$

(second order potential for f)

$$\Rightarrow \quad A_{i+1}^* y_f = x_f$$

Let $\phi \in D(\mathcal{A}_{i+1}^*)$: $\leftarrow$ ($\mathcal{A}_{i+1}^*$ for coercivity)

$$< A_{i+1}^* y_f , A_{i+1}^* \phi > = < x_f , A_{i+1}^* \phi > = < f, \phi >$$

Observe:

- $\varphi \mapsto < f, \varphi > \in D(\mathcal{A}_{i+1}^*)' \,/\, H_{i+2}' \cong H_{i+2}$
- $(\psi, \varphi) \mapsto < A_{i+1}^* \psi , A_{i+1}^* \varphi >$ is a continuous bilinearform on $D(\mathcal{A}_{i+1})^*$ and coercive, since:

$$\forall \varphi \in D(\mathcal{A}_{i+1}^*) : \quad |\varphi| \leq c_{i+1} |A_{i+1}^* \varphi|$$

Formulation of the problem:

Find $\tilde{y}_F \in D(\mathcal{A}_{i+1}^*)$ such that $\forall \phi \in D(\mathcal{A}_{i+1}^*)$

(*)

$$< A_{i+1}^* \tilde{y}_F , A_{i+1}^* \phi > = < f , \phi >$$

Riesz / Lax-Milgram $\Rightarrow \exists! \; \hat{y}_F \in D(\mathcal{A}_{i+1}^*)$

still unclear: $x_F \overset{?}{=} \mathcal{A}_{i+1}^* y_F$

$\hookrightarrow$ numerically the space $D(\mathcal{A}_{i+1}^*)$ is ugly and should be replaced:

1. observation: (*) holds for all $\phi \in D(A_{i+1}^*)$, since

$$D(A_{i+1}^*) = N(A_{i+1}^*) \oplus D(\mathcal{A}_{i+1}^*)$$

$$\overset{\Downarrow}{\phi} = \phi_N + \hat{\phi}$$

$\Rightarrow$ for $\phi \in D(A_{i+1}^*)$:

$$\begin{aligned} &< A_{i+1}^* \tilde{y}_F , A_{i+1}^* \phi > \\ &= < A_{i+1}^* \hat{y}_F , A_{i+1}^* \hat{\phi} > \\ &= < f , \hat{\phi} > \\ &= < f , \phi > - \underbrace{< f , \phi_N >}_{\substack{=0, \text{ since} \\ f \in R(A_{i+1}) = N(A_{i+1}^*)^\perp}} = < f , \phi > \end{aligned}$$

2. observation:

$D(\mathcal{A}_{i+1}^*) = D(A_{i+1}^*) \cap R(A_{i+1})$;

put the second condition in a second equation.

↳ <u>Reformulate $(*)$:</u>

$(*) \iff$ Find $\widehat{y}_F \in D(A_{i+1}^*)$ such that

$$\forall \phi \in D(A_{i+1}^*):$$
$$< A_{i+1}^* \widehat{y}_F , A_{i+1}^* \phi > = < f , \phi >$$

$y_F \in R(A_{i+1}) \overset{\shortparallel}{} N(A_{i+1}^*)^\perp$ ↘

$$\forall \psi \in N(A_{i+1}^*):$$
$$< \widetilde{y}_F , \psi > = 0$$

Now: $N(A_{i+1}^*) = R(A_{i+2}^*) \oplus K_{i+2}$, so
if we <u>assume</u>: $K_{i+2} = \{0\}$, we get
$$N(A_{i+1}^*) = R(A_{i+2}^*) = R(\mathcal{A}_{i+2}^*)$$

↳ <u>Reformulate $(*)$:</u>

$(*) \iff$ Find $\widehat{y}_F \in D(A_{i+1}^*)$ such that

$$\forall \phi \in D(A_{i+1}^*):$$
$(*_2)$
$$< A_{i+1}^* \widehat{y}_F , A_{i+1}^* \phi > = < f , \phi >$$
$$\forall \Theta \in D(A_{i+2}^*):$$
$$< \widetilde{y}_F , A_{i+2}^* \Theta > = 0$$

Here it has to be $\mathcal{A}_{i+2}^*$.
↓

$(*) \iff$ Find $(\widehat{y}_F , v_f) \in D(A_{i+1}^*) \times D(\mathcal{A}_{i+2}^*)$ s.t.

$$\forall \phi \in D(A_{i+1}^*):$$
$$< A_{i+1}^* \widetilde{y}_F , A_{i+1}^* \phi >$$
$(*_3)$
$$+ < \phi , A_{i+2}^* \phi > = < f , \phi >$$
$$\forall \Theta \in D(A_{i+2}^*): \longleftarrow$$
Here we can also put $\mathcal{A}_{i+2}^*$
$$< \widehat{y}_F , A_{i+2}^* \Theta > = 0$$

<u>Remark:</u>

$\widetilde{y}_F$ solution to $(*_2) \Rightarrow (\widetilde{y}_F , 0)$ solution to $(*_3)$

We will show:

If $(\tilde{y}_f, v_f)$ is a solution to $(*_3)$, then:

$$v_f = 0$$

Proof:

Put $\phi := A_{i+2}^* v_f$

$$\Rightarrow \| A_{i+2}^* v_f \|^2 = 0$$

$$\Rightarrow A_{i+2}^* v_f = 0$$

$$\Rightarrow v_f = 0$$

(Here you see why $D(\mathcal{A}_{i+2}^*)$ is needed)

$v_f \in D(\mathcal{A}_{i+2}^*)$, $\mathcal{A}_{i+2}^*$ is injective

$\boxed{\lightning}$

For numerics one additionally needs

① coercivity:

$\langle A_{i+1}^* \cdot , A_{i+1}^* \cdot \rangle$ is coercive on

$$\underbrace{N(A_{i+2})}_{= R(A_{i+1}) \oplus K_{i+2} = R(A_{i+1}) \oplus \{0\}} \cap D(A_{i+1}^*) = D(\mathcal{A}_{i+1}^*)$$

⇝ clear, according to our formulations.

② inf-sup-condition:

$$\inf_{0 \neq \psi \in D(\mathcal{A}_{i+2}^*)} \; \sup_{0 \neq \phi \in D(A_{i+1}^*)} \frac{\langle \phi, A_{i+2}^* \psi \rangle}{|\phi|_{D(A_{i+1}^*)} \, |\psi|_{D(\mathcal{A}_{i+2}^*)}}$$

$$\geq \inf_{0 \neq \psi \in D(\mathcal{A}_{i+2}^*)} \frac{|A_{i+2} \psi|}{|\psi|_{D(\mathcal{A}_{i+2}^*)}} \geq (1 + c_{i+2}^2)^{-1/2}$$

$\phi := A_{i+2}^* \psi \in N(A_{i+1}^*)$

Lecture 6 14.11.16

Theorem 9: (solution theory for (ESP))

Let $\mathring{\operatorname{rot}}\,\mathring{R}$, $\mathring{\nabla}\mathring{H}^1$ be closed. Then (ESP) is uniquely solvable in $\mathring{R} \cap \varepsilon^{-1}D$ iff $F \in \mathring{\operatorname{rot}}\,\mathring{R}$, $g \in L^2$, $D \in \mathcal{D}_\varepsilon$. The unique solution is given by

$$E = E_F + E_f + D \in (\mathring{R} \cap \varepsilon^{-1}\operatorname{rot} R) \oplus (\varepsilon^{-1}D \cap \mathring{\nabla}\mathring{H}^1) \oplus \mathcal{D}_\varepsilon$$

where

$$E_F := \mathring{\operatorname{rot}}^{-1} F \in \mathring{R} \cap \varepsilon^{-1}\operatorname{rot} R = \mathring{R} \cap \varepsilon^{-1}D_0 \cap \mathcal{D}_\varepsilon^{\perp_{L^2_\varepsilon}}$$
$$E_f := (-\operatorname{div}\varepsilon)^{-1} f \in \varepsilon^{-1}D \cap \mathring{\nabla}\mathring{H}^1 = \varepsilon^{-1}D \cap \mathring{R}_0 \cap \mathcal{D}_\varepsilon^{\perp_{L^2_\varepsilon}}$$

and depends continuously on the data, i.e.

$$|E|^2_{L^2_\varepsilon} \leq c_m^2 |F|^2_{L^2} + c_F^2 |f|^2_{L^2} + |D|_{L^2_\varepsilon}$$
$$|E|^2_{\mathring{R} \cap \varepsilon^{-1}D} \leq (1 + c_m^2)|F|^2_{L^2} + (1 + c_F^2)|f|^2_{L^2} + |D|^2_{L^2_\varepsilon}.$$

Furthermore: $E_F = \pi_{\varepsilon^{-1}\operatorname{rot}} E$, $E_f = \pi_{\mathring{\nabla}} E_0$

Remark:

① E_F, E_f solve

$$\begin{aligned} \mathring{\operatorname{rot}}\, E_F &= F & \qquad \mathring{\operatorname{rot}}\, E_f &= 0 \\ -\operatorname{div}\varepsilon\, E_F &= 0 & \qquad -\operatorname{div}\varepsilon\, E_f &= f \\ \pi_D E_F &= 0 & \qquad \pi_D E_f &= 0 \end{aligned}$$

② Again:

$\mathring{\operatorname{rot}}\,\mathring{R}$ is closed $\Leftrightarrow$ $\operatorname{rot} R$ is closed

$\mathring{\nabla}\mathring{H}^1$ is closed $\Leftrightarrow$ $\operatorname{div} \mathbb{D}$ is closed

Both we get e.g. from

$$\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow L^2_\varepsilon$$

$$\Rightarrow \left\{ \begin{array}{l} |\mathcal{D}_\varepsilon| < \infty \\ \mathring{\operatorname{rot}}^{-1}, \operatorname{rot}^{-1}, \mathring{\nabla}^{-1}, \nabla^{-1}, \operatorname{div}^{-1}, \dots \\ \text{are compact.} \end{array} \right.$$

„Variational formulation" for (ESP)

$$E_F = A_1^* H_F = \varepsilon^{-1} \operatorname{rot} H_F \,,$$

$$\Rightarrow \quad H_F = (A_1^*)^{-1} E_F$$

$$= (\varepsilon^{-1} \operatorname{rot})^{-1} E_F \in R \cap \mathring{\operatorname{rot}} \mathring{R}$$

can be found by:

Find $H_F \in R \cap \mathring{\operatorname{rot}} \mathring{R}$ such that

$\forall \phi \in R \cap \mathring{\operatorname{rot}} \mathring{R}$:

$$\langle \varepsilon^{-1} \operatorname{rot} H_F , \varepsilon^{-1} \operatorname{rot} \phi \rangle_{L^2_\varepsilon} = \langle F , \phi \rangle_{L^2}$$

$$\shortparallel$$

$$\langle \operatorname{rot} H_F , \operatorname{rot} \phi \rangle_{L^2_{\varepsilon^{-1}}}$$

$\mathring{\operatorname{rot}} \mathring{R} = R_0^\perp$

$\downarrow$

$\Leftrightarrow$ Find $H_F \in R$ such that

$$\forall \phi \in R: \ \langle \operatorname{rot} H_F , \operatorname{rot} \phi \rangle_{L^2_{\varepsilon^{-1}}} = \langle F , \phi \rangle_{L^2}$$

$$\forall \psi \in R_0: \ \langle H_F , \psi \rangle_{L^2} = 0$$

Now assume: $\mathcal{N} = \{0\}$

$\Leftrightarrow$ Ω is simply connected.

$R_0 = \nabla H^1 \oplus \mathcal{N}$

$\downarrow$

$\Leftrightarrow$ Find $(H_f, u_f) \in R \times H^1_\perp$ such that

$$\forall \phi \in R : \quad \langle \operatorname{rot} H_F , \operatorname{rot} \phi \rangle_{L^2_{\varepsilon^{-1}}} + \langle \phi , \nabla u_F \rangle_{L^2} = \langle F, \phi \rangle_{L^2}$$

$$\forall \psi \in H^1 : \quad \langle H_F , \nabla \psi \rangle_{L^2} = 0$$

One can also put $H^1_\perp$.

So : $E_f \rightsquigarrow$ rot-rot-problem

$E_f = A_0 u_f = \overset{\circ}{\nabla} u_f$, $u_f = A_0^{-1} E_f = \overset{\circ}{\nabla}^{-1} E_f \in \overset{\circ}{H}{}^1$
can be found by

Find $u_f \in \overset{\circ}{H}{}^1$ such that
$$\forall \phi \in \overset{\circ}{H}{}^1 \quad \langle \nabla u_F , \nabla \phi \rangle_{L^2_\varepsilon} = \langle f, \phi \rangle_{L^2}$$

So : $u_f \rightsquigarrow -\Delta_0$-problem

"Variational formulation" for (MSP)

$H_G = A_1 E_G = \operatorname{rot} E_G$,
$$\Rightarrow E_G = (A_1)^{-1} H_G = \operatorname{rot}^{-1} H_G \in \overset{\circ}{R} \cap \operatorname{rot} R$$

can be found by

Find $(E_G, v_G) \in \overset{\circ}{R} \times \overset{\circ}{H}{}^1$ such that
$$\forall \phi \in \overset{\circ}{R} : \langle \operatorname{rot} E_G , \operatorname{rot} \phi \rangle_{L^2_{\mu^{-1}}} + \langle \phi , \nabla v_G \rangle_{L^2} = \langle G, \phi \rangle_{L^2}$$

$$\forall \psi \in \overset{\circ}{H}{}^1 : \quad \langle E_G , \nabla \psi \rangle_{L^2} = 0$$

$$H_G = A_2^* u_g = -\nabla u_g \,,$$
$$\Rightarrow \quad u_g = (A_2^*)^{-1} H_g$$
$$= (-\nabla)^{-1} H_g \in H^1$$

can be found by

Find $u_g \in H^1_\perp$ such that

$$\forall \phi \in H^1_\perp : \langle \nabla u_g, \nabla \phi \rangle_{L^2_\mu} = \langle g, \phi \rangle_{L^2}$$

So far we can handle:

$A_i x = f$	$A_i^* A_i \, x = f$	$A_i^* A_i x = f$
$A_{i-1}^* x = g$	$A_{i-1}^* x = g$	$A_{i-1} A_{i-1}^* x = g$
$\pi_i x = h$	$\pi_i x = h$	$\pi_i x = h$
kernel: K_i	kernel: K_i $\uparrow$ since:	kernel: K_i

$$0 = \langle A_i^* A_i \, x, x \rangle = \| A_i x \|^2$$
$$\Rightarrow A_i x = 0$$

now: <u>case ②</u>

solution theory for: $(M - \lambda) x = f, \ \lambda \in \mathbb{C}$?

<u>$\lambda = 0$:</u> ✓ $\rightsquigarrow$ case ③ (static case)

<u>$\lambda \in \mathbb{C} \setminus \mathbb{R}$:</u> ✓ $\rightsquigarrow$ M is selfadjoint

$$\Rightarrow \sigma(M) \subset \mathbb{R}$$
$$\Rightarrow \mathbb{C} \setminus \mathbb{R} \subset \rho(M)$$

so we are left with: $\boxed{\lambda \in \mathbb{R} \setminus \{0\}}$

recall:

$$* \quad A = \begin{pmatrix} 0 & -\mathring{\operatorname{rot}}^* \\ \mathring{\operatorname{rot}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{pmatrix}$$

$$* \quad \Lambda = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$$

$$* \quad M = i\Lambda^{-1}A = i\begin{pmatrix} 0 & -\varepsilon^{-1}\operatorname{rot} \\ \mu^{-1}\mathring{\operatorname{rot}} & 0 \end{pmatrix}$$

$M : D(M) = \mathring{R} \times R \subset L^2_\Lambda = L^2_\varepsilon \times L^2_\mu \longrightarrow L^2_\Lambda$
is selfadjoint for all Ω (open) $\subset \mathbb{R}^3$

$\mathcal{M} : D(\mathcal{M}) = D(M) \cap \overline{R(M)} \subset \overline{R(M)} \longrightarrow \overline{R(M)}$

<u>assume</u>: $\boxed{D(\mathcal{M}) \hookrightarrow\hookrightarrow R(M) \,/\, L^2_\Lambda \,/\, L^2}$ (*)

($\rightsquigarrow$ critical point for Fredholm-alternative)

$D(\mathcal{M}) = (\mathring{R} \cap \varepsilon^{-1}\overline{\operatorname{rot} R}) \times (R \cap \mu^{-1}\overline{\mathring{\operatorname{rot}}\mathring{R}}) \hookrightarrow\hookrightarrow L^2_\Lambda$

$\Leftrightarrow \quad \mathring{R} \cap \varepsilon^{-1}\overline{\operatorname{rot} R} \hookrightarrow\hookrightarrow L^2_\varepsilon \;\wedge\; R \cap \mu^{-1}\overline{\mathring{\operatorname{rot}}\mathring{R}} \hookrightarrow\hookrightarrow L^2_\mu$

$$\Uparrow$$

$$\boxed{\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow\hookrightarrow L^2_\varepsilon \;\wedge\; R \cap \mu^{-1}\mathring{D} \hookrightarrow\hookrightarrow L^2_\mu}$$

(Weck's selection theorem; true for e.g. Ω bounded, weak Lipschitz)

$\Rightarrow R(\mathcal{M}) = R(M)$ is closed
$\wedge \;\; \forall x \in D(\mathcal{M}) : \; |x| \leq c_m |Mx|$
$\wedge \;\; \mathcal{M}^{-1} : R(M) \longrightarrow D(\mathcal{M})$
is continuous.

$$\wedge \quad \boxed{\begin{array}{l} \mathcal{M}^{-1} : R(\mathcal{M}) \to R(\mathcal{M}) / L^2_\Lambda \\ \text{is compact} \end{array}}$$

Now: $\mathcal{M}_\lambda := \mathcal{M} - \lambda , \; \lambda \in \mathbb{R} \setminus \{0\}$

$\rightsquigarrow$ $\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) = D(\mathcal{M}) \subset L^2_\Lambda \to L^2_\Lambda$ is linear, densely defined, closed and self-adjoint.

$\ulcorner$ $\mathcal{M}_\lambda$ is closed, since:

$$x_n \to x \text{ in } L^2_\Lambda , \; \mathcal{M}_\lambda x_n \to y \text{ in } L^2_\Lambda$$
$$\Rightarrow x_n \to x \text{ in } L^2_\Lambda$$
$$\wedge \; \mathcal{M} x_n - \lambda x_n \to y \text{ in } L^2_\Lambda$$
$$\Rightarrow x_n \to x \text{ in } L^2_\Lambda$$
$$\wedge \; \mathcal{M} x_n \to y + \lambda x \text{ in } L^2_\Lambda$$
$$\overset{\mathcal{M} \text{ is closed}}{\Rightarrow} x \in D(\mathcal{M}) = D(\mathcal{M}_\lambda)$$
$$\wedge \; \mathcal{M} x = y + \lambda x \Leftrightarrow \mathcal{M}_\lambda x = y .$$

$\mathcal{M}_\lambda$ is selfadjoint, since:

$$(\mathcal{M} - \lambda)^* = \mathcal{M}^* - \bar{\lambda} \underset{\substack{\uparrow \\ \mathcal{M} \text{ is selfadjoint}}}{=} \mathcal{M} - \lambda$$

$\lrcorner$

$\rightsquigarrow$ $\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) = D(\mathcal{M}_\lambda) \cap \overline{R(\mathcal{M}_\lambda)}$
$= D(\mathcal{M}) \cap \overline{R(\mathcal{M}_\lambda)} \subset \overline{R(\mathcal{M}_\lambda)} \to \overline{R(\mathcal{M}_\lambda)}$

Let's solve $(\mathcal{M} - \lambda) x = f \in L^2_\Lambda , \; x \in D(\mathcal{M})$:

Suppose x solves the system, then:

$$L^2 = N(\mathcal{M}) \oplus \overline{R(\mathcal{M})}$$

We decompose f, x:

$$f = f_n + f_r \in \mathcal{N}(\mathcal{M}) \oplus \overline{R(\mathcal{M})}$$

$$x = x_n + x_r \in \mathcal{N}(\mathcal{M}) \oplus D(\mathcal{M}) \quad (x \in D(\mathcal{M}))$$

Then:

$$\begin{aligned}
&(\mathcal{M} - \lambda)\, x = f \\
&\Leftrightarrow \; (\mathcal{M} - \lambda)(x_n + x_r) = f_n + f_r \\
&\Leftrightarrow \; -\lambda x_n + (\mathcal{M} - \lambda) x_r = f_n + f_r \\
&\Leftrightarrow \; \underbrace{(\mathcal{M} - \lambda) x_r - f_r}_{\in \overline{R(\mathcal{M})}} = \underbrace{f_n + \lambda x_n}_{\in \mathcal{N}(\mathcal{M})}
\end{aligned}$$

$$\mathcal{N}(\mathcal{M}) = \overline{R(\mathcal{M})}^{\perp} \longrightarrow \Downarrow$$

$$\mathcal{M}_\lambda x_r = f_r \in \overline{R(\mathcal{M})}, \; x_r \in D(\mathcal{M})$$

$$-\lambda x_n = f_n$$

So, for solving we take: $f \in L^2_\Delta$

<u>1. step:</u> decompose

$$f = f_n + f_r \in \mathcal{N}(\mathcal{M}) \oplus \overline{R(\mathcal{M})}$$

$\rightsquigarrow$ define: $x_n := -\lambda^{-1} f_n \in \mathcal{N}(\mathcal{M})$

<u>2. step:</u> solve (?)

find $x_r \in D(\mathcal{M})$: $\mathcal{M}_\lambda x_r = f_r \in \overline{R(\mathcal{M})}$

We end up with a reduced problem:

$$\boxed{\begin{array}{l}\text{Find } x \in D(\mathcal{M}) \text{ such that} \\ \qquad \mathcal{M}_\lambda x = f \in \overline{R(\mathcal{M})}\end{array}}$$

$\rightsquigarrow$ $\mathcal{M} : \mathring{R} \times R \subset L^2_\varepsilon \times L^2_\mu \to L^2_\varepsilon \times L^2_\mu$

$$\rightsquigarrow \quad \mathcal{M} : (\mathring{R} \times R) \cap (\varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}}\,\mathring{R}})$$

$$\subset \varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}}\,\mathring{R}} \to \varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}}\,\mathring{R}}$$

$$\overline{R(\mathcal{M})} = \varepsilon^{-1}\overline{\operatorname{rot} R} \times \mu^{-1}\overline{\mathring{\operatorname{rot}}\,\mathring{R}}$$

$$= (\varepsilon^{-1} D_0 \cap \mathcal{D}_\varepsilon^{\perp_{L^2_\varepsilon}}) \times (\mu^{-1}\mathring{D}_0 \cap \mathcal{M}_\mu^{\perp_{L^2_\mu}})$$

$$\mathcal{N}(\mathcal{A}_0^*) = K_1 \oplus \overline{R(\mathcal{A}_1^*)}$$

$$\mathcal{N}(\mathcal{A}_2) = K_2 \oplus \overline{R(\mathcal{A}_1)}$$

$(*) \Rightarrow \mathcal{M}_\lambda x = (\mathcal{M} - \lambda)x = f \in \overline{R(\mathcal{M})} = R(\mathcal{M})$

$\wedge$ $\mathcal{M}^{-1} : R(\mathcal{M}) \to R(\mathcal{M})$ is compact

(since: $\mathcal{M}^{-1} : R(\mathcal{M}) \to D(\mathcal{M}) \hookrightarrow R(\mathcal{M})$)

So

$$\mathcal{M}_\lambda x = (\mathcal{M} - \lambda)x = f \in R(\mathcal{M})$$

$$\Updownarrow \ \mathcal{M}^{-1}$$

$$\underbrace{\mathcal{M}^{-1}\mathcal{M}x}_{= x} - \lambda \mathcal{M}^{-1}x = \mathcal{M}^{-1} f$$

⌜ $\mathcal{M}^{-1}\mathcal{M}x = x$, since $\mathcal{M}$ is injective &

$\mathcal{M}(\underbrace{\mathcal{M}^{-1}\mathcal{M}x - x}_{\in D(\mathcal{M})}) = 0$ $\uparrow$ $\mathcal{M}^{-1}$ is right inverse ⌟

Now: $x - \lambda \mathcal{M}^{-1} x = \mathcal{M}^{-1} f$

$\Leftrightarrow \left(\frac{1}{\lambda} I - \mathcal{M}^{-1}\right) x = \frac{1}{\lambda}\mathcal{M}^{-1} f \quad (*_2)$

Apply <u>Fredholm - alternative</u>:

$\Rightarrow \sigma(\mathcal{M}^{-1}) \subset \sigma_p(\mathcal{M}^{-1}) \subset J := [-|\mathcal{M}^{-1}|, |\mathcal{M}^{-1}|]$

$$\land \quad \forall\, \omega = 1/\lambda \in \sigma_p(\mathcal{M}^{-1}) :$$

$$|\,\mathcal{N}(\mathcal{M}^{-1} - \omega)\,| < \infty$$

$\land \quad \sigma_p(\mathcal{M}^{-1})$ has no accumulation point in $J \setminus \{0\}$

Furthermore:

$$(\mathcal{M}^{-1} - 1/\lambda)\, x = 0 \,, \quad x \in D(\mathcal{M})$$

$$\Leftrightarrow \quad (\mathcal{M} - \lambda)\, x = 0 \,, \quad x \in D(\mathcal{M})$$

so: $\boxed{\begin{array}{l} x \text{ is eigenfunction of } (\mathcal{M}^{-1}, 1/\lambda) \\ \Leftrightarrow x \text{ is eigenfunction of } (\mathcal{M}, \lambda) \end{array}}$

$$\Rightarrow \quad |\,\mathcal{N}(\mathcal{M} - \lambda)\,| < \infty \quad \land \quad \sigma(\mathcal{M}) \subset \underset{\sigma_p(\mathcal{M}^{-1})^{-1}}{\underset{\parallel}{\sigma_p(\mathcal{M})}}$$

We therefore can solve $(*_2)$, iff

$$\mathcal{M}^{-1} f \perp \mathcal{N}(\mathcal{M}^{-1} - 1/\lambda) = \mathcal{N}(\mathcal{M} - \lambda)$$

$$\Leftrightarrow \quad \langle \mathcal{M}^{-1} f, y \rangle = 0 \quad \forall\, y \in \mathcal{N}(\mathcal{M} - \lambda)$$

$$\begin{array}{c} \langle \mathcal{M}^{-1} f, y \rangle \\ \parallel \\ 1/\lambda \, \langle \mathcal{M}^{-1} f, \mathcal{M} y \rangle \\ \parallel \\ 1/\lambda \, \langle f, y \rangle \end{array} \qquad \left(\begin{array}{l} y \in \mathcal{N}(\mathcal{M} - \lambda) \\ \Leftrightarrow \mathcal{M} y = \lambda y \end{array} \right)$$

$$\Leftrightarrow \quad \boxed{f \perp \mathcal{N}(\mathcal{M} - \lambda)}$$

Lecture 7 28.11.16

Reminder:

- $A := \begin{pmatrix} 0 & -\operatorname{rot} \\ \mathring{\operatorname{rot}} & 0 \end{pmatrix} : \mathring{R} \times R \subset L^2_\Lambda \to L^2_\Lambda$

  is linear, densely defined and closed,
- $L^2_\Lambda := L^2_\varepsilon \times L^2_\mu$, $\operatorname{rot} = \mathring{\operatorname{rot}}^*$
- $M := i\Lambda^{-1}A : \mathring{R} \times R \subset L^2_\Lambda \to L^2_\Lambda$ is linear, densely defined, closed and selfadjoint
- $\mathcal{M} : D(\mathcal{M}) = D(M) \cap \overline{R(M)} \subset \overline{R(M)} \to \overline{R(M)}$,

  $R(M) = \varepsilon^{-1}\operatorname{rot} R \times \mu^{-1}\mathring{\operatorname{rot}}\mathring{R}$

We are still interested in solving:

$$(M - \lambda)x = f$$

$\hookrightarrow$ $\lambda = 0$: (static case) ✓

$\hookrightarrow$ $\lambda \in \mathbb{C}\setminus\mathbb{R}$: ✓, since $\mathbb{C}\setminus\mathbb{R} \subset \rho(M)$.

Left with: $\boxed{\lambda \in \mathbb{R}\setminus\{0\}}$

Starting with $f \in L^2_\Lambda$ we are looking for $x \in D(M)$ with $(M-\lambda)x = f$.

idea: decompose x and f

- $L^2_\Lambda = N(M) \oplus \overline{R(M)}$

  $\rightsquigarrow f = f_0 + f_R \in N(M) \oplus \overline{R(M)}$
- $D(M) = N(M) \oplus D(\mathcal{M})$

  $\rightsquigarrow x = x_0 + x_R \in N(M) \oplus D(\mathcal{M})$

Then we get:

$$(\mathcal{M}-\lambda)x = f$$

$$\iff (\mathcal{M}-\lambda)x_0 + (\mathcal{M}-\lambda)x_R = f_0 + f_R$$

$$\iff -\lambda x_0 - f_0 = f_R - (\mathcal{M}-\lambda)x_R$$

$\overline{R(\mathcal{M})} = N(\mathcal{M})^{\perp_\Lambda}$

$$\iff \begin{cases} -\lambda x_0 = f_0 \in N(\mathcal{M}), & x_0 \in N(\mathcal{M}) \\ (\mathcal{M}-\lambda)x_R = f_R \in \overline{R(\mathcal{M})}, & x_R \in D(\mathfrak{M}) \end{cases}$$

Since $x_0 := -\lambda^{-1} f_0 \in N(\mathcal{M})$ is a solution to the first problem, we just have to deal with the second one:

$$(\mathcal{M}-\lambda)x_R = f_R \in \overline{R(\mathcal{M})}, \quad x_R \in D(\mathfrak{M})$$

<u>crucial assumption</u>: $\boxed{D(\mathfrak{M}) \hookrightarrow L^2}$

$\Rightarrow$ $R(\mathcal{M})$ is closed, Poincaré-estimates and $\mathfrak{M}^{-1}: R(\mathcal{M}) \to D(\mathfrak{M})$ is continuous

<u>first idea</u>: apply $\mathfrak{M}^{-1}$

$$(\mathcal{M}-\lambda)x_R = f_R$$
$$\iff (1/\lambda \, \mathrm{id} - \mathfrak{M}^{-1})x_R = 1/\lambda \, \mathfrak{M}^{-1} f_R$$

Now $\mathfrak{M}^{-1}: R(\mathcal{M}) \to R(\mathcal{M})$ is compact and we can use Fredholms alternative.

<u>alternative idea</u>: use toolbox

Look at: $\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) = D(\mathcal{M}) \subset H \to H$,
$\mathcal{M}_\lambda := \mathcal{M} - \lambda$

still we assume: $\boxed{D(\mathcal{M}) \hookrightarrow H}$

and again we have:

$$\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) \subset R(M_\lambda) \to R(M_\lambda)$$
$$\text{with } D(\mathcal{M}_\lambda) = D(\mathcal{M}) \cap R(M_\lambda)$$

<u>But</u>: Is $D(\mathcal{M}_\lambda) \hookrightarrow H$?

$\rightsquigarrow$ Yes it is, but it has to be proven!

To avoid this „gap", we instead start with another operator:

$$M_\lambda : D(M_\lambda) = D(\mathcal{M}) \subset R(M) \to R(M)$$
$$\Rightarrow \mathcal{M}_\lambda : D(\mathcal{M}_\lambda) \subset R(M_\lambda) \to R(M_\lambda)$$
$$\text{with } D(\mathcal{M}_\lambda) = D(\mathcal{M}) \cap R(M_\lambda)$$

Of course we have to check if M_λ is of the „<u>right kind</u>":

* M_λ is linear: clear!
* M_λ is densely defined:
  Take $f \in R(M) \subset H$. Since $D(M)$ is dense in H there exists $(f_n)_{n\in\mathbb{N}} \subset D(M)$, $f_n \to f$ in H. By $D(M) = N(M) \oplus D(\mathcal{M})$ we get for $n \in \mathbb{N}$:

$$f_n = f_n^1 + f_n^2 \in N(M) \oplus D(\mathcal{M})$$

Then clearly $f_n^1 \to 0$ in H and $f_n^2 \to f$ in H.

* $\mathcal{M}_\lambda$ is closed:

Take $(x_n)_n \subset D(\mathcal{M})$, $x_n \to x$ in $R(\mathcal{M})$

and $\mathcal{M}_\lambda x_n \to f$ in $R(\mathcal{M})$

$\Rightarrow \mathcal{M} x_n = \mathcal{M}_\lambda x_n + \lambda x_n \to f + \lambda x$

Since $\mathcal{M}$ is closed we have

$x \in D(\mathcal{M})$ and $\mathcal{M} x = f + \lambda x$

$\Rightarrow x \in D(\mathcal{M})$ and $\mathcal{M}_\lambda x = f$.

Since $D(\mathcal{M}_\lambda) = D(\mathcal{M}) \cap R(\mathcal{M}_\lambda) \subset D(\mathcal{M})$

we have:

$$\boxed{D(\mathcal{M}_\lambda) \hookrightarrow R(\mathcal{M})}$$

$\Rightarrow R(\mathcal{M}_\lambda)$ is closed and with

$R(\mathcal{M}) = \mathcal{N}(\mathcal{M}_\lambda) \oplus R(\mathcal{M}_\lambda)$

$\hookrightarrow D(\mathcal{M}) = D(\mathcal{M}_\lambda)$

$= \mathcal{N}(\mathcal{M}_\lambda) \oplus [D(\mathcal{M}) \cap R(\mathcal{M}_\lambda)]$

we have:

$\forall x \in D(\mathcal{M}_\lambda) = D(\mathcal{M}) \cap R(\mathcal{M}_\lambda)$:

$$|x|_H \leq c\, |\mathcal{M}_\lambda x|_H$$

Furthermore:

$\mathcal{M}_\lambda : D(\mathcal{M}_\lambda) \subset R(\mathcal{M}_\lambda) \cap R(\mathcal{M}) \to R(\mathcal{M}_\lambda) \cap R(\mathcal{M})$

$\wedge$ $\mathcal{M}_\lambda^{-1} : R(\mathcal{M}_\lambda) \cap R(\mathcal{M}) \to D(\mathcal{M}_\lambda)$

is continuous.

$\wedge$ $\mathcal{M}_\lambda^{-1} : R(\mathcal{M}_\lambda) \cap R(\mathcal{M}) \to R(\mathcal{M}_\lambda) \cap R(\mathcal{M})$

is compact.

Thus we just proved:

Theorem: (Fredholms alternative)

Let $f \in R(\mathcal{M})$. Then $\mathcal{M}_\lambda x = f$ is solvable in $D(\mathcal{M})$, iff $f \perp N(\mathcal{M}_\lambda)$. Choosing e.g. $x \perp N(\mathcal{M}_\lambda)$ makes x unique.

Lemma:

$\sigma_p(\mathcal{M}) \setminus \{0\} = \sigma(\mathcal{M}) \setminus \{0\} = \sigma(\mathcal{M}) = \sigma_p(\mathcal{M}) \subset \mathbb{R} \setminus \{0\}$ is discrete and $\sigma(\mathcal{M})$ can only accumulate at ∞. All $N(\mathcal{M}_\lambda)$ are finite dimensional.

Proof:

$0 \in \varrho(\mathcal{M})$, since $\mathcal{M}^{-1}$ is continuous. Then $(0 \in \sigma_p(\mathcal{M})$ or $0 \in \varrho(\mathcal{M}))$ we have $0 \notin \sigma_p(\mathcal{M})$. Furthermore:

$$\begin{aligned} 0 \neq \lambda \notin \sigma_p(\mathcal{M}) &\Rightarrow N(\mathcal{M}_\lambda) = 0 \\ &\Rightarrow R(\mathcal{M}_\lambda) = R(\mathcal{M}) \\ \mathcal{M}_\lambda^{-1} \text{ is contin.} &\Rightarrow \mathcal{M}_\lambda = \mathcal{M}_\lambda \\ &\Rightarrow \lambda \in \varrho(\mathcal{M}) \end{aligned}$$

If $|N(\mathcal{M}_\lambda)| = \infty$ or there is an accum. point of $\sigma(\mathcal{M})$ in $\mathbb{R}$ we find $(x_n)_{n \in \mathbb{N}}$ orthonormalsequence in $N(\mathcal{M}_{\lambda_n})$ for $(\lambda_n)_{n \in \mathbb{N}}$, $\lambda_n \to \lambda$.

⌈ clear for $|N(\mathcal{M}_\lambda)| = \infty$. If $\sigma(\mathcal{M})$ has an accumulation point λ, we can choose a sequence $(\lambda_n)_{n \in \mathbb{N}}$, $\lambda_n \neq \lambda_m$ for $n \neq m$ with $\lambda_n \to \lambda$, and a corresponding sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in N(\mathcal{M}_{\lambda_n})$. Then:

$$\lambda_n \langle x_n, x_m \rangle$$
$$= \langle \mathcal{M} x_n, x_m \rangle$$
$$= \langle x_n, \mathcal{M} x_m \rangle = \lambda_m \langle x_n, x_m \rangle$$
$$\Rightarrow (\lambda_n - \lambda_m) \langle x_n, x_m \rangle = 0$$
$$\Rightarrow \langle x_n, x_m \rangle = 0$$

$\Rightarrow$ $(x_n)_n$ is orthogonalsequence. $\lrcorner$

As an orthonormalsequence $x_n \to 0$ in H resp. $R(\mathcal{M})$. Now $x_n \in D(\mathcal{M})$, $|x_n|_H = 1$ and therefore:

$$|\mathcal{M} x_n| = |\mathcal{M}_\lambda x_n + \lambda_n x_n|$$
$$\leq |\lambda_n| \, |x_n| \leq |\lambda|,$$

which means $(x_n)_{n\in\mathbb{N}}$ is bounded in $D(\mathcal{M})$. $D(\mathcal{M}) \hookrightarrow\hookrightarrow H \Rightarrow \exists$ subsequence $(x_{\pi(n)})_{n\in\mathbb{N}}$ with $x_{\pi(n)} \to x = 0$ in H. ↯

$\boxed{\xi}$

Weck's selection theorem or the Maxwell compactness property:

Theorem:

$\Omega \subset \mathbb{R}^3$ bounded and strong lipschitz, $\varepsilon : \Omega \to \mathbb{R}^{3\times3}$ uniformly positive definit, symmetric and with L^∞-coefficients.

Then:

$$\mathring{R}(\Omega) \cap \varepsilon^{-1} D(\Omega) \hookrightarrow L^2(\Omega)$$
$$R(\Omega) \cap \varepsilon^{-1} \mathring{D}(\Omega) \hookrightarrow L^2(\Omega)$$

<u>Ideas for a proof</u>:

'50/'60: $\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow H^1 \hookrightarrow L^2$

need: Ω, ε smooth

'74: Weck for strong Lipschitz (in $\mathbb{R}^n$ or manifolds)

'84: Weber for strong Lipschitz in $\mathbb{R}^3$ with potentials

We will follow the idea of Weber.

<u>Lemma n</u>:

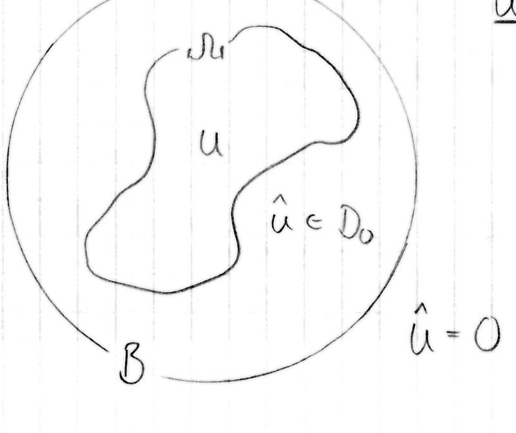
$\Omega \subset \mathbb{R}^3$ bd., str. Lipschitz and topological trivial, i.e. simply connected and $\partial\Omega$ is connected ($\partial\Omega$ connected is enough). Then $D_0 = \operatorname{rot} H^1$ with continuous potential, i.e. $\forall\, u \in D_0\ \exists\, \phi \in H^1$:

$$\operatorname{rot} \phi = u \quad \text{and} \quad |\phi|_{H^1} \le c\,|u|_{L^2}$$

<u>Proof</u>:

„$\supset$": $\operatorname{rot} H^1 \subset \operatorname{rot} R \subset D_0$

„$\subset$": let $u \in D_0(\Omega)$



<u>wish</u>: Extend u by $\hat{u} \in D_0(\mathbb{R}^3)$

<u>idea</u>: Extend u by $\hat{u} \in \mathring{D}_0(B)$

Assume we have constructed $\hat{u} \in \mathring{D}_0(B)$.
Then for all $\varphi \in H^1(B)$ we have

$$
\begin{aligned}
0 &= \langle \hat{u}, \nabla\varphi \rangle_{L^2(B)} \\
&= \langle u, \nabla\varphi \rangle_{L^2(\Omega_i)} + \langle \hat{u}, \nabla\varphi \rangle_{L^2(B\setminus\overline{\Omega_i})}
\end{aligned}
$$

So we make the <u>ansatz</u>:

$\hat{u} = \nabla v$, $v \in H^1_\perp(B\setminus\overline{\Omega_i})$:

$\forall \psi \in H^1_\perp(B\setminus\overline{\Omega_i})$:

$$\langle \nabla v, \nabla\psi \rangle_{L^2(B\setminus\overline{\Omega_i})} = -\langle u, \nabla(\mathcal{E}\psi)\rangle_{L^2(\Omega_i)}$$

where $\mathcal{E}: H^1(B\setminus\overline{\Omega_i}) \to H^1(B)$ is an extension operator (Calderon / Stein, ...)

⌜ need: $B\setminus\overline{\Omega_i}$ strong Lipschitz, connected. But this is given, since Ω_i is topological trivial (resp. $\partial\Omega_i$ is connected) and therefore $B\setminus\overline{\Omega}$ a domain. ⌟

$\Rightarrow$ By Riesz we get: $\exists!\, v \in H^1_\perp(B\setminus\overline{\Omega_i})$.

Then we define:

$$\hat{u} := \begin{cases} u &, \text{ in } \Omega_i \\ \nabla v &, \text{ in } B\setminus\overline{\Omega_i} \\ 0 &, \text{ in } \mathbb{R}^3\setminus\overline{B} \end{cases}$$

Now pick $\varphi \in \mathring{C}^\infty(\mathbb{R}^3)$ and define

$\psi := \varphi - |\Omega_i|^{-1}\langle \varphi, 1\rangle_{L^2(B\setminus\overline{\Omega_i})} \in H^1_\perp(B\setminus\overline{\Omega_i})$

Then $\nabla\psi = \nabla\varphi$ and we have:

$$\langle \hat{u}, \nabla\varphi \rangle_{L^2(\mathbb{R}^3)}$$
$$= \langle \nabla v, \nabla\varphi \rangle_{L^2(B \setminus \overline{\Omega_1})} + \langle u, \nabla\varphi \rangle_{L^2(\Omega_1)}$$
$$= \langle \nabla v, \nabla\psi \rangle_{L^2(B \setminus \overline{\Omega_1})} + \langle u, \nabla\psi \rangle_{L^2(\Omega_1)}$$
$$= \langle u, \nabla(\psi - \mathcal{E}\psi) \rangle_{L^2(\Omega_1)}$$
$$= - \langle \operatorname{div} u, \psi - \mathcal{E}\psi \rangle_{L^2(\Omega_1)} = 0$$

⌜ beachte:

$\psi - \mathcal{E}\psi \in H^1(B)$ und

$\psi - \mathcal{E}\psi = 0$ in $B \setminus \overline{\Omega_1}$

$\Rightarrow \psi - \mathcal{E}\psi \in \mathring{H}^1(\Omega_1)$ ⌟

$\Rightarrow$ $\hat{u} \in D_0(\mathbb{R}^3)$ and furthermore $\hat{u} \in D_0(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ since $\operatorname{supp} \hat{u} \subset \overline{B}$

Now we define $\Phi := \xi / r^2 \times \mathcal{F}(\hat{u})$ with $\xi(x) = x$, $r(x) = |x|$ and

$$\mathcal{F}^{\pm 1}(v)(x) = \int_{\mathbb{R}^3} e^{\mp ixy} v(y)\, dy$$

Now: $\mathcal{F}(\hat{u}) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, since

$$|\mathcal{F}(\hat{u})|(x) \leq c\, |\hat{u}|_{L^1(\mathbb{R}^3)}$$

$\Rightarrow$ $\Phi \in L^2(\mathbb{R}^3 \setminus \overline{B}_1) \cap L^2(B_1) = L^2(\mathbb{R}^3)$, since:

$$|\Phi|_{L^2(B_1)} \leq c \int_0^1 r^{-2} r^2 dr \leq c.$$

$$\Rightarrow r\Phi \in L^2(\mathbb{R}^3) \Leftrightarrow \mathcal{F}^{-1}(\Phi) \in H^1(\mathbb{R}^3).$$

We define: $y := -i\mathcal{F}^{-1}(\Phi)$

Now remember that:

$$\partial_n(\mathcal{F}^{\pm 1}(v)) = \mp i \mathcal{F}^{\pm 1}(\varrho_n v)$$

$$\varrho_n(\mathcal{F}^{\pm 1}(v)) = \mp i \mathcal{F}^{\pm 1}(\partial_n u)$$

$$\Rightarrow \quad \mathcal{F}^{\pm 1}(\operatorname{rot} v) = \pm i \varrho \times \mathcal{F}^{\pm 1}(v),$$

$$\mathcal{F}^{\pm 1}(\operatorname{div} v) = \pm i \varrho \cdot \mathcal{F}^{\pm 1}(v),$$

$$\mathcal{F}^{\pm 1}(\varrho \times v) = \pm i \operatorname{rot} \mathcal{F}^{\pm 1}(v),$$

$$\mathcal{F}^{\pm 1}(\varrho \cdot v) = \pm i \operatorname{div} \mathcal{F}^{\pm 1}(v)$$

and therefore

$$* \quad \operatorname{div} \mathcal{F}^{-1}(\Phi) = i\mathcal{F}^{-1}(\varrho \cdot \Phi) \underset{\underset{\varrho \cdot \Phi = 0}{\uparrow}}{=} 0$$

$$\begin{aligned} * \quad \operatorname{rot} y &= -i \operatorname{rot} \mathcal{F}^{-1}(\Phi) \\ &= \mathcal{F}^{-1}(\varrho \times \Phi) \\ &= \mathcal{F}^{-1}(\varrho \times (\varrho/r^2 \times \mathcal{F}(\hat{u}))) \\ &= \mathcal{F}^{-1}(\mathcal{F}(\hat{u}) - (\varrho \cdot \mathcal{F}(\hat{u}))\varrho/r^2) \\ &= \hat{u} - i\mathcal{F}^{-1}(\mathcal{F}(\operatorname{div} \hat{u})\varrho/r^2) \\ &= \hat{u} \end{aligned}$$

$$\Rightarrow y \in H^1(\mathbb{R}^3) \cap D_0(\mathbb{R}^3), \quad \operatorname{rot} y = \hat{u}$$

in Ω_1: $y \in H^1(\Omega_1) \wedge \operatorname{rot} y = \hat{u} = u$

Finally:

$$T : D_0(\Omega_1) \to H^1(\mathbb{R}^3) \cap D_0(\mathbb{R}^3)$$
$$u \mapsto -i\,\mathcal{F}^{-1}\left(\varrho/r^2 \times \mathcal{F}(\hat{u}) \right)$$

suffices rot $Tu = u$ in Ω_1 and

$$
\begin{aligned}
|Tu|_{H^1(\mathbb{R}^3)}
&\leq |Tu|_{L^2(\mathbb{R}^3)} + |\nabla Tu|_{L^2(\mathbb{R}^3)} \\
&\leq |\varrho/r^2 \times \mathcal{F}(\hat{u})|_{L^2(\mathbb{R}^3)} \\
&\qquad + |\nabla \mathcal{F}^{-1}(\varrho/r^2 \times \mathcal{F}(\hat{u}))|_{L^2(\mathbb{R}_3)} \\
&\leq |\varrho/r^2 \times \mathcal{F}(\hat{u})|_{L^2(\mathbb{R}^3)} \\
&\qquad + |\varrho/r \times \mathcal{F}(\hat{u})|_{L^2(\mathbb{R}^3)} \\
&\leq |\varrho/r^2 \times \mathcal{F}(\hat{u})|_{L^2(B_1)} + |\hat{u}|_{L^2(\mathbb{R}^3)} \\
&\leq c\,|\mathcal{F}(\hat{u})|_{L^\infty(\mathbb{R}_3)} + |\hat{u}|_{L^2(\mathbb{R}^3)} \\
&\leq c\,|\hat{u}|_{L^1(\mathbb{R}^3)} + |\hat{u}|_{L^2(\mathbb{R}^3)} \\
&\leq c\,|\hat{u}|_{L^2(B)} \\
&\leq c\left(|u|_{L^2(\Omega_1)} + |\nabla v|_{L^2(B\setminus\overline{\Omega_1})} \right) \\
&\leq c\,|u|_{L^2(\Omega_1)}
\end{aligned}
$$

$\ulcorner$

$$\langle \nabla v, \nabla \varphi \rangle_{L^2(B\setminus\overline{\Omega_1})} = -\langle u, \nabla(\mathcal{E}\varphi) \rangle_{L^2(\Omega_1)}$$

$$\begin{aligned}
\Rightarrow |\nabla v|^2_{L^2(B\setminus\overline{\Omega})} &\leq |u|_{L^2(\Omega)} |\nabla(\mathcal{E}v)|_{L^2(\Omega)} \\
&< |u|_{L^2(\Omega)} |\mathcal{E}v|_{H^1(B)} \\
\text{continuous extension} \searrow \quad &\leq |u|_{L^2(\Omega)} |v|_{H^1(B\setminus\overline{\Omega})} \\
v \in H^1_\perp(B\setminus\overline{\Omega}) \searrow \quad &\leq |u|_{L^2(\Omega)} |\nabla v|_{L^2(B\setminus\overline{\Omega})} \quad \rfloor
\end{aligned}$$

$\boxed{\lightning}$

Lecture 8 28.11.16

Lemma n+1:

$\Omega \subset \mathbb{R}^3$ bd., str. Lipschitz and topological trivial (simply connected is enough). Then $\mathring{D}_0 = \operatorname{rot} \mathring{H}^1$ with continuous potential, i.e. $\forall u \in \mathring{D}_0 \ \exists \phi \in \mathring{H}^1$:

$\operatorname{rot} \phi = u \ \wedge \ |\phi|_{H_1} \leq c\,|u|_{L^2}$

Proof:

„$\supset$": $\operatorname{rot} \mathring{H}^1 \subset \operatorname{rot} \mathring{R} \subset \mathring{D}_0$

„$\subset$": Take $u \in \mathring{D}_0(\Omega)$. We define

$$\hat{u} := \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^3\setminus\overline{\Omega} \end{cases} \Rightarrow \hat{u} \in D_0(\mathbb{R}^3)$$

(cf. exercise 3)

As in the proof of lemma n we get $\psi \in H^1(\mathbb{R}^3)$, $\operatorname{rot}\psi = \hat{u}$ in $\mathbb{R}^3$,

$|\psi|_{H^1(\mathbb{R}^3)} \leq c\,|u|_{L^2(\Omega)}$

In $B\setminus\overline{\Omega}$ we have: $\operatorname{rot}\psi = 0$

Since $B\setminus\overline{\Omega_i}$ is simply connected, we have:

$$\psi = \nabla v,\ v \in H^1_1(B\setminus\overline{\Omega_i})$$

$$\Rightarrow v \in H^2(B\setminus\overline{\Omega_i})$$

$\psi \in H^1(\mathbb{R}^3)$ ↗

Now we extend v to $\hat{v} \in H^2(\mathbb{R}^3)$ (again with Calderon / Stein ...)

⌜ now: $B\setminus\overline{\Omega_i}$ is simply connected since Ω_i is simply connected. ⌟

We define: $\phi := \psi - \nabla\hat{v} \in H^1(\mathbb{R}^3)$

$\Rightarrow$ $\operatorname{rot}\phi = \operatorname{rot}\psi = \hat{u}$ in $\mathbb{R}^3$

and in $B\setminus\overline{\Omega_i}$:

$$\phi = \psi - \nabla\hat{v} = \psi - \nabla v = 0$$

$\Rightarrow$ $\phi \in H^1(B)$, $\phi = 0$ in $B\setminus\overline{\Omega_i}$

$\Rightarrow$ $\phi \in \mathring{H}^1(\Omega_i)$

Finally

$$S: \mathring{D}_0(\Omega_i) \to H^1(\mathbb{R}^3) \cap \mathring{H}^1(\Omega_i)$$
$$u \mapsto \psi - \nabla\hat{v}$$

suffices $\operatorname{rot} Su = u$ in Ω_i and

$$|Su|_{H^1(\mathbb{R}^3)}$$
$$\le c\left(|\psi|_{H^1(\mathbb{R}^3)} + |\hat{v}|_{H^2(\mathbb{R}^3)}\right)$$
$$\le c\left(|\psi|_{H^1(\mathbb{R}^3)} + |v|_{H^2(B\setminus\overline{\Omega_i})}\right)$$

$$\overset{v \in H^1_\perp(B\setminus\overline{\Omega})}{\leq} c \left(|\psi|_{H^1(\mathbb{R}^3)} + |\nabla v|_{H^1(B\setminus\overline{\Omega})} \right)$$

$$\leq c\, |\psi|_{H^1(\mathbb{R}^3)} \leq c \cdot |u|_{L^2(\Omega)} .$$

$\square$

Corollary:

$\Omega \subset \mathbb{R}^3$ bd., strong lipschitz and topological trivial. Then:

$$\mathcal{D}_\varepsilon = \mathring{R}_0 \cap \varepsilon^{-1} D_0 = \{0\}$$
$$\mathcal{N}_\mu = R_0 \cap \mu^{-1} \mathring{D}_0 = \{0\}$$

Furthermore:

$$\varepsilon^{-1} D_0 = \varepsilon^{-1} \operatorname{rot} R \oplus_{L^2_\varepsilon} \mathcal{D}_\varepsilon = \varepsilon^{-1} \operatorname{rot} H^1$$
$$\mu^{-1} \mathring{D}_0 = \mu^{-1} \operatorname{rot} \mathring{R} \oplus_{L^2_\mu} \mathcal{N}_\mu = \mu^{-1} \operatorname{rot} \mathring{H}^1$$

and

$$L^2_\varepsilon = \nabla \mathring{H}^1 \oplus_{L^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} H^1$$
$$= \mathring{R}_0 \oplus_{L^2_\varepsilon} \varepsilon^{-1} D_0$$
$$L^2_\mu = \nabla H^1 \oplus_{L^2_\mu} \mu^{-1} \operatorname{rot} \mathring{R}$$
$$= R_0 \oplus_{L^2_\mu} \mu^{-1} \mathring{D}_0$$

Remark:

$\Omega \subset \mathbb{R}^3$ topological trivial. Then $\mathcal{D}_\varepsilon = \mathcal{N}_\mu = \{0\}$ can be shown by elementary calculations.

Lemma:

$\Omega_1 \subset \mathbb{R}^3$ bd., str. Lipschitz and topological trivial. Then we have:

$$\mathring{R}(\Omega_1) \cap \varepsilon^{-1} D(\Omega_1) \hookrightarrow\hookrightarrow L^2(\Omega_1)$$
$$R(\Omega_1) \cap \varepsilon^{-1} \mathring{D}(\Omega_1) \hookrightarrow\hookrightarrow L^2(\Omega_1)$$

Again with $\varepsilon : \Omega_1 \to \mathbb{R}^{3\times 3}$ uniformly positive definite, symmetric and with L^∞-coefficients.

Proof:

Observe that:

$$L^2_\varepsilon = \nabla \mathring{H}^1 \oplus_{L^2_\varepsilon} \varepsilon^{-1} D_0$$
$$\Rightarrow \mathring{R} \cap \varepsilon^{-1} D = (\nabla \mathring{H}^1 \cap \varepsilon^{-1} D) \oplus_{L^2_\varepsilon} (\mathring{R} \cap \varepsilon^{-1} D_0)$$

⌜ $\nabla \mathring{H}^1$ is closed by Rellich's selection theorem. ⌟

Let $(u_n)_n \subset \mathring{R} \cap \varepsilon^{-1} D$ bd. in $R \cap \varepsilon^{-1} D$

$$\Rightarrow u_n = u_{\nabla,n} + u_{0,n} \in (\nabla \mathring{H}^1 \cap \varepsilon^{-1} D) \oplus_{L^2_\varepsilon} (\mathring{R} \cap \varepsilon^{-1} D_0),$$
$$|u_{\nabla,n}|^2_{L^2_\varepsilon} + |u_{0,n}|^2_{L^2_\varepsilon} = |u_n|^2_{L^2_\varepsilon},$$
$$\operatorname{rot} u_n = \operatorname{rot} u_{0,n} \wedge \operatorname{div} u_n = \operatorname{div} u_{\nabla,n}$$

Therefore $(u_{0,n})_n \subset \mathring{R} \cap \varepsilon^{-1} D$ is bounded in $R \cap \varepsilon^{-1} D$ and also $(u_{\nabla,n})_n \subset \nabla \mathring{H}^1 \cap \varepsilon^{-1} D$ is bounded in $R \cap \varepsilon^{-1} D$. Then $u_{\nabla,n} = \nabla v_n$ for $v_n \in \mathring{H}^1$ and we have:

$$|v_n|_{L^2_\varepsilon} \leq c\, |\nabla v_n|_{L^2_\varepsilon}$$
$$= c\, |u_{\nabla,n}|_{L^2_\varepsilon} \leq c < \infty$$

Therefore $(v_n)_n$ is bounded in H^1 and by Rellich's selection theorem we can choose a subsequence $(v_{\pi(n)})_n$ converging in L^2. Then by

$$u_{\nabla,n,m} := u_{\nabla,n} - u_{\nabla,m}$$
$$u_{0,n,m} := u_{0,n} - u_{0,m}$$

and

$$|u_{\nabla,n,m}|^2_{L^2_\varepsilon} = \langle u_{\nabla,n,m} , \nabla v_{n,m} \rangle_{L^2_\varepsilon}$$
$$= - \langle \operatorname{div} \varepsilon u_{\nabla,n,m} , v_{n,m} \rangle_{L^2}$$
$$= - \langle \operatorname{div} \varepsilon u_{n,m} , v_{n,m} \rangle_{L^2}$$
$$\leq c\, |v_{n,m}|_{L^2}$$

we get that $(u_{\nabla,\pi(n)})_n$ is a cauchy-sequence in L^2 and therefore converging. Additionally we get from Lemma n:

$$\varepsilon u_{0,n} = \operatorname{rot} \phi_n ,\ \phi_n \in H^1 \text{ with}$$
$$|\phi_n|_{H_1} \leq c\, |u_{0,n}|_{L^2_\varepsilon} < c$$

By Rellich's selection theorem we again get a converging subsequence

$(\phi_{\pi_2(n)})_n$ of $(\phi_{\pi(n)})_n$. Then:

$$\begin{aligned} |u_{0,\pi_2(n,m)}|^2_{L^2_\varepsilon} &= \langle u_{0,\pi_2(n,m)}, \varepsilon^{-1} \operatorname{rot} \phi_{\pi_2(n,m)} \rangle_{L^2_\varepsilon} \\ &= \langle \operatorname{rot} u_{0,\pi_2(n,m)}, \phi_{\pi_2(n,m)} \rangle_{L^2} \\ &= \langle \operatorname{rot} u_{\pi_2(n,m)}, \phi_{\pi_2(n,m)} \rangle_{L^2} \\ &\leq c\, |\phi_{\pi_2(n,m)}| \end{aligned}$$

and therefore $(u_{0,\pi_2(n)})_n$ is a cauchy-sequence in L^2. In total $(u_{\pi_2(n)})_n$ is a cauchysequence in L^2 and therefore converging in L^2.

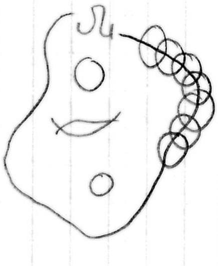
$\boxed{k}$

Remark:

The second assertion follows analogously with Lemma n+1.

Proof of the theorem:

Take $(E_n)_n \subset \mathring{R} \cap \varepsilon^{-1} D$ bounded in $R \cap \varepsilon^{-1} D$.



Now $\forall x \in \overline{\Omega}\ \exists\ U_x := B_{r_x}(x)$ open with $x \in U_x$ and $\Omega_x := U_x \cap \Omega$ is topological trivial.

$$\Rightarrow \overline{\Omega} \subset \bigcup_{x \in \overline{\Omega}} \Omega_x \underset{\substack{\uparrow \\ \overline{\Omega} \text{ is compact}}}{\Rightarrow} \Omega \subset \bigcup_{i=1}^{k} \Omega_{x_i}$$

Now let $\varphi_i \in \mathring{C}^\infty(U_{x_i})$ with

$$\sum_{i=1}^{k} \varphi_i = 1 \text{ in } \overline{\Omega_1}. \Rightarrow E_n = \sum_{i=1}^{k} \varphi_i E_n$$

Then $(\varphi_i E_n)_n \subset \overset{\circ}{R}(\Omega_{1x_i}) \cap \varepsilon^{-1} D(\Omega_{1x_i})$ is bounded in $L^2(\Omega_{1x_i})$. Furthermore

$$\operatorname{rot}(\varphi_i E_n) = \varphi_i \operatorname{rot} E_n - \nabla\varphi_i \times E_n$$
$$\operatorname{div}(\varphi_i \varepsilon E_n) = \varphi_i \operatorname{div} \varepsilon E_n + \nabla\varphi_i \cdot \varepsilon E_n$$

$\Rightarrow$ $(\varphi_i E_n)_n$ is bounded in $R(\Omega_{1x_i}) \cap \varepsilon^{-1} D(\Omega_{1x_i})$

$\ulcorner$ since: $\operatorname{supp}(\varphi_i E_n) \subset \operatorname{supp}(\varphi_i) \subset U_{x_i}$ $\lrcorner$

From Lemma above we can choose a subsequence $(\varphi_1 E_{\pi_1(n)})_n$ of $(\varphi_1 E_n)_n$ converging in $L^2(\Omega_{1x_1})$. Then $(\varphi_2 E_{\pi_1(n)})_n$ is also bounded in $R(\Omega_{1x_2}) \cap \varepsilon^{-1} D(\Omega_{1x_2})$ such that we can extract a subsequence $(\varphi_2 E_{\pi_2(n)})_n$ converging in $L^2(\Omega_{1x_2})$. Continuing this procedure until $i = k$ we end up with a subsequence $(E_{\pi_k(n)})_n$ such that $(\varphi_i E_{\pi_k(n)})_n$ is converging in $L^2(\Omega_{1x_i})$ for all $i \in \{1, \ldots, k\}$.

$\Rightarrow$ $(E_{\pi_k(n)})_n$ is converging in $L^2(\Omega_1)$.

The assertion for $R \cap \varepsilon^{-1} \overset{\circ}{D}$ follows analogously.

$\square$