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"Picard's theory"

lecture 1

Aim: Prove well-posedness, i.e. uniqueness, existence and continuous dependence on the data for a large class of (linear) partial differential equations that involve time.

For instance:

① Maxwell's equations:

$$\left[\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix}\right] \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -J \\ K \end{pmatrix}$$

② Heat equation:

$$\partial_0 \Theta + \operatorname{div} q = F$$
$$q = -a \operatorname{\mathring{grad}} \Theta$$

③ Poisson's equation:

$$(1 - \Delta_0) u = F$$

structure in all this equations:

$$(\partial_0 M_0 + M_1 + A) U = F$$

M_0, M_1 : bounded

A : unbounded, skew-selfadjoint

$\rightsquigarrow$ identify:

① $M_0 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$, $M_1 = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 0 & -\mathrm{curl} \\ \mathring{\mathrm{curl}} & 0 \end{pmatrix}$, $U = \begin{pmatrix} E \\ H \end{pmatrix}$, $F = \begin{pmatrix} -j \\ k \end{pmatrix}$

② $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_1 = \begin{pmatrix} 0 & 0 \\ 0 & a^{-1} \end{pmatrix}$

$A = \begin{pmatrix} 0 & \mathrm{div} \\ \mathring{\mathrm{grad}} & 0 \end{pmatrix}$, $U = \begin{pmatrix} \Theta \\ q \end{pmatrix}$, $F = \begin{pmatrix} f \\ 0 \end{pmatrix}$

③ Substitute $v := \mathring{\mathrm{grad}}\, u$

$M_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 0 & -\mathrm{div} \\ -\mathring{\mathrm{grad}} & 0 \end{pmatrix}$, $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $F = \begin{pmatrix} f \\ 0 \end{pmatrix}$

Find conditions on M_0, M_1 such that $(\partial_0 M_0 + M_1 + A)$ is continuously invertible!

↳ __Task 0:__ *Find appropriate function space!*

__Task 1:__ *"Understand" the operator sum!*

__On task 0:__

(First deal with $M_0 = 1$, $M_1 = A = 0$)

__Definition:__

For $\nu > 0$ we define

$$L^2_\nu(\mathbb{R};H) := \{ f : \mathbb{R} \to H \text{ measurable} :$$

$$\|f\|^2_{L^2_\nu} := \int_{\mathbb{R}} \|f(t)\|^2_H \exp(-2t\nu)\, dt < \infty \}$$

where H is a Hilbert-space.

We put:

$$\partial_0 : H^1_\nu(\mathbb{R};H) \subseteq L^2_\nu(\mathbb{R};H) \to L^2_\nu(\mathbb{R};H)$$
$$\phi \mapsto \phi'$$

where:

$$H^1_\nu(\mathbb{R};H) = \{ f \in L^2_\nu(\mathbb{R};H) \mid \forall \varphi \in \overset{\circ}{C}{}^\infty(\mathbb{R}) :$$
$$-\int_{\mathbb{R}} f\varphi' = \int_{\mathbb{R}} g\varphi \text{ for some } g \in L^2_\nu(\mathbb{R};H) \}$$

<u>Theorem</u>: (the case $M_1 = 1$, $M_0 = A = 0$)
The operator ∂_0 (defined as above) is continuously invertible and

$$\partial_0^{-1} f(t) = \int_{-\infty}^{t} f(s)\, ds$$

for all $f \in L^2_\nu(\mathbb{R};H)$.

<u>Proof</u>:
∂_0 is one-to-one:
Let $f, g \in D(\partial_0)$ s.t. $\partial_0 f = \partial_0 g$. Hence $f - g$

is a constant. (Weyl's lemma) But $f-g \in L^2_\nu(\mathbb{R}; H)$ implies $f-g=0$.

Next, for $h := 1\!\!1_{[0,\infty)}$ we observe:

$$\int_{-\infty}^{t} f(s)\,ds = \int_{\mathbb{R}} h(t-s) f(s)\,ds = h * f\,(t)$$

Take $f \in L^2_\nu(\mathbb{R}; H)$. Then

$$\|(h*f)\|^2_{L^2_\nu}$$

$$= \int_{\mathbb{R}} \Big\| \int_{\mathbb{R}} h(t-s) f(s)\,ds \Big\|^2_H \exp(-2\nu t)\,dt$$

$$\leq \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} h(t-s)\, \|f(s)\|_H\, e^{-(t-s)\nu} e^{-s\nu}\,ds \Big)^2 dt$$

$$\overset{CSI}{\leq} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} h(t-s) e^{-(t-s)\nu}\,ds \Big)$$

$$\cdot \Big(\int_{\mathbb{R}} h(t-s) e^{-(t-s)\nu} \|f(s)\|^2_H e^{-2s\nu}\,ds \Big)\,dt$$

Fubini-theorem

$$\leq \|1\!\!1_{[0,\infty)} \cdot e^{-(\cdot)\nu}\|^2_{L^1(\mathbb{R})}\, \|f\|^2_{L^2_\nu}$$

$$= 1/\nu^2\, \|f\|^2_{L^2_\nu}.$$

Hence: $h*\cdot \in L(L^2_\nu)$, $\|h*\cdot\| \leq 1/\nu$

and for given $\phi \in L^2_\nu$:

$$\partial_0 (h * \phi) = \phi$$
(according to Fubini's theorem);
if, in addition, $\phi \in D(\partial_0)$:
$$h * (\partial_0 \phi) = \phi$$
(according to Fubini's theorem)

$\boxtimes$

Going into more details of task 0:
$\rightsquigarrow$ the spectral representation for ∂_0.

<u>Definition:</u>
For $f \in L^2(\mathbb{R}; H) \cap L^1(\mathbb{R}; H)$ we define

$$\mathcal{F} f(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\xi} \, dt, \quad (\xi \in \mathbb{R}),$$

the <u>Fourier-transform</u>.

<u>Reminder:</u>
(a) $\mathcal{F}$ extends unitarily to
$$\mathcal{F} : L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H)$$
(vector-valued Plancherel's theorem)
(b) Given $g \in L^1(\mathbb{R})$, $f \in L^2(\mathbb{R}; H)$
$$\mathcal{F}(g * f) = \sqrt{2\pi}\, M_{\hat{g}} \mathcal{F}(f),$$
where $M_{\hat{g}}$ is the multiplication operator of multiplying by $\mathcal{F}g$
(Multiplication theorem)

Definition:

We define

$$m_\nu := L^2_\nu(\mathbb{R}; H) \to L^2(\mathbb{R}; H)$$
$$\phi \mapsto \left(t \mapsto e^{-t\nu} \phi(t) \right)$$

(unitary, obvious (!)) and $\mathcal{L}_\nu := \mathcal{F} m_\nu$, the Fourier-laplace-transform

Theorem: (spectral representation of ∂_0^{-1})

We have

$$\partial_0^{-1} = \mathcal{L}_\nu^* \mathbb{1}_{\hat{h}_\nu} \mathcal{L}_\nu,$$

where

$$\hat{h}_\nu(\xi) = \frac{1}{i\xi + \nu}, \quad (\xi \in \mathbb{R}).$$

In particular, ∂_0 is unitarily equivalent to multiplication by $\xi \mapsto i\xi + \nu$ and $\mathrm{Re}\, \partial_0 = \nu$.

Proof:

Let $h := \mathbb{1}_{[0,\infty)}$. From the theorem above:

$$\partial_0^{-1} = h *$$

$$\Rightarrow \partial_0^{-1} = h * \mathcal{L}_\nu^* \mathcal{L}_\nu$$

$$= h * m_\nu^* \mathcal{F}^* \mathcal{L}_\nu$$

m_ν is unitary

$$= h * m_{-\nu} \mathcal{F}^* \mathcal{L}_\nu$$

Now:

$$\Big(h*(m_{-\nu}f)\Big)(t)$$

$$=\int_{\mathbb{R}} h(t-s)e^{s\nu}f(s)\,ds$$

$$=e^{t\nu}\int_{\mathbb{R}} h(t-s)e^{-(t-s)\nu}f(s)\,ds$$

so: (with $(g*\mathcal{F}^*f)=\mathcal{F}^*\sqrt{2\pi}\,\mathcal{M}_{\hat{g}}\mathcal{F}\mathcal{F}^*f$)

$$\partial_0^{-1}=h*m_{-\nu}\mathcal{F}^*\mathcal{L}_\nu$$

$$=m_{-\nu}\Big((m_\nu h)*\Big)\mathcal{F}^*\mathcal{L}_\nu$$

$g=m_\nu h$

$$=m_{-\nu}\mathcal{F}^*\sqrt{2\pi}\,\mathcal{M}_{\hat{g}}\mathcal{F}\mathcal{F}^*\mathcal{L}_\nu$$

$$=\mathcal{L}_\nu^*\sqrt{2\pi}\,\mathcal{M}_{\hat{g}}\mathcal{L}_\nu$$

But,

$$\sqrt{2\pi}\,\mathcal{M}_{\hat{g}}(\xi)=\sqrt{2\pi}\,\mathcal{F}m_\nu h(\xi)$$

$$=\int_{\mathbb{R}} e^{-\nu t}h(t)e^{-i\xi t}\,dt$$

$$=\int_0^\infty e^{-(\nu+i\xi)t}\,dt=\frac{1}{i\xi+\nu}$$

$$=\mathcal{M}_{\hat{h}_\nu}$$

$\boxed{\xi}$

Lecture 2 03.11.16

Recap:

$\partial_0 : H_\nu^1(\mathbb{R};H) \subseteq L_\nu^2(\mathbb{R};H) \to L_\nu^2(\mathbb{R};H),\ \phi \mapsto \phi'$

$\hookrightarrow\ 0 \in \varrho(\partial_0),\ \|\partial_0^{-1}\| \leq 1/\nu,$

$$\partial_0^{-1} = \mathcal{L}_\nu^* \, M_{\hat{h}_\nu} \mathcal{L}_\nu,\ h_\nu(\xi) = \frac{1}{i\xi + \nu}$$

<u>Now</u>, <u>task 1:</u> $\left(\overline{\partial_0 M_0 + M_1 + A}\right) u = F$

<u>Proposition</u>: (solution strategy)

Let $B : D(B) \subseteq H \to H$ densely defined, closable, linear operator. Assume that there is $c > 0$ s.t.:

$\forall \phi \in D(B) : \mathrm{Re} \langle B\phi, \phi\rangle_H \geq c \langle \phi, \phi\rangle_H \quad (*_1)$

$\forall \psi \in D(B^*) : \mathrm{Re} \langle B^*\psi, \psi\rangle_H \geq c \langle \psi, \psi\rangle_H \quad (*_2)$

Then:

$$0 \in \varrho(\overline{B}),\ \|\overline{B}^{-1}\| \leq 1/c$$

<u>Proof:</u>

(1): $\overline{B}$ is one-to-one and continuously invertible on its (closed) range:

By $(*_1)$ we obtain for $\phi \in D(\overline{B})$:

$$\|\overline{B}\phi\|_H \|\phi\|_H \geq \mathrm{Re}\langle \overline{B}\phi, \phi\rangle_H \geq c \langle \phi, \phi\rangle_H = c\|\phi\|_H^2,$$

Hence, $\|\overline{B}\phi\|_H \geq c\|\phi\|_H \quad (\phi \in D(\overline{B}))$

In particular, $R(\bar{B})$ is closed and $\bar{B}^{-1}: R(\bar{B}) \to H$ is continuous with norm bounded by $1/c$.

(2) $\bar{B}$ is onto:

From spectral theorem:

$$
\begin{aligned}
H &= \mathcal{N}(\bar{B}^*) \oplus_H R(\bar{B}) \\
&= \mathcal{N}(B^*) \oplus_H R(\bar{B}) \\
&= \{0\} \oplus_H R(\bar{B}) = R(\bar{B})
\end{aligned}
$$

$(*_2)$ $\Downarrow$ B^* is one-to-one

$\square$

<u>Theorem:</u> (solution theory, R. Picard '09)

Let $\mathcal{M}_0, \mathcal{M}_1 \in L(H)$, $\mathcal{A}: D(\mathcal{A}) \subseteq H \to H$ skew-selfadjoint, $\mathcal{M}_0 = \mathcal{M}_0^* \geq 0$ and assume there exists $\nu > 0$:

$$
\nu \mathcal{M}_0 + \underbrace{\tfrac{1}{2}(\mathcal{M}_1 + \mathcal{M}_1^*)}_{= \operatorname{Re}(\mathcal{M}_1)} \geq c \cdot \mathrm{id}_H \qquad (*_3)
$$

Then

$$
\underbrace{\partial_0 \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{A}}_{=: B} : \underbrace{D(\partial_0) \cap L^2_\nu(\mathbb{R}; D(\mathcal{A}))}_{\subseteq L^2_\nu(\mathbb{R}; H)} \to L^2_\nu(\mathbb{R}; H)
$$

satisfies the assumptions on B in the proposition above.

Proof:

① B is densely defined:

Lemma: (Hille's theorem)
let $C : D(C) \subseteq H \to H$ closed, densely defined operator and

$$\hat{C} : L^2_\nu(\mathbb{R}; D(C)) \subset L^2_\nu(\mathbb{R}; H) \to L^2_\nu(\mathbb{R}; H)$$
$$\phi \mapsto (t \mapsto C\phi(t))$$

Then for all $\varepsilon > 0$:

$$(1+\varepsilon\partial_0)^{-1}\hat{C}\psi = \hat{C}(1+\varepsilon\partial_0)^{-1}\psi \quad (\psi \in D(\hat{C}))$$

Proof:
Note that $\hat{C}$ is a closed operator:
Take $(\phi_n)_n \subset D(\hat{C})$, $\phi_n \to \phi$ in $L^2_\nu(\mathbb{R}; H)$ and $\hat{C}\phi_n \to \psi$ in $L^2_\nu(\mathbb{R}; H)$. Then without loss of generality:

$$(\hat{C}\phi_n)(t) \to \psi(t), \; \phi_n(t) \to \phi(t)$$

for a.e. $t \in \mathbb{R}$, $\phi_n(t) \in D(C)$. But

$$(\hat{C}\phi_n)(t) = C\phi_n(t) \to \psi(t),$$

hence, for a.e. $t \in \mathbb{R}$: $\phi(t) \in D(C)$ and $C\phi(t) = \psi(t)$. Thus, $\phi \in D(\hat{C})$ and $\hat{C}\phi = \psi$.

Next, for the proof of the lemma:

By the density of elementary functions with values in $D(C)$ in $L^2_\nu(\mathbb{R}; D(C))$ and the closedness of $\hat{C}$ it suffices to prove

$$(1+\varepsilon\partial_0)^{-1}\hat{C}\psi = \hat{C}(1+\varepsilon\partial_0)^{-1}\psi$$

For $\psi = 1\!\!1_E\,\phi$ with $\phi \in D(C)$, $E \subseteq \mathbb{R}$ measurable with finite measure.
But

$$(1+\varepsilon\partial_0)^{-1}\hat{C}\psi$$

$\hat{C}$ acts pointwise
$$= (1+\varepsilon\partial_0)^{-1}\hat{C}\,1\!\!1_E\,\phi$$
$$= (1+\varepsilon\partial_0)^{-1}1\!\!1_E\,\hat{C}\phi$$

$\hat{C}$ is linear, $(1+\varepsilon\partial_0)^{-1}1\!\!1_E$ is pointwise a constant
$$= \hat{C}(1+\varepsilon\partial_0)^{-1}1\!\!1_E\,\phi = \hat{C}(1+\varepsilon\partial_0)^{-1}\psi$$

$\boxtimes$

For $\varepsilon, \delta > 0$; $\phi \in L^2_\nu(\mathbb{R}; H)$:

$$(1+\varepsilon\partial_0)^{-1}(1+\delta A)^{-1}\phi$$

Lemma
$$= (1+\delta A)^{-1}(1+\varepsilon\partial_0)^{-1}\phi$$
$$\in D(A) \cap D(\partial_0)$$

But:

$$(1+\varepsilon\partial_0)^{-1}\phi \to \phi \quad \text{for } \varepsilon \to 0$$
$$(1+\delta A)^{-1}\phi \to \phi \quad \text{for } \delta \to 0$$

$(\phi \in L^2_\nu(\mathbb{R}; H))$

$$\Gamma((1+\delta A)^{-1} - 1)\phi$$

$$= \left((1+\delta A)^{-1} - (1+\delta A)^{-1}(1+\delta A) \right)\phi$$
$$= (1+\delta A)^{-1}(\phi - \phi - \delta A\phi)$$
$$= \underbrace{-(1+\delta A)^{-1}}_{\|\cdot\| \le 1} \delta A \longrightarrow 0, \quad \delta \to 0$$

for $\phi \in D(A)$ $\lrcorner$

Hence $(1+\delta A)^{-1} - 1 \to 0$ on the dense subspace $D(A)$ and

$$\sup_{\delta > 0} \| (1+\delta A)^{-1} - 1 \| \le 2.$$

Thus we get convergence on $L^2_\nu(\mathbb{R}; H)$.

② <u>B is closable:</u>

Note that $(M_0 \partial_0^* + M_1^* - A)|_{D(\partial_0) \cap D(A)}$ is a restriction of B^*.
But $D(\partial_0) \cap D(A)$ is dense, hence, $\bar{B} = (B^*)^*$ is well defined.

③ <u>B satisfies $(*_1)$:</u>

Take $\phi \in D(B)$ and compute:

$$\mathrm{Re} \langle \partial_0 M_0 \phi + M_1 \phi + A\phi, \phi \rangle_H$$

$\mathrm{Re}\langle A\phi, \phi\rangle_H = 0$ $= \mathrm{Re} \langle \partial_0 M_0 \phi + M_1 \phi, \phi \rangle_H$

What is the real part of $\langle \partial_0 M_0 \phi, \phi \rangle_H$?
↳ For $t \in \mathbb{R}$, $\phi \in \mathring{C}^\infty(\mathbb{R}; H)$

$$\int_{-\infty}^{t} \langle (M_0 \phi)'(s), \phi(s) \rangle_H \, e^{-2s\nu} \, ds$$

$$\overset{\text{part. Integration}}{=} \langle (M_0 \phi)(s), \phi(s) \rangle_H \Big|_{-\infty}^{t}$$

$$- \int_{-\infty}^{t} \langle (M_0 \phi)(s), \phi'(s) \rangle_H \, e^{-2s\nu} \, ds$$

$$+ 2\nu \cdot \int_{-\infty}^{t} \langle (M_0 \phi)(s), \phi(s) \rangle_H \, e^{-2s\nu} \, ds$$

$$\overset{t \text{ large}}{=} - \int_{-\infty}^{t} \langle (M_0 \phi)(s), \phi'(s) \rangle_H \, e^{-2s\nu} \, ds$$

$$+ 2\nu \int_{-\infty}^{t} \langle (M_0 \phi)(s), \phi(s) \rangle_H \, e^{-2\nu s} \, ds,$$

Hence:

$$\langle \partial_0 M_0 \phi, \phi \rangle + \langle M_0 \phi, \partial_0 \phi \rangle$$
$$= 2\nu \langle M_0 \phi, \phi \rangle$$

$$\overset{M_0 = M_0^*}{\Rightarrow} \operatorname{Re} \langle \partial_0 M_0 \phi, \phi \rangle = \nu \langle M_0 \phi, \phi \rangle, \ (\phi \in D(\partial_0))$$

So:

$$\operatorname{Re} \langle \partial_0 M_0 \phi + M_1 \phi + A\phi, \phi \rangle$$
$$= \operatorname{Re} \langle \partial_0 M_0 \phi + M_1 \phi, \phi \rangle$$

$$= \nu \langle M_0 \phi, \phi \rangle + \langle \operatorname{Re} M_1 \phi, \phi \rangle$$

$$\overset{(*_3)}{\geq} c \langle \phi, \phi \rangle$$

④ <u>B satisfies $(*_2)$:</u>

Let $f \in D(B^*)$. Next we proof that for $\varepsilon > 0$:

$$f_\varepsilon := (1 + \varepsilon \partial_0^*)^{-1} f \in D(B^*)$$

and $B^* f_\varepsilon = (1 + \varepsilon \partial_0^*)^{-1} B^* f$,

$f_\varepsilon \in D(\partial_0) \cap D(A)$, $B^* f_\varepsilon = (M_0 \partial_0^* + M_1^* - A) f_\varepsilon$.

⌜ Let $u \in D(B) \; (= D(\partial_0) \cap D(A))$. Then

$\langle Bu, f_\varepsilon \rangle$

$= \langle Bu, (1 + \varepsilon \partial_0^*)^{-1} f \rangle$

$(1+\varepsilon\partial_0^*)^{-1}$ bd, linear $\to$ $= \langle (1 + \varepsilon \partial_0)^{-1} Bu, f \rangle$

$= \langle (1 + \varepsilon \partial_0)^{-1} (\partial_0 M_0 + M_1 + A) u, f \rangle$

$= \langle \big(\partial_0 (1 + \varepsilon \partial_0)^{-1} M_0 + M_1 (1 + \varepsilon \partial_0)^{-1}$
$\quad + A (1 + \varepsilon \partial_0)^{-1} \big) u, f \rangle$

$= \langle B (1 + \varepsilon \partial_0)^{-1} u, f \rangle$

$= \langle (1 + \varepsilon \partial_0)^{-1} u, B^* f \rangle$

$= \langle u, (1 + \varepsilon \partial_0^*)^{-1} B^* f \rangle$

Hence, $f_\varepsilon \in D(B^*)$ and $B^* f_\varepsilon = (1 + \varepsilon \partial_0^*)^{-1} B^* f$.
Furthermore: $f_\varepsilon \in D(\partial_0)$. Now note that for all $u \in D(\partial_0) \cap D(A)$:

$\langle Au, (1 + \varepsilon \partial_0^*)^{-1} f$

see computation above $\to$ $= - \langle u, M_0 \partial_0^* f_\varepsilon \rangle - \langle u, M_1 f_\varepsilon \rangle \qquad (*_4)$
$\quad - \langle u, (1 + \varepsilon \partial_0^*)^{-1} B^* f$

But $D(\partial_0) \cap D(A)$ is dense in $D(A)$

$\ulcorner (1+\varepsilon\partial_0)^{-1} u \to u$ for $\varepsilon \to 0$

$A(1+\varepsilon\partial_0)^{-1} u = (1+\varepsilon\partial_0)^{-1} A u \to A u$

for $\varepsilon \to 0$ $\lrcorner$

Hence $(*_4)$ holds for $u \in D(A)$. Thus, $f_\varepsilon \in D(A^*) = D(-A)$ and

$$-A f_\varepsilon = -M_0 \partial_0^* f_\varepsilon - M_1 f_\varepsilon + \underbrace{(1+\varepsilon\partial_0^*)^{-1} B^* f}_{= B^* f_\varepsilon}$$

Thus, $f_\varepsilon \in D(\partial_0) \cap D(A)$ and

$B^* f_\varepsilon = (M_0 \partial_0^* + M_1^* - A) f_\varepsilon$. $\lrcorner$

Finally, for $(*_2)$ we compute:

$$\begin{aligned} & \mathrm{Re} \langle B^* f_\varepsilon, f_\varepsilon \rangle \\ &= \mathrm{Re} \langle (M_0 \partial_0^* + M_1^* - A) f_\varepsilon, f_\varepsilon \rangle \\ A^* = -A \quad &= \mathrm{Re} \langle (M_0 \partial_0^* + M_1^*) f_\varepsilon, f_\varepsilon \rangle \\ &= \mathrm{Re} \langle f_\varepsilon, (\partial_0 M_0 + M_1) f_\varepsilon \rangle \\ \text{cf. } ③ \quad &\geq c \langle f_\varepsilon, f_\varepsilon \rangle \end{aligned}$$

Now, let $\varepsilon \to 0$:

$\Rightarrow B^* f_\varepsilon \to B^* f$, $f_\varepsilon \to f$

$\Rightarrow \mathrm{Re} \langle B^* f, f \rangle \geq c \langle f, f \rangle$

$\boxed{\text{q.e.}}$

Lecture 3 03.11.16

Application of the general theory:

* Maxwell's equations: (in $\Omega \subseteq \mathbb{R}^3$ open)

$$\partial_0 \underbrace{\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}}_{=:M_0} + \underbrace{\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}}_{=:M_1} + \underbrace{\begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix}}_{=:A}$$

$\hookrightarrow$ $H = L^2(\Omega)^{3+3}$

$$A = \begin{pmatrix} 0 & -\mathring{\operatorname{curl}}^* \\ \overline{\operatorname{curl}|_{C_c^\infty(\Omega)}} & 0 \end{pmatrix}$$

If $\varepsilon, \mu \in L^\infty(\Omega; \mathbb{R}_{\geq 0})$, then $M_0^* = M_0 \geq 0$.
If $\sigma \in L^\infty(\Omega; \mathbb{C})$, then $M_1 \in L(L^2(\Omega)^{3+3})$

$\rightsquigarrow$ In order to warrant $(*_3)$ we observe:

$$(*_3) \Leftrightarrow \begin{cases} \nu\varepsilon + \operatorname{Re}\sigma \geq c & (a) \\ \nu\mu \geq c & (b) \end{cases}$$

for some $\nu, c > 0$.

$$(b) \Leftrightarrow \mu \geq d \quad \text{for } d > 0$$

$\rightsquigarrow$ For (a): If $\varepsilon = 0$ on $D \subseteq \Omega$, then $\operatorname{Re}\sigma \geq c$ on D

Remark:

Take $D \subseteq \Omega$ measurable and $\varepsilon = \mathbb{1}_D$, $\sigma = \mathbb{1}_{D^c}$, $\mu = \mathbb{1}_\Omega$. Then $(*_3)$ is satisfied, implying well-posedness for a system with change of types.

⇝ What about transmission conditions?

<u>Lemma:</u>

Let the conditions of the theorem be satisfied. Then $\overline{B}^{-1}\phi \in D(B)$ for $\phi \in D(\partial_0) = H^1_\nu(\mathbb{R}; H)$, $D(B) = D(\partial_0) \cap D(A)$

<u>Proof:</u>

For $\phi \in D(B)$:

$$(\partial_0 M_0 + M_1 + A)\,\partial_0^{-1}\phi$$
$$= \partial_0^{-1}(\partial_0 M_0 + M_1 + A)\,\phi \qquad (*_5)$$

see ①

Next, let $(\phi_n)_n$ in $D(B)$ with $\phi_n \to \psi$ in $D(\overline{B})$. From $(*_5)$ it follows:

$$\overline{B}\,\partial_0^{-1}\psi = \partial_0^{-1}\,\overline{B}\,\psi. \qquad (*_6)$$

Take $f \in D(\partial_0)$, $u := \overline{B}^{-1} f \in D(\overline{B})$. Now, $(*_6)$ implies:

$$\overline{B}\,\partial_0^{-1}\,\overline{B}^{-1} f = \partial_0^{-1}\,\overline{B}\,\overline{B}^{-1} f$$
$$= \partial_0^{-1} f,$$

hence, $\overline{B}^{-1}\partial_0^{-1} = \partial_0^{-1}\overline{B}^{-1}$. So

$$\overline{B}^{-1}[D(\partial_0)] \subseteq D(\partial_0)$$

For $\overline{B}^{-1}[D(\partial_0)] \subseteq D(A)$ let

$u \in D(\partial_0) \cap D(\bar{B})$. Then there is $(u_n)_n \subseteq D(B)$: $u_n \to u$ in $D(\bar{B})$ and we have

$$(\partial_0 M_0 + M_1 + A)(1 + \varepsilon \partial_0)^{-1} u_n$$
$$= (1 + \varepsilon \partial_0)^{-1} (\partial_0 M_0 + M_1 + A) u_n .$$

Now

$$\partial_0 M_0 (1 + \varepsilon \partial_0)^{-1} u_n \to \partial_0 M_0 (1 + \varepsilon \partial_0)^{-1} u$$
$$\| $$
$$(1 + \varepsilon \partial_0)^{-1} \partial_0 M_0 u ,$$

hence, as $n \to \infty$, we obtain

$$(1 + \varepsilon \partial_0)^{-1} \partial_0 M_0 u + \lim_{n \to \infty} (M_1 + A)(1 + \varepsilon \partial_0)^{-1} u_n = (1 + \varepsilon \partial_0)^{-1} \bar{B} u .$$

Thus, $(1 + \varepsilon \partial_0)^{-1} u \in D(A)$ and

$$(1 + \varepsilon \partial_0)^{-1} \partial_0 M_0 u + (M_1 + A)(1 + \varepsilon \partial_0)^{-1} u = (1 + \varepsilon \partial_0)^{-1} \bar{B} u$$

Again, by closedness of A, we get for $\varepsilon \to 0$:

$$\partial_0 M_0 u + M_1 u + A u = \bar{B} u ,$$
$$u \in D(A)$$

$\boxed{\text{ }}$

$\rightsquigarrow$ What about initial conditions?

For $u_0 \in D(A)$ we model the initial data u_0 by saying

$$\underset{\text{"solution"}}{u} - \mathbb{1}_{[0,\infty)} \cdot u_0 \in H^1_\nu(\mathbb{R}; H)$$

is "solution" for:

$$(\partial_0 M_0 + M_1 + A)\, u = 0 \text{ on } (0,\infty).$$

Let v solve

$$(\partial_0 M_0 + M_1 + A)\, v = -\mathbb{1}_{[0,\infty)} (M_1 + A)\, u_0$$

$\rightsquigarrow$ Put $u := v + \mathbb{1}_{[0,\infty)} u_0$

Then:

$$\begin{aligned}
(\partial_0 M_0 + M_1 + A)\, u &= (\partial_0 M_0 + M_1 + A)\, v \\
&\qquad + \partial_0 M_0 \mathbb{1}_{[0,\infty)} u_0 + M_1 \mathbb{1}_{[0,\infty)} u_0 \\
&\qquad + A \mathbb{1}_{[0,\infty)} u_0 \\
&= \delta_0 M_0 u_0 = 0 \quad \text{on } (0,\infty)
\end{aligned}$$

$\rightsquigarrow$ What about maximal regularity?

For simplicity: $M_0 = 1$

$$A = \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix}$$

Recall: everybody knows that the heat equation admits maximal L^2-regularity.

$(\partial_0 - \operatorname{div} a \operatorname{grad}_0) u = f$, $a \in L^\infty(\Omega)^{d\times d}$
strictly positive-definite,
$f \in L^2(0,T; L^2(\Omega))$. Then

$$u \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; D(\operatorname{div} a \operatorname{grad}_0))$$

$\rightsquigarrow$ Maxwell regularity.

If $f_1 \in L^2_\nu(\mathbb{R}; H)$, $f_2 \in H^1_\nu(\mathbb{R}; H)$,
then $u \in D(\partial_0) \cap D(A)$, where

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} + \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix}\right) u = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$k^* \geq d$$

$(f_2 = 0,\ C = \operatorname{grad}_0,\ k = a^{-1})$