Maxwell - School, Linz 2016

Lecture 1, 10.10.16

Ω (open, connected) $\subset \mathbb{R}^3$,
$I \subset \mathbb{R}$ (time interval)

$E: I \times \Omega \to \mathbb{C}^3$	electric field
$D: I \times \Omega \to \mathbb{C}^3$	electric displacement field
$B: I \times \Omega \to \mathbb{C}^3$	magnetic field
$H: I \times \Omega \to \mathbb{C}^3$	magnetizing field
$\varepsilon: I \times \Omega \to \mathbb{R}^{3\times 3}$	permittivity
$\mu: I \times \Omega \to \mathbb{R}^{3\times 3}$	permeability
$\rho: I \times \Omega \to \mathbb{C}$	charge density
$P: I \times \Omega \to \mathbb{C}$	polarization
$M: I \times \Omega \to \mathbb{C}^3$	magnetization
$\sigma: I \times \Omega \to \mathbb{R}^{3\times 3}$	conductivity
$J: I \times \Omega \to \mathbb{C}^3$	current density

$\partial_t D - \operatorname{rot} H = -J$ in Ω (Ampere's law)
$\partial_t B + \operatorname{rot} E = 0$ in Ω (Faraday's law)
$\operatorname{div} D = \rho$ in Ω
$\operatorname{div} B = 0$ in Ω (Gauss law)

$D = \varepsilon E + P$
$B = \mu H + M$
$J = \sigma E - F$

Assumptions:
$P = M = \sigma = 0$

$\rightsquigarrow$ end up with:

$$\partial_t \varepsilon E - \operatorname{rot} H = F$$
$$\partial_t \mu H + \operatorname{rot} E = G \quad (\text{usually } G = 0)$$
$$\operatorname{div} \varepsilon E = \rho$$
$$\operatorname{div} \mu H = \tau \quad (\text{usually } \tau = 0)$$

(E, H) Fields of interest

$\rightsquigarrow$ initial conditions:

$$E(0) = E_0 \quad \& \quad H(0) = H_0$$

$\rightsquigarrow$ boundary conditions:

later on ...

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some identities:

$$\varphi, \psi \in C^\infty(\Omega, \mathbb{C}) \; ; \; \phi \in C^\infty(\Omega, \mathbb{C}^3)$$

$$\rightsquigarrow \nabla \varphi = \begin{bmatrix} \partial_1 \varphi \\ \partial_2 \varphi \\ \partial_3 \varphi \end{bmatrix}, \quad \operatorname{rot} \phi = \begin{bmatrix} \partial_2 \phi_3 - \partial_3 \phi_2 \\ \partial_3 \phi_1 - \partial_1 \phi_3 \\ \partial_1 \phi_2 - \partial_2 \phi_1 \end{bmatrix}$$

$$\operatorname{div} \phi = \partial_1 \phi_1 + \partial_2 \phi_2 + \partial_3 \phi_3$$

$\rightsquigarrow$ $\operatorname{rot} \nabla \varphi = 0$ (Schwarz lemma)

$$\operatorname{div} \operatorname{rot} \phi = 0$$
$$\Delta \varphi = \operatorname{div} \nabla \varphi = \partial_1^2 \varphi + \partial_2^2 \varphi + \partial_3^2 \varphi$$

$$\rightsquigarrow \operatorname{rot} \operatorname{rot} \phi - \nabla \operatorname{div} \phi = -\Delta \phi = -\begin{bmatrix} \Delta \phi_1 \\ \Delta \phi_2 \\ \Delta \phi_3 \end{bmatrix}$$

$$\rightsquigarrow \nabla(\varphi \psi) = \psi \nabla \varphi + \varphi \nabla \psi$$
$$\operatorname{rot}(\varphi \phi) = \varphi \operatorname{rot} \phi + \nabla \varphi \times \phi$$
$$\operatorname{div}(\varphi \phi) = \varphi \operatorname{div} \phi + \nabla \varphi \cdot \phi$$

$\Rightarrow$ commutators are related to algebraic operations.

Suppose ε, μ are time independent:

$$\begin{aligned}\partial_t^2 \varepsilon E &= \partial_t (\partial_t \varepsilon E) \\ &= \partial_t (\operatorname{rot} H) + \partial_t F \\ &= \operatorname{rot} (\partial_t H) + \partial_t F \\ &= \operatorname{rot} (-\mu^{-1} \operatorname{rot} E + \mu^{-1} G) + \partial_t F\end{aligned}$$

$\Rightarrow$ $\partial_t^2 \varepsilon E + \operatorname{rot} \mu^{-1} \operatorname{rot} E = \tilde{F}$
with $\tilde{F} := \operatorname{rot} \mu^{-1} G + \partial_t F$

$\varepsilon = \mu = id$: (wave equation)

$$\Rightarrow \partial_t^2 E + \operatorname{rot} \operatorname{rot} E = \tilde{F}$$
$$\begin{aligned}\Rightarrow \partial_t^2 E - \Delta E &= \tilde{F} + \nabla \operatorname{div} E \\ &= \tilde{F} + \nabla \rho\end{aligned}$$

$\rightsquigarrow$ similar with H: $\partial_t^2 H - \Delta H = \tilde{G}$

Applying div to equations we get:

$$\partial_t \operatorname{div} \varepsilon E = \operatorname{div} F$$
$$\partial_t \operatorname{div} \mu H = \operatorname{div} G$$

$$\Rightarrow \rho = \operatorname{div} \varepsilon E = \operatorname{div} \varepsilon E_0 + \int_0^t \operatorname{div} F$$

$$\tau = \operatorname{div} \mu H = \operatorname{div} \mu H_0 + \int_0^t \operatorname{div} G$$

$\Rightarrow$ second two equations are already included.

Abstract framework, spectral - theory

$$\left(\partial_t \underbrace{\begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}}_{=:\Lambda} + \underbrace{\begin{bmatrix} 0 & -\mathrm{rot} \\ \mathrm{rot} & 0 \end{bmatrix}}_{=:A} \right) \underbrace{\begin{bmatrix} E \\ H \end{bmatrix}}_{=:x} = \underbrace{\begin{bmatrix} F \\ G \end{bmatrix}}_{=:\widehat{F}}$$

$$(\partial_t \Lambda + A)\, x = \widehat{F}$$
$$\Leftrightarrow (\partial_t + \Lambda^{-1} A)\, x = \Lambda^{-1} \widehat{F} =: f$$

$$\Rightarrow \partial_t^2 x = \partial_t (f - \Lambda^{-1} A x)$$
$$= - \Lambda^{-1} A\, \partial_t x + \partial_t f$$
$$= \Lambda^{-1} A \Lambda^{-1} A + \partial_t f + \Lambda^{-1} A f$$

$$\Rightarrow (\partial_t^2 - \Lambda^{-1} A \Lambda^{-1} A)\, x = \partial_t f + \Lambda^{-1} A f$$
(generalized wave equation)

$\phi, \psi \in C^\infty(\Omega, \mathbb{C}^3)$

$$\Rightarrow \operatorname{div}(\phi \times \psi)$$
$$= \operatorname{div} \begin{bmatrix} \phi_2 \psi_3 - \phi_3 \psi_2 \\ \phi_3 \psi_1 - \phi_1 \psi_3 \\ \phi_1 \psi_2 - \phi_2 \psi_1 \end{bmatrix}$$
$$= (\partial_1 \phi_2) \psi_3 + \phi_2 (\partial_1 \psi_3) - (\partial_1 \phi_3) \psi_2$$
$$- \phi_3 (\partial_1 \psi_2) + (\partial_2 \phi_3) \psi_1 + \phi_3 (\partial_2 \psi_1)$$
$$- (\partial_2 \phi_1) \psi_3 - \phi_1 (\partial_2 \psi_3) + \dots$$
$$= \psi \cdot \mathrm{rot}\, \phi - \phi \cdot \mathrm{rot}\, \psi$$

$\Rightarrow \forall \phi, \psi \in C^\infty(\overline{\Omega}, \mathbb{C}^3) := C^\infty(\mathbb{R}^3, \mathbb{C}^3)|_\Omega$

$$\int_\Omega \operatorname{rot}\phi \cdot \psi - \phi \cdot \operatorname{rot}\psi \, dx$$

$$= \int_\Omega \operatorname{div}(\phi \times \psi) \, dx$$

$$= \int_{\partial\Omega} n \cdot (\phi \times \psi) \, dS$$

$$= \int_{\partial\Omega} (n \times \phi) \cdot \psi \, dS = \int_{\partial\Omega} (n \times \phi)(n \times \psi \times n) \, dS$$

since $\psi = (\psi \cdot n) \cdot n$

Notation:

$\langle \phi, \psi \rangle_\Omega := \int_\Omega \phi \cdot \overline{\psi} \, dx$ (L^2 inner product)

$\langle \phi, \psi \rangle_\Gamma := \int_\Gamma \phi \cdot \overline{\psi} \, dS$ (on boundary)

$\rightsquigarrow \langle \operatorname{rot}\phi, \psi \rangle_\Omega - \langle \phi, \operatorname{rot}\psi \rangle_\Omega$
$= \langle n \times \phi, \psi \rangle_\Gamma$

$\operatorname{div}(\varphi \phi) = \varphi \operatorname{div}\phi + \nabla\varphi \cdot \phi$

$\rightsquigarrow \langle \nabla\varphi, \phi \rangle_\Omega + \langle \varphi, \operatorname{div}\phi \rangle_\Omega = \langle \varphi, n \cdot \phi \rangle_\Gamma$

$\phi, \psi \in C^\infty(\overline{\Omega}, \mathbb{C}^3)$, $\psi \in \mathring{C}^\infty(\Omega, \mathbb{C}^3)$

$$< \begin{bmatrix} E \\ H \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix} >_D := < E, \phi >_D + < H, \psi >_D$$

$$< A \begin{bmatrix} E \\ H \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix} >_\Omega$$

$$= < -\operatorname{rot} H, \phi >_\Omega + < \operatorname{rot} E, \psi >_\Omega$$

$$= < H, -\operatorname{rot} \phi >_\Omega + < E, \operatorname{rot} \psi >_\Omega$$

$$= - < \begin{bmatrix} E \\ H \end{bmatrix}, A \begin{bmatrix} \phi \\ \psi \end{bmatrix} >$$

$\Rightarrow$ A is skew-symmetric operator

$\Rightarrow$ $\Delta^{-1} A$ is skew-symmetric (with respect to <u>weighted</u> inner product)

$\Rightarrow$ $\boxed{i \Delta^{-1} A}$ is symmetric and a good candidate for a „self-adjoint"-operator.

rewrite:

$$\Rightarrow \left(\partial_t - i \underbrace{i \Delta^{-1} A}_{=: M} \right) x = f$$

$=: M$ Maxwell-operator is self-adjoint.

spectral theorem

$$\Rightarrow x(t) = e^{itM} x_0 + \int_0^t e^{-i(t-s)M} f(s) \, ds$$

boundary conditions:

$\varphi, \phi \in C^\infty(\bar{\Omega})$:

$\langle \operatorname{rot} \phi, \nabla \varphi \rangle_\Omega$

$\overset{①}{=} - \langle \underbrace{\operatorname{div} \operatorname{rot} \phi}_{=0}, \varphi \rangle_\Omega + \int_\Gamma \bar{\varphi}\, n \cdot \operatorname{rot} \phi \, dS$

$\overset{②}{=} \langle \phi, \underbrace{\operatorname{rot} \nabla \varphi}_{=0} \rangle + \int_\Gamma (n \times \phi) \cdot \nabla \bar{\varphi} \, dS$

$$\Rightarrow \int_\Gamma \bar{\varphi} (n \cdot \operatorname{rot} \phi) \, dS = \int_\Gamma \nabla \bar{\varphi} \cdot (n \times \phi) \, dS$$

$$= \int_\Gamma -(n \times \nabla \bar{\varphi}) \cdot \phi \, dS$$

Therefore:

* $\varphi = 0$ at Γ $\Rightarrow \int_\Gamma (n \times \nabla \bar{\varphi}) \cdot \phi \, dS = 0$

since $C^\infty(\bar{\Omega})|_\Gamma$ dense in $L^2(\Gamma)$ $\Rightarrow$ $n \times \nabla \varphi = 0$ at Γ

* $n \times \phi = 0$ at Γ $\Rightarrow$ $n \cdot \operatorname{rot} \phi = 0$ at Γ

$\rightsquigarrow$ so: apply $n \times E|_\Gamma = 0$

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time-harmonic ansatz:

$$E(t,x) = e^{i\omega t} \tilde{E}(x), \quad H(t,x) = e^{i\omega t} \hat{H}(x)$$

$\Rightarrow \quad (\partial_t - i\mathcal{M})x = f \Leftrightarrow i(\omega - \mathcal{M})\hat{x} = \hat{f}$

$\Leftrightarrow \quad (\mathcal{M} - \omega)\hat{x} = -i\hat{f}$

$\hookrightarrow$ apply Fredholm alternative

$$-i\varepsilon^{-1}\operatorname{rot} H - \omega E = F \quad \text{in } \Omega$$
$$i\mu^{-1}\operatorname{rot} E - \omega H = G \quad \text{in } \Omega$$

with homogeneous boundary condition:

$$n \times E_{|\Gamma} = 0$$

① $\underline{\omega \neq 0}$:

$$\rho = \operatorname{div} \varepsilon E = 1/\omega \operatorname{div} \varepsilon F$$
$$\tau = \operatorname{div} \mu H = 1/\omega \operatorname{div} \mu G$$

$\rightsquigarrow$ ρ, τ already given by initial data

② $\underline{\omega = 0}$: (static case)

$$\operatorname{rot} E = F \qquad n \times E = 0 \quad \text{on } \Gamma$$
$$\operatorname{rot} H = G$$

now we have to add:

$$\operatorname{div} \varepsilon E = \rho \qquad n \cdot \mu H = 0 \quad \text{on } \Gamma$$
$$\operatorname{div} \mu H = \tau$$

after proj.: $n\mu G = 0$

observe:

$n \times E = 0 \Rightarrow n \cdot \operatorname{rot} E = 0$

$\Rightarrow n(\omega\mu H + \mu G) = 0$ on Γ

kernels: $\mathcal{H}_D$ (for E), $\mathcal{H}_N$ (for H)
* uniqueness by: $E \perp \mathcal{H}_D$ or $\pi_{\mathcal{H}_D} E = K$ for K given.

Lecture 2, 10.10.16

∇, rot, div ; $\operatorname{rot} \nabla = 0$, $\operatorname{div} \operatorname{rot} = 0$

$\Rightarrow R(\nabla) \subset N(\operatorname{rot})$, $R(\operatorname{rot}) \subset N(\operatorname{div})$

$$\mathbb{R} \xrightarrow{\iota_{\mathbb{R}}} C^\infty(\bar{\Omega}) \xrightarrow{\nabla} C^\infty(\bar{\Omega}) \xrightarrow{\operatorname{rot}} C^\infty(\bar{\Omega}) \xrightarrow{\operatorname{div}} C^\infty(\bar{\Omega}) \xrightarrow{0}$$

sequence / complex

Sobolev spaces:

* $\underbrace{L^2(\Omega)}_{\text{skipped if clear}}$ either scalar/vector/tensor-valued
* $u \in L^2(\Omega)$, $\partial_m u \in L^2(\Omega)$
  $\Leftrightarrow \exists\, v_m \in L^2\ \forall \varphi \in \mathring{C}^\infty: \langle u, \partial_m \varphi \rangle_\Omega = -\langle v_m, \varphi \rangle_\Omega$
  (if v_m exists, it is uniquely defined)
  $\rightsquigarrow$ set $\partial_m u := v_m$
* $H^1 := \{ u \in L^2 : \partial_m u \in L^2\ \forall\, m = 1,2,3 \}$
  $H^l := \{ u \in L^2 : \forall\, |\alpha| \leq l\ \ \partial^\alpha u \in L^2 \}$
* $R := \{ E \in L^2 : \operatorname{rot} E \in L^2 \}$

  $\exists\, F \in L^2\ \forall \phi \in \mathring{C}^\infty: \langle E, \operatorname{rot} \phi \rangle_\Omega = \langle F, \phi \rangle_\Omega$
* $D := \{ E \in L^2 : \operatorname{div} E \in L^2 \}$

  $\exists\, F \in L^2\ \forall \phi \in \mathring{C}^\infty: \langle E, \nabla \phi \rangle_\Omega = -\langle F, \phi \rangle_\Omega$

$$\mathbb{R} \xrightarrow{\iota_{\mathbb{R}}} H^1 \xrightarrow{\nabla} R \xrightarrow{\operatorname{rot}} D \xrightarrow{\operatorname{div}} L^2 \xrightarrow{0}$$

$$\langle u,v\rangle_{L^2} := \langle u,v\rangle_{\Omega}$$
$$\langle u,v\rangle_{H^1} := \langle u,v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{L^2}$$
$$\langle u,v\rangle_{H^l} := \sum_{|\alpha|\le l} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2}$$
$$\langle u,v\rangle_{R} := \langle u,v\rangle_{L^2} + \langle \operatorname{rot} u, \operatorname{rot} v\rangle_{L^2}$$
$$\langle u,v\rangle_{D} := \langle u,v\rangle_{L^2} + \langle \operatorname{div} u, \operatorname{div} v\rangle_{L^2}$$

$\leadsto$ spaces are complete.

$(u_n)_n$ cauchy-sequence in R

$\Rightarrow$ $(u_n)_n$, $(\operatorname{rot} u_n)_n$ are cauchy-sequences in L^2

L^2 complete $\Rightarrow$ $u_n \to u \in L^2$

$\operatorname{rot} u_n \to v \in L^2$

$\Rightarrow \forall \phi \in \mathring{C}^\infty$:

$$\begin{aligned}\langle u, \operatorname{rot}\phi\rangle_{L^2} &= \lim_{n\to\infty} \langle u_n, \operatorname{rot}\phi\rangle_{L^2} \\ &= \lim_{n\to\infty} \langle \operatorname{rot} u_n, \phi\rangle_{L^2} \\ &= \langle v, \phi\rangle_{L^2}\end{aligned}$$

$\Rightarrow u \in R$, $\operatorname{rot} u = v$

$\Rightarrow R$ complete. $\boxtimes$

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* $\mathring{H}^1 := \overline{\mathring{C}^\infty}^{H^1} \ni u : \langle \nabla u, \phi\rangle_{L^2} = -\langle u, \operatorname{div}\phi\rangle_{L^2}$ for all $\phi \in D$.
* $\mathring{R} := \overline{\mathring{C}^\infty}^{R} \ni u : \langle \operatorname{rot} u, \phi\rangle_{L^2} = \langle u, \operatorname{rot}\phi\rangle_{L^2}$ for all $\phi \in R$

$* \quad \mathring{D} := \overline{\mathring{C}^\infty}^{D} \ni u : \langle \operatorname{div} u, \phi \rangle_{L^2} = -\langle u, \nabla \phi \rangle_{L^2}$

for all $\phi \in H^1$

<u>Proof</u>:

$u \in \mathring{C}^\infty, \phi \in C^\infty(\bar{\Omega}) : \langle \nabla u, \phi \rangle_{L^2} = -\langle u, \operatorname{div} \phi \rangle_{L^2}$

now:

$u \in \mathring{H}^1, \phi \in D$

$\Rightarrow \exists (u_n)_n \subset \mathring{C}^\infty, u_n \to u$ in H^1

$\Rightarrow \langle \nabla u, \phi \rangle_{L^2}$

$= \lim_{n \to \infty} \langle \nabla u_n, \phi \rangle_{L^2}$

by defin. of weak ∇ $\rightsquigarrow$ $= \lim_{n \to \infty} -\langle u_n, \operatorname{div} \phi \rangle_{L^2}$

$= -\langle u, \operatorname{div} \phi \rangle_{L^2}$

$\square$

($u \in H^1, \phi \in \mathring{D}$ also possible since you only need $\mathring{C}^\infty$ dense in the space)

<u>so</u>:

$u \in \mathring{H}^1 \rightsquigarrow$ „ $u_{|\Gamma} = 0$ "

$u \in \mathring{H}^1 \cap C^\infty(\bar{\Omega})$

$\Rightarrow \langle u, n \cdot \phi \rangle_{L^2(\Gamma)} = 0 \quad \forall \phi \in C^\infty(\bar{\Omega})$

$n \cdot C^\infty(\bar{\Omega})_{|\Gamma}$ dense in $L^2(\Gamma)$

$u \in \mathring{R} \rightsquigarrow$ „ $n \times u_{|\Gamma} = 0$ "

$u \in \mathring{D} \rightsquigarrow$ „ $n \cdot u_{|\Gamma} = 0$ "

Remarks:

* so far: no smoothness needed
* also holds in unbounded case
* can say $(u - u_0) \in \mathring{R}$ to „speak" about traces.

$$\mathring{H}^1 \xrightarrow{\nabla} \mathring{R} \xrightarrow{\mathrm{rot}} \mathring{D} \xrightarrow{\mathrm{div}} L^2 \xrightarrow{0}$$

<u>Lemma:</u>

(i) $\overline{\nabla H^1} \subset R_0$ (i') $\overline{\nabla \mathring{H}^1} \subset \mathring{R}_0$

(ii) $\overline{\mathrm{rot}\, R} \subset D_0$ (ii') $\overline{\mathrm{rot}\, \mathring{R}} \subset \mathring{D}_0$

<u>Proof of (ii):</u>

Enough to show: $\mathrm{rot}\, R \subset D_0$

(because D_0 is closed - being a kernel)

$u \in R$, $\varphi \in \mathring{C}^\infty$:

$$\langle \mathrm{rot}\, u, \nabla \varphi \rangle_{L^2} = \langle u, \mathrm{rot}\, \nabla \varphi \rangle = 0$$

$$\Rightarrow \mathrm{rot}\, u \in D \wedge \mathrm{div}\, \mathrm{rot}\, u = 0$$

<u>Proof of (ii'):</u>

$u \in \mathring{R} \Leftrightarrow \exists (u_n)_n \subset \mathring{C}^\infty,\ u_n \to u$ in R

$\Rightarrow u_n \to u$ in L^2

$\quad \mathrm{rot}\, u_n \to \mathrm{rot}\, u$ in L^2

$\Rightarrow \mathrm{rot}\, u_n \to \mathrm{rot}\, u$ in D

$\Rightarrow \mathrm{rot}\, u \in \mathring{D}_0$

$\boxed{\not\downarrow}$

$(\mathring{H}^1 = \mathbb{R}$ (or $\mathbb{C}$), $\mathring{H}^1_0 = \{0\})$

Lemma:

$\varphi \in C^\infty(\overline{\Omega}, \mathbb{C})$

(i) $u \in \overset{(\circ)}{H}{}^1 \Rightarrow \varphi u \in \overset{(\circ)}{H}{}^1$,

$\nabla(\varphi u) = u \nabla \varphi + \varphi \nabla u$

(ii) $u \in \overset{(\circ)}{R} \Rightarrow \varphi u \in \overset{(\circ)}{R}$,

$\operatorname{rot}(\varphi u) = \varphi \operatorname{rot} u + \nabla \varphi \times u$

(iii) $u \in \overset{(\circ)}{D} \Rightarrow \varphi u \in \overset{(\circ)}{D}$,

$\operatorname{div}(\varphi u) = \varphi \operatorname{div} u + \nabla \varphi \cdot u$

(property of first order operators)

Proof:

* $u \in R, \varphi \in C^\infty(\overline{\Omega})$:

$\Rightarrow \forall \psi \in \overset{\circ}{C}{}^\infty(\Omega)$:

$$\begin{aligned}
&\langle \varphi u, \operatorname{rot} \psi \rangle_{L^2} \\
&= \langle u, \varphi \operatorname{rot} \psi \rangle_{L^2} \\
&= \langle u, \operatorname{rot}(\varphi \psi) \rangle_{L^2} - \langle u, \nabla \varphi \times \psi \rangle_{L^2} \\
&= \langle \operatorname{rot} u, \varphi \psi \rangle_{L^2} + \langle \nabla \varphi \times u, \psi \rangle_{L^2} \\
&= \langle \varphi \operatorname{rot} u, \psi \rangle_{L^2} + \langle \nabla \varphi \times u, \psi \rangle_{L^2}
\end{aligned}$$

* $u \in \overset{\circ}{R} \Rightarrow \exists (u_n)_n \subset \overset{\circ}{C}{}^\infty$, $u_n \to u$ in R

$\Rightarrow \varphi u_n \to \varphi u$ in L^2

$$\operatorname{rot}(\varphi u_n) = \varphi \operatorname{rot} u_n + \nabla \varphi \times u_n \;\Big\downarrow \text{ in } L^2 \; \varphi \operatorname{rot} u + \nabla \varphi \times u$$

$\Rightarrow \varphi u_n \to \varphi u$ in R $\Rightarrow \varphi u \in \overset{\circ}{R}$

Definition:

A sequence is called closed, iff all ranges are closed.

$$\left.\begin{array}{l} \nabla : \mathring{C}^\infty \subset L^2 \to L^2 \\ \mathrm{rot} : \mathring{C}^\infty \subset L^2 \to L^2 \\ \mathrm{div} : \mathring{C}^\infty \subset L^2 \to L^2 \end{array}\right\}$$ linear operators, densly defined, poss. unbounded

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Functional Analysis Toolbox

$A : D(A) \subset H_1 \to H_2$ linear

$D(A)$: domain of definition

H_1, H_2: Hilbert spaces

$G(A) := \{ (x, Ax) \in H_1 \times H_2 : x \in D(A) \}$

"graph of A"

* A is densly defined $:\Leftrightarrow \overline{D(A)}^{H_1} = H_1$
* A is closed $:\Leftrightarrow G(A)$ closed

$\Leftrightarrow \forall (x_n)_n \subset D(A),\ x_n \to x$ in H_1,

$Ax_n \to y$ in H_2

$\Rightarrow x \in D(A) \wedge Ax = y$

(i.e. ∇, rot, div as above are not closed)

* A is closable $:\Leftrightarrow \exists B : \overline{G(A)} = G(B)$,

$B = \bar{A}$

$\Leftrightarrow \forall (x_n)_n \subset D(A),\ x_n \to 0$ in H_1,

$Ax_n \to y$ in $H_2 \Rightarrow y = 0$.

Definition:

$A : D(A) \subset H_1 \to H_2$ densly defined, linear
$\Rightarrow \exists A^* : D(A^*) \subset H_2 \to H_1$ „adjoint"

$y \in D(A^*) \wedge A^* y = f \in H_1$
$\Leftrightarrow y \in H_2 \wedge \exists f \in H_1 \; \forall \varphi \in D(A):$
$\langle A\varphi, y \rangle_{H_2} = \langle \varphi, f \rangle_{H_1}$, $A^* y := f$

$\leadsto A^*$ is unique, since
$\langle \varphi, f_1 - f_2 \rangle = 0 \quad \forall \varphi \in D(A)$
$\Rightarrow f_1 - f_2 = 0$ ($D(A)$ is dense in H_1)

$\overline{A} : D(\overline{A}) \subset H_1 \to H_2$, $D(\overline{A}) = \overline{D(A)}^{G(A)}$

$\|.\|_{G(A)} = (|.|^2 + |A.|^2)^{1/2}$ graph norm

$\leadsto A$ densly defined, closed: $A = \overline{A} = A^{**}$
then: (A, A^*) is called dual pair

General assumptions:

A linear, densly defined, closed
($\Rightarrow A^*$ closed , $A^* = \overline{A}^* = \overline{A^*}$)

Lecture 3, 10.10.16

projection theorem

$\Rightarrow H_2 = \overline{R(A)} \oplus N(A^*)$, $H_1 = \overline{R(A^*)} \oplus N(A)$

$A^* : D(A^*) \subset H_2 \to H_1$ densly defined, closed and (A, A^*) dual pair.

reduced operators: (project kernels away)

$$\mathcal{A} : \underbrace{D(A) \cap \overline{R(A^*)}}_{=: D(\mathcal{A})} \subset \overline{R(A^*)} \to R(A)$$

$$\mathcal{A}^* : \underbrace{D(A^*) \cap \overline{R(A)}}_{=: D(\mathcal{A}^*)} \subset \overline{R(A)} \to R(A^*)$$

* $(\mathcal{A}, \mathcal{A}^*)$ dual pair
* $\mathcal{A}, \mathcal{A}^*$ injective (possibly unbounded)

$\Rightarrow \exists\, \mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$

$(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$

If the ranges $R(A), R(A^*)$ are closed

$\Rightarrow$ inverses are continuous

closed range theorem:

$R(A)$ closed $\Leftrightarrow R(A^*)$ closed

<u>Lemma 1</u>:

A densly defined, closed linear operator. Then the following assertions are equivalent:

(i) $\forall x \in D(\mathcal{A}) : |x|_{H_1} \le c_A |Ax|_{H_2}$

(i') $\forall y \in D(\mathcal{A}^*) : |y|_{H_2} \le c_A |A^* y|_{H_1}$

(ii) $R(A) = R(\mathcal{A})$ closed

(ii') $R(A^*) = R(\mathcal{A}^*)$ closed

(iii) $\mathcal{A}^{-1} : R(A) \to D(\mathcal{A})$ bounded

(iii*) $(\mathcal{A}^*)^{-1} : R(A^*) \to D(\mathcal{A}^*)$ bounded

Then:

$$|\mathcal{A}^{-1}|_{R(A),\,R(A^*)} \leq c_A$$

$$|\mathcal{A}^{-1}|_{R(A),\,D(A)} \leq (1 + c_A^2)^{1/2}$$

$$|(\mathcal{A}^*)^{-1}|_{R(A^*),\,R(A)} \leq c_{A^*}$$

$$|(\mathcal{A}^*)^{-1}|_{R(A^*),\,D(A^*)} \leq (1 + c_{A^*}^2)^{1/2}$$

<u>idea of the proof:</u>

$$D(A) = \overline{R(A^*)} \cap D(A) \oplus N(A)$$

$$\Rightarrow R(A) = R(\mathcal{A})$$

(same for the adjoint: $R(A^*) = R(\mathcal{A}^*)$)

(ii) $\Leftrightarrow$ (ii') by closed range theorem.

Show:

(i) $\Rightarrow$ (ii') $\Rightarrow$ (iii') $\Rightarrow$ (i')

$\boxed{4}$

$$1/c_A := \inf_{\substack{x \in D(\mathcal{A}) \\ x \neq 0}} \frac{|Ax|}{|x|} \; ; \quad 1/c_{A^*} := \inf_{\substack{y \in D(\mathcal{A}^*) \\ y \neq 0}} \frac{|A^* y|}{|y|}$$

$$|\mathcal{A}^{-1}|_{R(A),\,R(A^*)} = \sup_{\substack{y \in R(A) \\ y \neq 0}} \frac{|\mathcal{A}^{-1} y|}{|y|}$$

$$x := \mathcal{A}^{-1} y \in D(\mathcal{A}) \quad = \sup_{\substack{x \in D(\mathcal{A}) \\ x \neq 0}} \frac{|x|}{|Ax|}$$

$$= \left(\inf_{\substack{x \in D(\mathcal{A}) \\ x \neq 0}} \frac{|Ax|}{|x|} \right)^{-1} = c_A$$

Lemma 2:

$$C_A = C_{A^*}$$

Proof:

Let $y \in D(\mathcal{A}^*) = D(A^*) \cap R(A)$.

$\Rightarrow \; y = Ax, \; x \in D(\mathcal{A})$

$$\Rightarrow |y|^2 = \langle y, Ax \rangle$$
$$= \langle A^* y, x \rangle$$
$$\leq |A^* y| \, C_A \, |Ax|$$
$$= C_A |A^* y| |y|$$

$\Rightarrow \; C_{A^*} \leq C_A$

The other way around: $C_A \leq C_{A^*}$

$\boxtimes$

Remark:

Even the whole spectrum coincides.

$\rightsquigarrow$ Need: e.g. $R(A)$ closed
(for solution theory)

Lemma 3:

If $D(\mathcal{A}) \hookrightarrow H_1$, then Lemma 1 holds.

Proof:

Like standard Poincaré proof with cauchy sequence.

$\boxtimes$

$\Rightarrow \mathcal{A}^{-1}: R(A) \to D(\mathcal{A}) \to R(A^*) / H_1$
is compact.

(solution operator is compact
$\Rightarrow$ discrete spectrum, Fredholm)

$$\begin{aligned} A_1 &: D(A_1) \subset H_1 \to H_2 \\ A_2 &: D(A_2) \subset H_2 \to H_3 \end{aligned} \quad \text{densly defined, closed}$$

$$\begin{aligned} A_2 A_1 = 0 \quad &\Leftrightarrow \quad R(A_1) \subset N(A_2) \\ &\Leftrightarrow \quad A_1^* A_2^* = 0 \\ &\Leftrightarrow \quad R(A_2^*) \subset N(A_1^*) \end{aligned}$$

<u>*Helmholtz decompositions*</u>:

$$\begin{aligned} H_1 &= \overline{R(A_1^*)} \oplus N(A_1) \\ H_2 &= \overline{R(A_1)} \oplus N(A_1^*) = \underset{\cap}{\overline{R(A_1)}} \oplus \underset{\cup}{N(A_1^*)} \\ &= \overline{R(A_2^*)} \oplus N(A_2) = N(A_2) \oplus \overline{R(A_2^*)} \\ H_3 &= \overline{R(A_2)} \oplus N(A_2^*) \end{aligned}$$

$\Rightarrow N(A_2) = \overline{R(A_1)} \oplus [N(A_1^*) \cap N(A_2)]$
$\rightsquigarrow K_2 = N(A_2) \cap N(A_1^*)$ cohom. group
(„harmonic fields" in Maxwell)

$$\Rightarrow H_2 = \underbrace{\overline{R(A_1)} \oplus K_2}_{N(A_2)} \oplus \overline{R(A_2^*)}$$

$$\phantom{\Rightarrow H_2 = \overline{R(A_1)} \oplus} \underbrace{K_2 \oplus \overline{R(A_2^*)}}_{N(A_1^*)}$$

$$\begin{aligned} \Rightarrow D(A_2) &= \overline{R(A_1)} \oplus K_2 \oplus [D(A_2) \cap \overline{R(A_2^*)}] \\ &= N(A_2) \oplus D(\mathcal{A}_2) \end{aligned}$$

$\Rightarrow \; D(\mathcal{A}_2) \cap D(\mathcal{A}_1^*) = D(\mathcal{A}_2) \oplus K_2 \oplus D(\mathcal{A}_1^*)$

<u>Lemma $2\frac{1}{2}$:</u>

$$D(\mathcal{A}) \hookrightarrow H_1 \iff D(\mathcal{A}^*) \hookrightarrow H_2$$

<u>Lemma 4:</u>

$$D(\mathcal{A}_1) \hookrightarrow H_1 \wedge D(\mathcal{A}_2) \hookrightarrow H_2 \wedge K_2 \hookrightarrow H_2$$
$$\iff D_2 := D(A_2) \cap D(\mathcal{A}_1^*) \hookrightarrow H_2$$

<u>Proof:</u>

$$D(\mathcal{A}_1^*) = D(\mathcal{A}_1^*) \cap \overline{R(\mathcal{A}_1)}$$
$$\subset D(\mathcal{A}_1^*) \cap N(A_2) \subset D_2$$

$$D(\mathcal{A}_2) = D(A_2) \cap \overline{R(\mathcal{A}_2^*)}$$
$$\subset D(A_2) \cap N(A_1^*) \subset D_2$$

$K_2 \subset D_2$

$\rightsquigarrow$ For other direction:
refined Helmholtz equation

$\boxed{q}$

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sequence:

$$D(\mathcal{A}_1) \xrightarrow{\mathcal{A}_1} D(\mathcal{A}_2) \xrightarrow{A_2} H_3 \qquad A_2 \mathcal{A}_1 = 0$$
$$H_1 \xleftarrow{A_1^*} D(A_1^*) \xleftarrow{A_2^*} D(\mathcal{A}_2^*) \qquad A_1^* \mathcal{A}_2^* = 0 \quad \Updownarrow$$

(one can also use $\mathcal{A} \leftrightarrow A$)

$\rightsquigarrow$ sequence is <u>exact</u>

$$:\iff K_2 = N(A_2) \ominus \overline{R(\mathcal{A}_1)} = \{0\}$$

A densly defined, linear, closed

$\Rightarrow$ A^*A self-adjoint

$D(A^*A) = \{ x \in D(A) : Ax \in D(A^*) \}$

$$\Rightarrow \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} : D(A) \times D(A^*) \subset H_1 \times H_2 \to H_1 \times H_2$$

is self-adjoint operator.

Maxwell-operator:

$\begin{bmatrix} 0 & -A^* \\ A & 0 \end{bmatrix}$ is skew-self-adjoint

$\rightsquigarrow$ $\mathcal{M} := i \begin{bmatrix} 0 & -A^* \\ A & 0 \end{bmatrix}$ is self-adjoint

$\Rightarrow$ $\mathcal{M}^{-1}$ exists

Assume: ε, μ Material-parameters

$\rightsquigarrow$ apply Functional-Analysis-Toolbox:

$\hat{A}_0 := \nabla : \overset{\circ}{C}{}^\infty \subset L^2 \to L^2$

$\hat{A}_1 := \operatorname{rot} : \overset{\circ}{C}{}^\infty \subset L^2 \to L^2$

$\hat{A}_2 := \operatorname{div} : \overset{\circ}{C}{}^\infty \subset L^2 \to L^2$

linear & densly defined

closable, since:

$(E_n) \subset \overset{\circ}{C}{}^\infty$, $E_n \to 0$ in L^2

$\operatorname{rot} E_n \to H$ in L^2

$$\Rightarrow \quad 0 = \lim_{n\to\infty} \langle E_n, \operatorname{rot} \varphi \rangle_{L^2}$$

$$= \lim_{n\to\infty} \langle \operatorname{rot} E_n, \varphi \rangle_{L^2}$$

$$= \langle H, \varphi \rangle_{L^2}$$

$$\Rightarrow \quad H = 0.$$

closures:

$A_0 := \mathring{\nabla} : \mathring{H}^1 \subset L^2 \to L^2_\varepsilon$

$A_1 := \mu^{-1} \operatorname{rot} : \mathring{R} \subset L^2_\varepsilon \to L^2_\mu$

$A_2 := \operatorname{div} \mu : \mu^{-1} \mathring{D} \subset L^2_\mu \to L^2$

densly defined, closed

adjoints:

* $E \in D(A_0^*) \wedge A_0^* E = f \in L^2$

  $\Leftrightarrow \quad E \in L^2 \wedge \exists f \in L^2 \ \forall \varphi \in D(A_0) = \mathring{H}^1 :$

  $\langle \varepsilon E, \nabla \varphi \rangle_{L^2} = \langle f, \varphi \rangle_{L^2}$

  $\Rightarrow \quad E \in \varepsilon^{-1} D \wedge f = -\operatorname{div} \varepsilon E$

  $\Rightarrow \quad A_0^* = -\operatorname{div} \varepsilon : \varepsilon^{-1} D \subset L^2_\varepsilon \to L^2$

* $E \in D(A_1^*) \wedge A_1^* E = f \in L^2$

  $\Leftrightarrow \quad E \in L^2 \wedge \exists f \in L^2 \ \forall \varphi \in D(A_1) :$

  $\langle E, \operatorname{rot} \varphi \rangle_{L^2} = \langle \varepsilon f, \varphi \rangle_{L^2}$

  $\Rightarrow \quad E \in R \wedge \operatorname{rot} E = \varepsilon f$

  $\Rightarrow \quad A_1^* = \varepsilon^{-1} \operatorname{rot} : R \subset L^2_\mu \to L^2_\varepsilon$

* $E \in D(A_2^*) \wedge A_2^* E = f \in L^2$

  $\Leftrightarrow \quad E \in L^2 \wedge \exists f \in L^2 \ \forall \varphi \in D(A_2) :$

  $\langle E, \operatorname{div} \mu \varphi \rangle_{L^2} = \langle \mu f, \varphi \rangle_{L^2}$

  $\Rightarrow \quad E \in H^1 \wedge -\nabla E = f$

  $\Rightarrow \quad A_2^* = -\nabla : H^1 \subset L^2 \to L^2_\mu$

reduced operators:

$$L^2 = \overline{\operatorname{div} D} \oplus \underbrace{N(\mathring{\nabla})}_{= \{ c \in \mathbb{R} : c = 0 \text{ auf } \Gamma \} = \{0\}}$$

$$L^2 = \overline{\operatorname{div} \mathring{D}} \oplus \underbrace{N(\nabla)}_{= \mathbb{R}}$$

$$\mathcal{A}^0 = \mathring{\nabla} : \mathring{H}^1 \cap \underbrace{\overline{\operatorname{div} D}}_{= L^2} \subset \underbrace{\overline{\operatorname{div} D}}_{L^2} \to \overline{\nabla \mathring{H}^1}$$

$$\mathcal{A}^1 = \mu^{-1} \operatorname{rot} : \mathring{R} \cap \overline{\varepsilon^{-1} \operatorname{rot} R} \subset \overline{\varepsilon^{-1} \operatorname{rot} R} \to \overline{\mu^{-1} \operatorname{rot} \mathring{R}}$$

$$\mathcal{A}^2 = \operatorname{div} \mu : \mu^{-1} \mathring{D} \cap \overline{\nabla H^1} \subset \overline{\nabla H^1} \to \mathbb{R}^\perp$$

$$\mathcal{A}_0^* = -\operatorname{div} \varepsilon : \varepsilon^{-1} D \cap \overline{\nabla \mathring{H}^1} \subset \overline{\nabla \mathring{H}^1} \to L^2$$

$$\mathcal{A}_1^* = \varepsilon^{-1} \operatorname{rot} : R \cap \overline{\mu^{-1} \operatorname{rot} \mathring{R}} \subset \overline{\mu^{-1} \operatorname{rot} \mathring{R}} \to \overline{\varepsilon^{-1} \operatorname{rot} R}$$

$$\mathcal{A}_2^* = -\nabla : H^1 \cap \mathbb{R}^\perp \subset \mathbb{R}^\perp \to \nabla H^1$$

Toolbox delivers: Poincaré estimates

$\forall u \in \mathring{H}^1 : \ |u|_{L^2} \le c_{\mathring{\nabla}} |\nabla u|_{L^2_\varepsilon} \quad (c_{\mathring{\nabla}} = c_F)$

$\forall E \in \mathring{R} \cap \varepsilon^{-1} \operatorname{rot} R : \ |E|_{L^2_\varepsilon} \le c_{\mathring{\operatorname{rot}}} |\operatorname{rot} E|_{L^2_{\mu^{-1}}}$

$\forall E \in \mu^{-1} \mathring{D} \cap \nabla H^1 : \ |E|_{L^2_\mu} \le c_{\mathring{\operatorname{div}}} |\operatorname{div} E|_{L^2}$

$\forall E \in \varepsilon^{-1} D \cap \nabla \mathring{H}^1 : \ |E|_{L^2_\varepsilon} \le c_{\nabla} |\operatorname{div} E|_{L^2}$

$\forall E \in R \cap \mu^{-1} \operatorname{rot} \mathring{R} : \ |E|_{L^2_\mu} \le c_\mu |\operatorname{rot} E|_{L^2_{\varepsilon^{-1}}}$

$\forall u \in H^1 \cap \mathbb{R}^\perp : \ |u|_{L^2} \le c_P |\nabla u|_{L^2_\mu}$

if we assume:

Rellich: $H^1 \hookrightarrow\hookrightarrow L^2$

∇H^1 closed, $\operatorname{rot} \mathring{R}$ closed, $\nabla \mathring{H}^1$ closed

$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow$

$\operatorname{div} \mathring{D}$ closed, $\operatorname{rot} R$ closed, $\operatorname{div} D$ closed

Weck's selection theorem:

$\mathring{R} \cap \varepsilon^{-1} D \hookrightarrow\hookrightarrow L^2$