Some combinatorial aspects of perfect codes.

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based on joint work with S.Costa

Special Days on Combinatorial Constructions using Finite Fields

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RICAM

Special Semester on
Applications of Algebra and Number Theory
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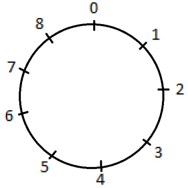

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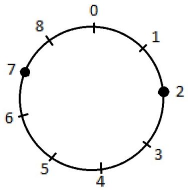

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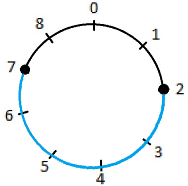

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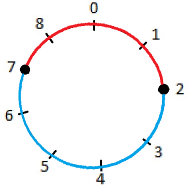

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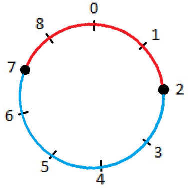

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$d(7, 2) =4$ (metric in the graph).

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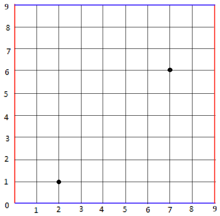

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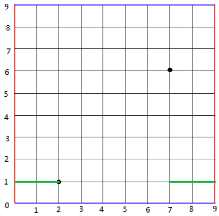

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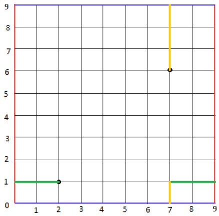

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The resurgence of Lee Codes

Engineering applications

- Constrained and partial-response channels.

  R. M. Roth and P. H. Siegel. Lee-metric BCH codes and their application to constrained and partial-response channels. IEEE Trans. on Inform. Theory, vol. IT-40, pp.1083-1096, July 1994.

- Interleaving schemes.

  M. Blaum, J. Bruck and A. Vardy. Interleaving schemes for multidimensional cluster errors. IEEE Trans. Inform. Theory, vol. IT-44, pp. 730-743, March 1998.

- Multidimensional burst-error-correction.

  T. Etzion and E. Yaakobi. Error-correction of multidimensional bursts. IEEE Trans. on Inform. Theory, vol. IT-55, pp. 961-976, March 2009.

- Error-correction for flash memories.

  A. Barg and A. Mazumdar. Codes in permutations and error correction for rank modulation. IEEE Trans. Inf. Theory, vol. 56, no. 7, pp.3158-3165, Jul. 2010.

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Theoretical research

- Enumerating and decoding perfect linear Lee codes.

  B. AlBdaiwi, P. Horak, L. Milazzo. Enumerating and decoding perfect linear Lee codes.

  Des. Codes. Crypt., vol. 52 no. 2, pp. 155-162, 2009.

- Dense Lee Codes.

  T. Etzion, A. Vardy, E. Yaakobi. Dense error-correcting codes in the Lee metric.

  Information Theory Workshop (ITW), 2010 IEEE.

- Special constructions for perfect Lee codes.

  T. Etzion. Product constructions for perfect Lee codes.

  IEEE Trans. Inform. Th.57(2011), no.11, 7473-7481.

- Diameter perfect Lee codes.

  P. Horak, B.F. AlBdaiwi. Diameter perfect Lee codes.

  IEEE Trans. Inform. Th.58(2012), no.8, 5490-5499.

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Let $C \subseteq \mathbb{Z}_q^n$ be a q-ary code.

Definitions

- As in the case of the Hamming metric, C is a perfect Lee code when $\mathbb{Z}_q^n = \biguplus_{c \in C} B(c, e)$, where e is the packing radius and the balls are Lee-balls.

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  - $PL(n, e, q) = \{C \subseteq \mathbb{Z}_q^n : C \text{ is } e\text{-perfect}\}$
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  - $PL(n, e) = \{C \subseteq \mathbb{Z}^n : C \text{ is } e\text{-perfect}\}, \left(d(x, y) = \sum_{i=1}^n |x_i - y_i|\right)$
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Main problem

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Golomb and Welch present the codes $D_e = \langle (e, e+1) \rangle \subset \mathbb{Z}_q^2$ for $q = 2e^2 + 2e + 1$ and prove that these codes are perfect, then $PL(2, e, q) \neq \emptyset$ for $q = 2e^2 + 2e + 1$ ($\Rightarrow PL(2, e) \neq \emptyset$).

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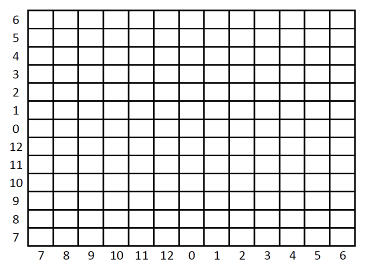
Example: $D_2 = \langle (2, 3) \rangle \subseteq \mathbb{Z}_{13}^2$



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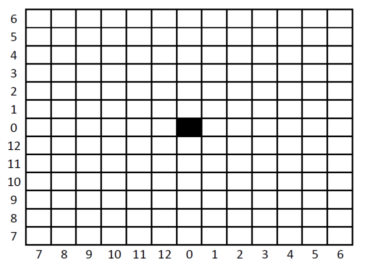

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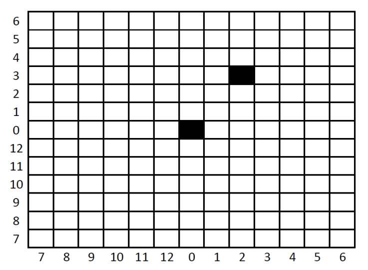

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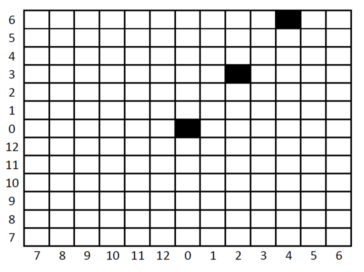

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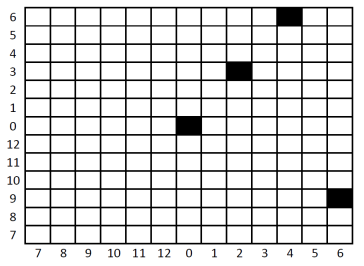

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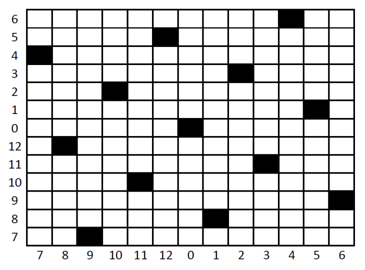

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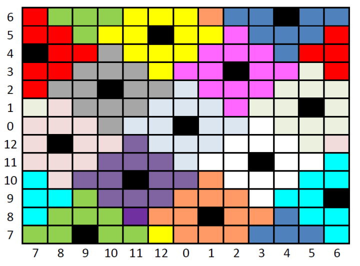

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- For which (e, q) we have $PL(2, e, q) \neq \emptyset$? In that case, is it possible to describe all these codes? (Remark: we are considering linear and non-linear codes.)
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We can use the geometry of polyominoes and combinatorial arguments.

S.Costa and C.Qureshi. Classification of the bidimensional codes in the Lee metric. XXXI Brazilian Symposium of Telecommunications, 2013.

Theorem

For $e, q \in \mathbb{Z}^+$ we define: $q_e = e^2 + (e+1)^2$, $\nu_1 = (e, e+1)$, $\nu_2 = (-(e+1), e), \eta_1 = (1, -(2e+1)), \eta_2 = (0, q_e) \in \mathbb{Z}_q^2$, $D_e = \nu_1\mathbb{Z} + \nu_2\mathbb{Z}$ and $\overline{v} = (-x, y)$ is the conjugate of $v = (x, y)$.

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1. (Existence) $PL(2, e, q) \neq \emptyset \Leftrightarrow q \equiv 0 \pmod{q_e}$.
2. (Characterization) $C \in PL(2, e, q) \Leftrightarrow C = c + D_e$ or $C = c + \overline{D_e}$ for any $c \in C$ (in particular $C - c$ is a group).

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1. (Existence) $PL(2, e, q) \neq \emptyset \Leftrightarrow q \equiv 0 \pmod{q_e}$.
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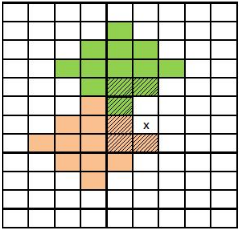
Sketch of the proof

Impossible configuration

Let $I = \{(-1,-1),(-1,0),(-1,1),(-1,2),(0,-1),(0,2)\} \subseteq \mathbb{Z}_q^2$.

If $C \in PL(2,e,q)$ and $c, c' \in C \Rightarrow \nexists\, x \in \mathbb{Z}_q^2$ such that

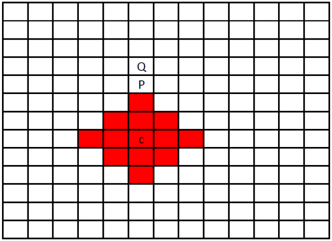
- $x \in B(c,e) \cup B(c',e)$.
- $x + C \subseteq B(c,e) \cup B(c',e)$



Sketch of the proof

Decoding of special points

If $C \in PL(2, e, q)$ and $c \in C$, the point $c + (0, e + 1)$ can only be decoded in two ways. These possibilities are $c + (-e, e + 1)$ and $c + (e, e + 1)$.



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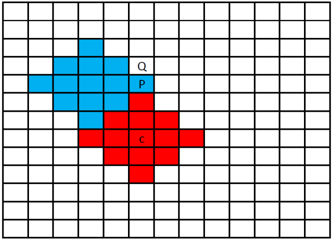

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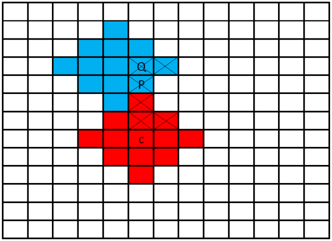

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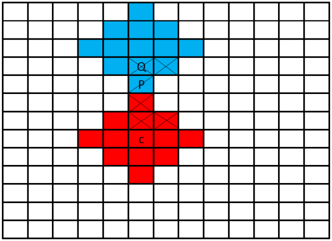

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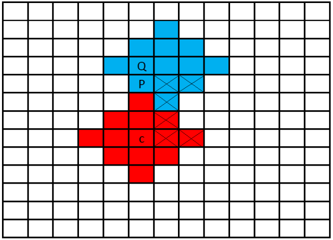

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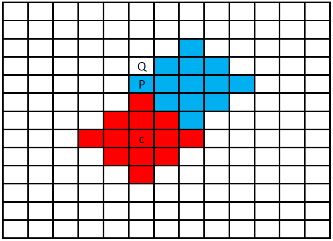

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Definition

For $C \in PL(2, e, q)$ and $c \in C$ we define the set $\omega(c) = \{v_1, \ldots, v_\tau\}$

where the adjacent balls of $B(c, e)$ are exactly $B(c + v_i, e)$ for $1 \leq i \leq \tau$.

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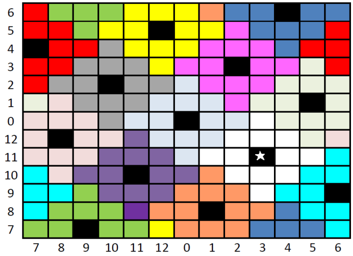

$D_2 = \langle (2,3) \rangle \subseteq \mathbb{Z}_{13}^2$

$c = (3, 11)$

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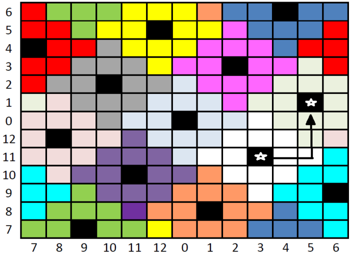

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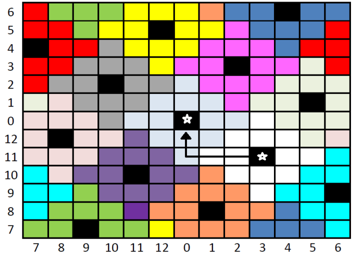

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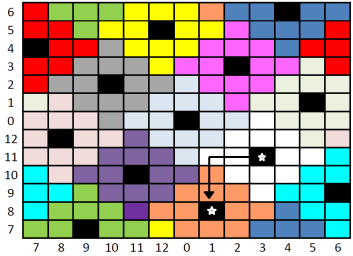

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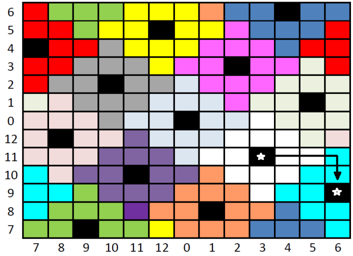

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Sketch of the proof

Kissing Lemma

If $C \in PL(2, e, q)$ the set $\omega(c)$ does not depend on c. Moreover we have only two possibilities: $\omega(c) = \{\pm\nu_1, \pm\nu_2\}$ (type 1) or $\omega(c) = \{\pm\overline{\nu_1}, \pm\overline{\nu_2}\}$ (type 2), where $\nu_1 = (e, e+1), \nu_2 = (-(e+1), e)$.

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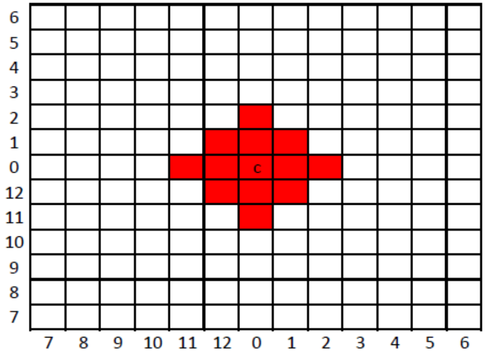

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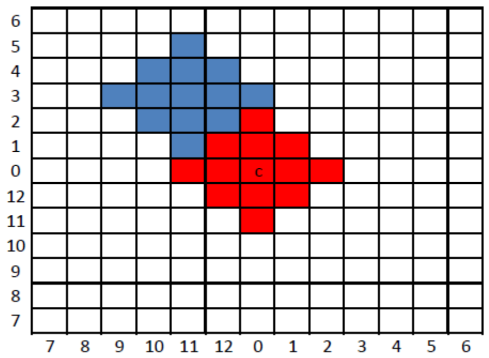

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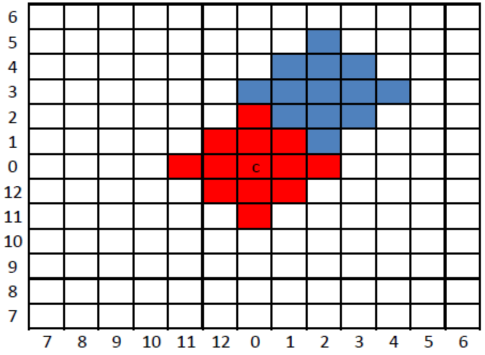

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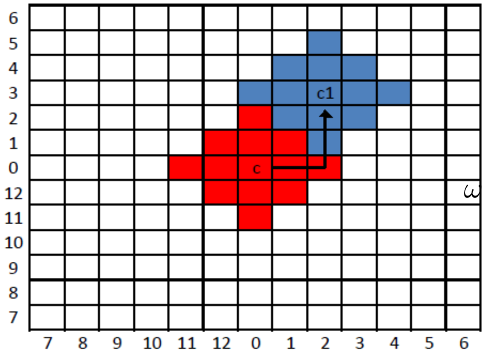

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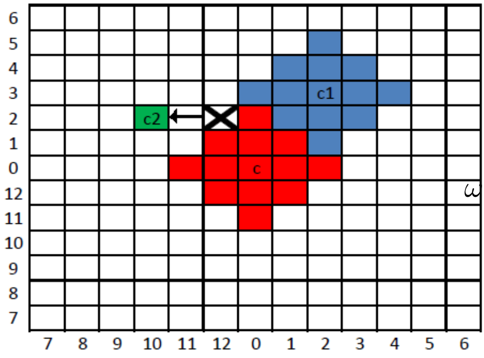

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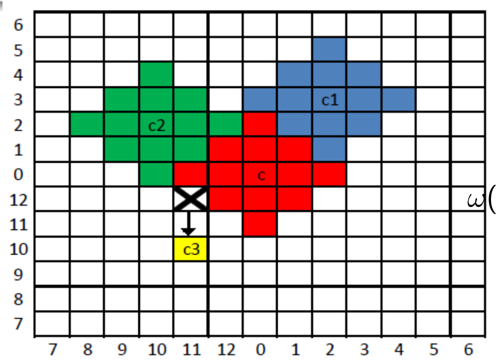

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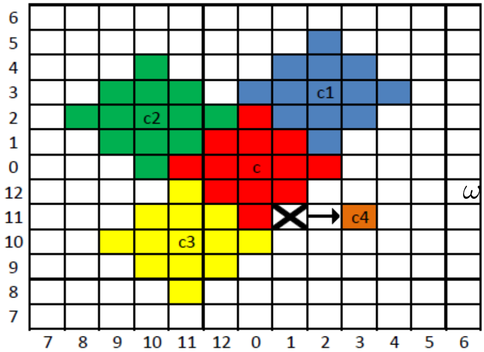

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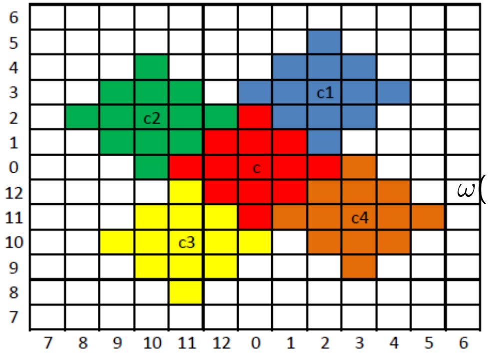

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Let $C \in PL(2, q, e)$ and fix any $c \in C$.

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For the other inclusion, if $c' \in C$ it is sufficient to consider a chain of adjacent balls from $B(c', e)$ to $B(c, e)$ and use kissing lemma again.

□

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$\#B(0, e) \cdot \#C = q^2 \Leftrightarrow q_e \cdot qh = q^2 \Leftrightarrow q = q_e h$ and so $q \equiv 0 \pmod{q_e}$.

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($\Leftarrow$) Golomb and Welch.

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   - ii) If $q = hq_e$ com $h > 1$ then $G_C \simeq \mathbb{Z}_q \times \mathbb{Z}_h$.

     Moreover, $G_C = \eta_1\mathbb{Z} \oplus \eta_2\mathbb{Z}$ if $G_C = D_e$ or $G_C = \overline{\eta_1}\mathbb{Z} \oplus \eta_2\mathbb{Z}$ if $G_C = \overline{D_e}$.

Sketch of the proof

For the last part...

We can suppose that C is linear and type 1, that is $C = \nu_1\mathbb{Z} + \nu_2\mathbb{Z}$ where

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As $\begin{pmatrix} -1 & -1 \\ e+1 & e \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ and $\det \begin{pmatrix} -1 & -1 \\ e+1 & e \end{pmatrix} = 1$,

then $C = \eta_1\mathbb{Z} + \eta_2\mathbb{Z}$ where $\eta_1 = (1, -(2e+1))$ and $\eta_2 = (0, q_e)$ (in $\mathbb{Z}_q^2$).

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Clearly $\eta_1\mathbb{Z} \cap \eta_2\mathbb{Z} = (0)$ (therefore $C = \eta_1\mathbb{Z} \oplus \eta_2\mathbb{Z}$)) and $|\eta_1\mathbb{Z}| = q$ e

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As $h|q$, it is clear that C is cyclic iff $h = 1$.

Example

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For $q = 1105$. There are exactly 5 codes in $\mathbb{Z}_{1105} \times \mathbb{Z}_{1105}$ up to translations and conjugation ($1105 = 5 \cdot 13 \cdot 17$). One of these codes is cyclic and the others are non-cyclic. These codes are given by:

- $C_1 = (1, -3)\mathbb{Z}_{1105} \oplus (0, 5)\mathbb{Z}_{1105}$ $(e = 1)$
- $C_2 = (1, -5)\mathbb{Z}_{1105} \oplus (0, 13)\mathbb{Z}_{1105}$ $(e = 2)$
- $C_3 = (1, -13)\mathbb{Z}_{1105} \oplus (0, 85)\mathbb{Z}_{1105}$ $(e = 6)$
- $C_4 = (1, -21)\mathbb{Z}_{1105} \oplus (0, 221)\mathbb{Z}_{1105}$ $(e = 10)$
- $C_5 = (23, 24)\mathbb{Z}_{1105}$ $(e = 23)$

More relevant results related to the Golomb-Welch conjecture

- $PL(2, e, q_e) \neq \emptyset$, $PL(n, 1, 2n+1) \neq \emptyset$, $PL(3, 2) = \emptyset$, $PL(n, e) = \emptyset$ for $e \geq e_n$.

  S. W. Golomb, L. R. Welch. Perfect Codes in the Lee metric and the packing of polynominoes, SIAM Journal Applied Math., vol. 18, pp. 302-317. 1970.

- $PL(n, e, q) = \emptyset$ for $3 \leq n \leq 5, e \geq n-1, q \geq 2e+1$ and for $n \geq 6, e \geq \frac{2n-3}{2\sqrt{2}} - \frac{1}{2}$

  K.A.Post. Nonexistence theorem on perfect Lee codes over large alphabets. Inf. and control 29, 369-380. 1975.

- $PL(n, 2, q) = \emptyset$ for $q = 13$, q not divisible by a prime $\dot{4}+1$, and $q = p^k$ with p prime, $p \neq 13$ and $p < \sqrt{n^2 + (n+1)^2}$.

  J.Astola. On perfect codes in the Lee metric. Ann. Univ. Turku (A) 176 (1), 56. 1978.

- $PL(3, e) = \emptyset$ for $e \geq 2$.

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- $PL(4, e) = \emptyset$ for $e \geq 2$.

  S.Spacapan. A complete proof of the nonexistence of regular four dimensional tilings in the Lee metric. Preprint Series, vol.42(993). 2004

More relevant results related to the Golomb-Welch conjecture

- $PL(n, 2) = \emptyset$ for $5 \le n \le 12$ for linear codes

  P.Horak. On perfect Lee codes. Discr. Math. 309. 5551-5561. 2009.

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Future work

- Approach the classification of n-dimensional perfect single-error-correcting Lee codes using the geometry of polyominoes and combinatorial arguments.
- Prove the non-existence of e-perfect Lee codes in some dimension $n > 3$ using these techniques.
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  - S. Gravier, M. Mollard, C. Payan succeed for $n = 3$.
    On the nonexistence of three-dimensional tiling in the Lee metric II. Discr.Math 235, 151-157. 2001.
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Some references in polyominoes

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Thanks for your attention!