k–Nets in a Projective Plane over a Field

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joint work with
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- 3-nets (in particular, 3-nets realizing groups)

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  - Tetrahedron type 3-nets

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- Classification of 3-nets realizing group
- Some recent result on k-nets, $k \geq 4$.

Projective Plane $\mathrm{PG}(2, \mathbb{K})$

Let $\mathbb{K}$ be a field

Points:

$$P : (x, y, z) \in \mathbb{K} \times \mathbb{K} \times \mathbb{K}, \quad (x, y, z) \neq (0, 0, 0)$$

$$(x, y, z) \sim (kx, ky, kz), \quad \text{for} \quad k \in \mathbb{K} \setminus \{0\}$$

Lines:

$$\ell : aX + bY + cZ = 0, \; a, b, c \in \mathbb{K}, \; (a, b, c) \neq (0, 0, 0)$$

Incidence Relation $\mathcal{I}$:

$$P \mathcal{I} \ell \iff ax + by + cz = 0$$

Projective Planes

Definition

A projective plane $\mathcal{P}$ is a set of points and lines, together with an incidence relation between the points and the lines such that

1. *Any two distinct points are incident with a unique line.*
2. *Any two distinct lines are incident with a unique point.*
3. *There exists four points no three of which are incident with one line.*

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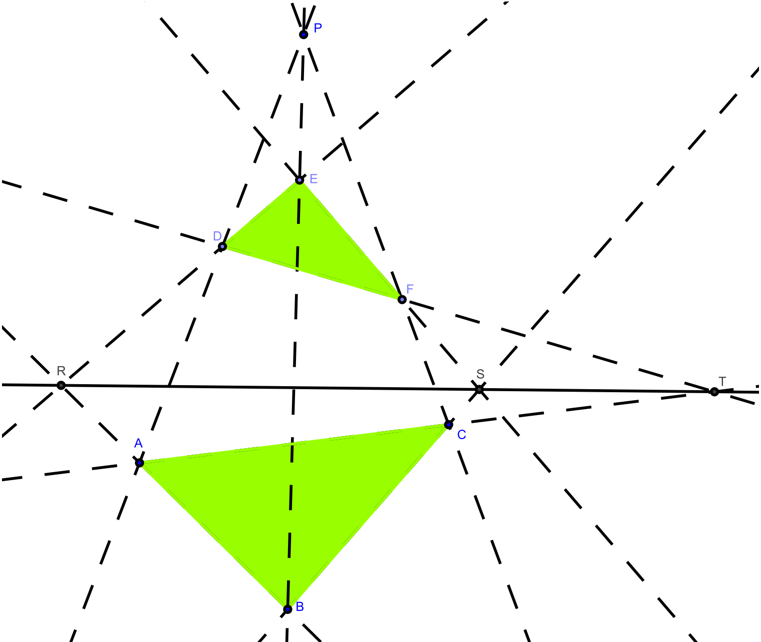
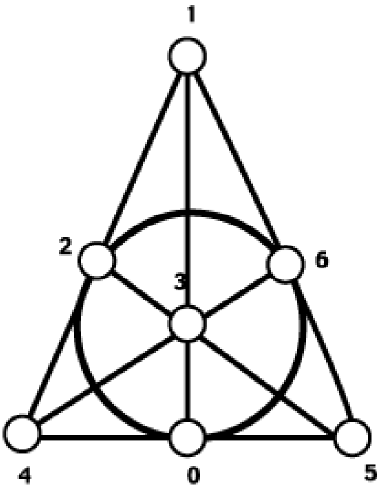
$PG(2, \mathbb{K})$ is a very particular projective plane. ... with a very special property.

Desargues' Theorem [the special property]

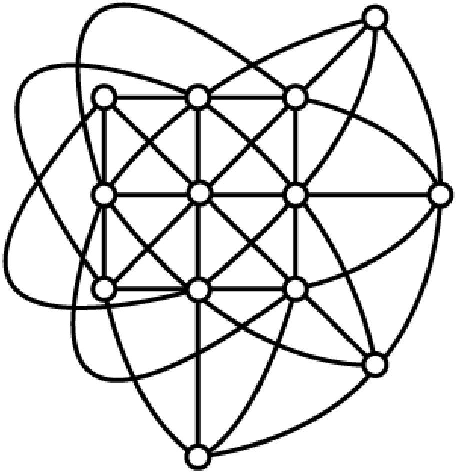


Fano Plane: $\mathrm{PG}(2, \mathrm{F}_2)$



(source: http://home.wlu.edu/∼mcraea/)

$PG(2, F_3)$



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3-nets

Definition

A 3-net in $PG(2,\mathbb{K})$ *is a pair* $(\mathcal{A},\mathcal{X})$ *where* $\mathcal{A}$ *is a finite set of lines partitioned into 3 subsets* $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ *and* $\mathcal{X}$ *is a finite set of points subject to the following conditions:*

- *for every* $i \neq j$ *and every* $\ell \in \mathcal{A}_i, \ell' \in \mathcal{A}_j$*, we have* $\ell \cap \ell' \in \mathcal{X}$
- *for every* $X \in \mathcal{X}$ *and every* i *(*$i \in \{1,2,3\}$*) there exists a unique line* $\ell \in \mathcal{A}_i$ *passing through* X*.*

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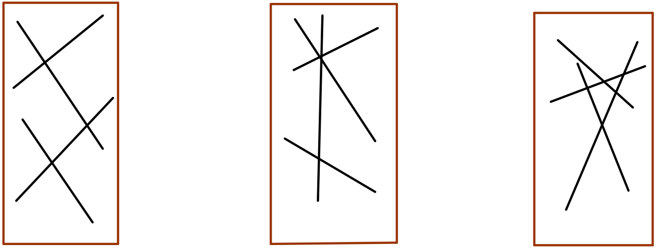
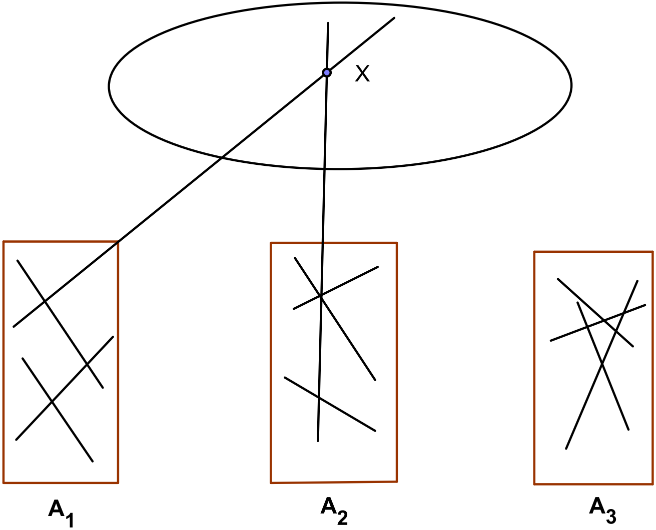
Note:
$|\mathcal{A}_1| = |\mathcal{A}_2| = |\mathcal{A}_3| = n, \ |\mathcal{X}| = n^2$
(n is the *order* of the 3-net)

3-nets



A_1 A_2 A_3

3-nets



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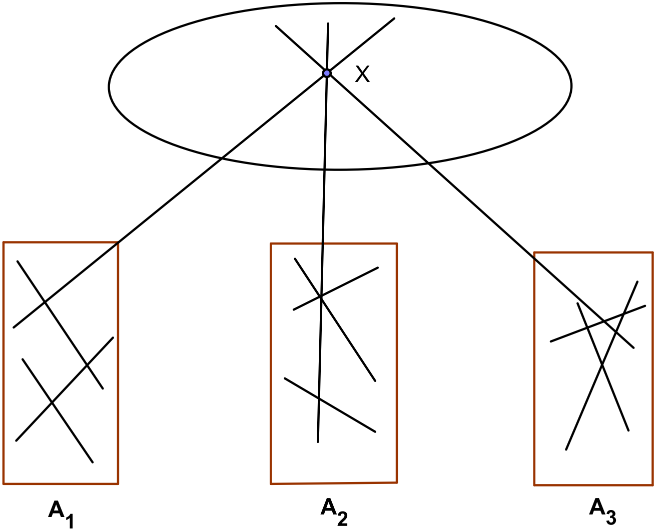

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- *for every $i \neq j$ and every $\ell \in \mathcal{A}_i, \ell' \in \mathcal{A}_j$, we have $\ell \cap \ell' \in \mathcal{X}$*
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1. *for every $i \neq j$ and every $P \in \mathcal{A}_i, P' \in \mathcal{A}_j$, we have that the line $\overline{PP'} \in \mathcal{X}$*
2. *for every $\omega \in \mathcal{X}$ and every i ($i \in \{1, 2, 3\}$) there exists a unique point $P \in \mathcal{A}_i$ passing through ω.*

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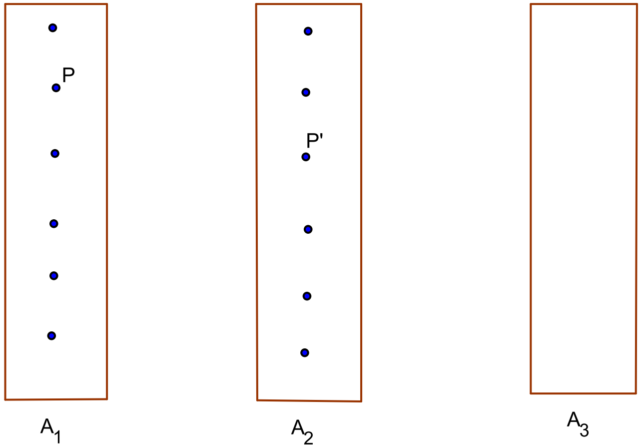

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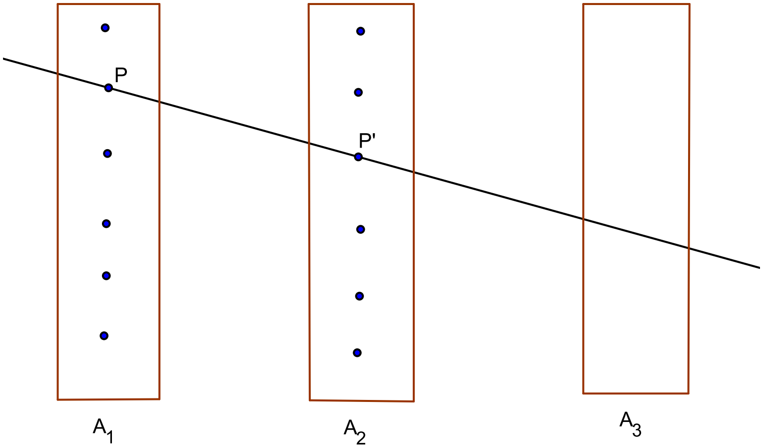

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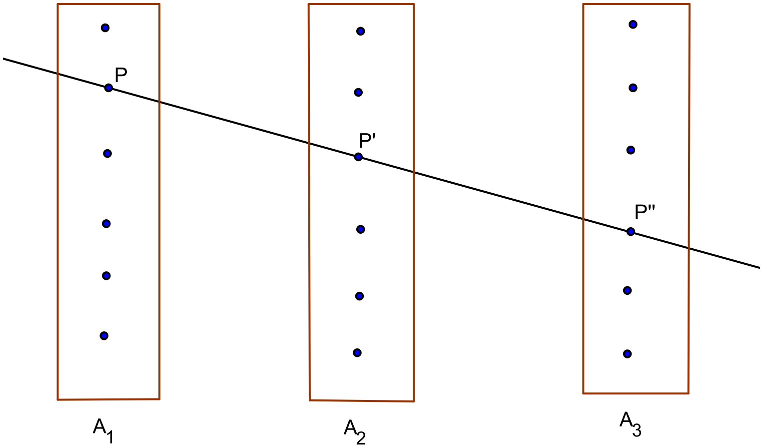
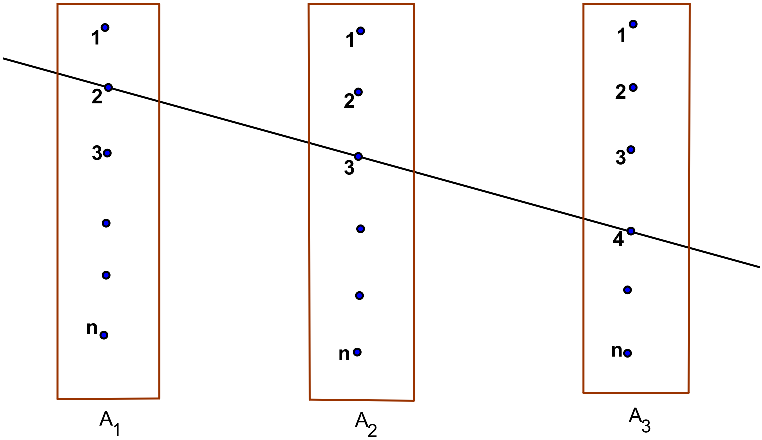


(dual) 3-nets, quasigroups, loops



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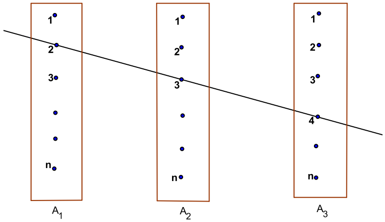


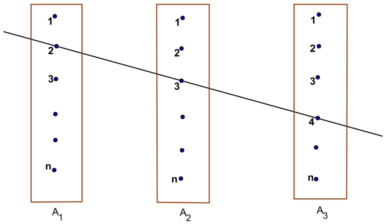
$\Rightarrow$

*	1	2	3	...	n
1			⋮		
2	...	...	4		
3					
⋮					
⋮					
n					

Definition

A quasigroup $(Q, *)$ *is a set* Q *with a binary operation* $*$*, such that for each* $a, b \in Q$*, there exist unique elements* x *and* y *in* Q *such that:*

$$a * x = b, \quad y * a = b.$$



$\Rightarrow$

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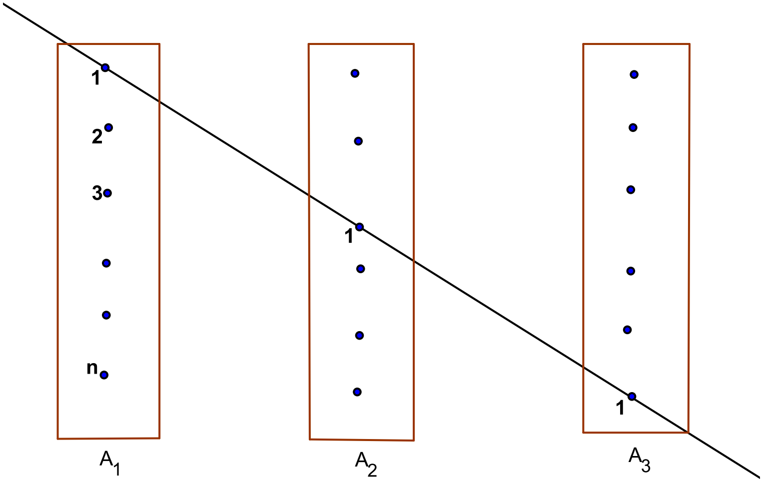
Definition

A loop is a quasigroup with an identity element e such that:

$$x * e = x = e * x$$

for all x in Q.

(dual) 3-nets, quasigroups, loops



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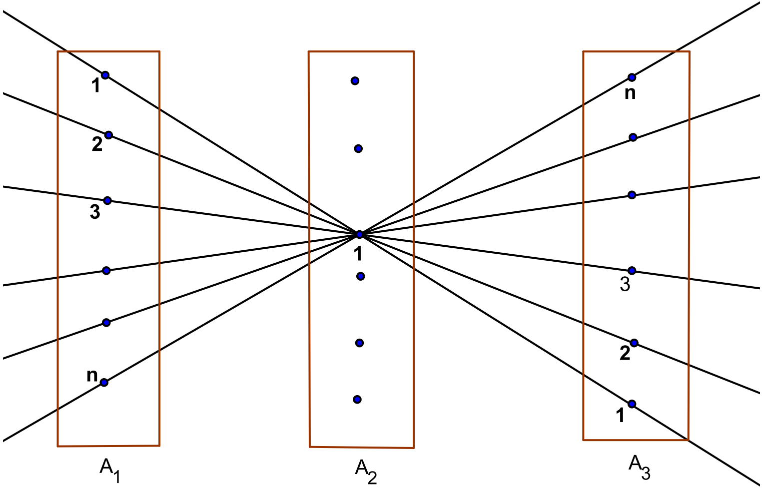

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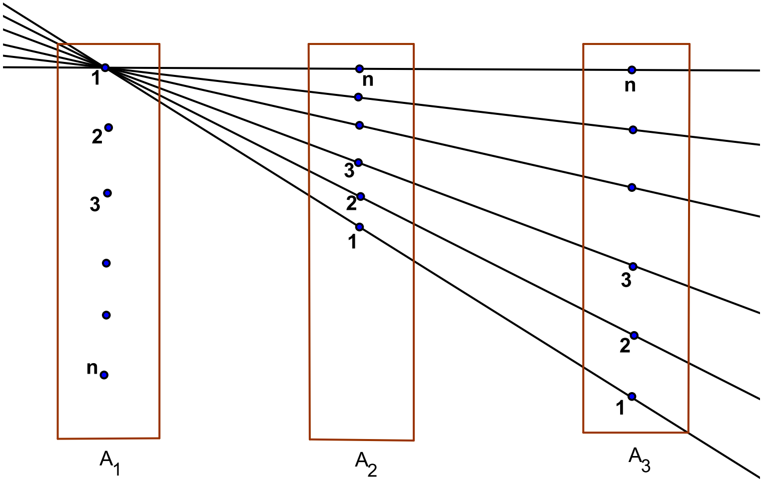
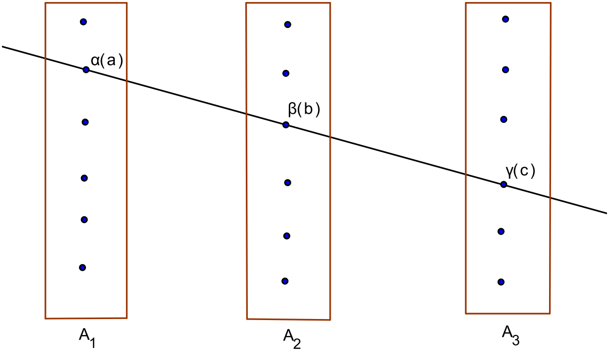


(dual) 3-nets realizing groups

A (dual) 3-net is said to realize a group $(G, \cdot)$ when it is coordinatized by G: if A_1, A_2, A_3 are the classes, there exists a triple of bijective maps from G to (A_1, A_2, A_3), say $\alpha : G \to A_1$, $\beta : G \to A_2$, $\gamma : G \to A_3$ such that $a \cdot b = c$ if and only if $\alpha(a)$, $\beta(b)$, $\gamma(c)$ are three collinear points, for any $a, b, c \in G$.



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    It depends on the characteristic of the field $\mathbb{K}$!
    If $n \geq 1$, $\mathrm{char}(\mathbb{K}) = 2$ and $\mathbb{K}$ "large enough", the group $(\mathbb{Z}_2)^n$ can be realized.
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- Some restrictions are needed. Our hypotheses are:
  (i) The 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizes a group G.
  (ii) $p > n$ or $p = 0$, where $|G| = n$ and p is the characteristic of the field.

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Algebraic dual 3-nets fall into subfamilies according as the plane cubic

- splits into three lines
- splits into an irreducible conic and a line
- is irreducible

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$P = (x_1, 0, 1)$, $Q = (1, -x_2, 0)$, $R = (0, x_3, 1)$, are collinear if and only if $x_1 x_2 = x_3$.

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We can assume $1 \in L_1$, $1 \in L_2$, $1 \in L_3$. Thus, $L = L_1 = L_2 = L_3$ is a finite multiplicative subgroup of $\mathbb{K}$. In particular, L is cyclic.

Pencil Type

Assume components of a dual 3-net (A_1, A_2, A_3) lie on three concurrent lines. These lines are assumed to be those with equations $Y = 0$, $X = 0$, $X - Y = 0$ respectively, so that the line of equation $Z = 0$ meets each component.

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The points in the components may be labeled such that
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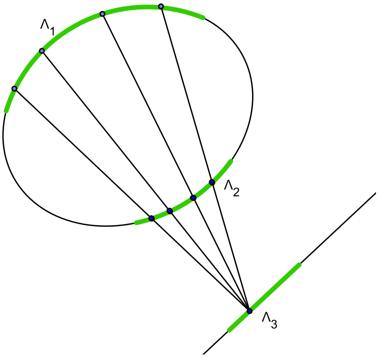
Note: n is a power of p, where p is the characteristic of the field $\mathbb{K}$
$\Rightarrow$ This case cannot occur if $p > n$.

Conic-Line Type



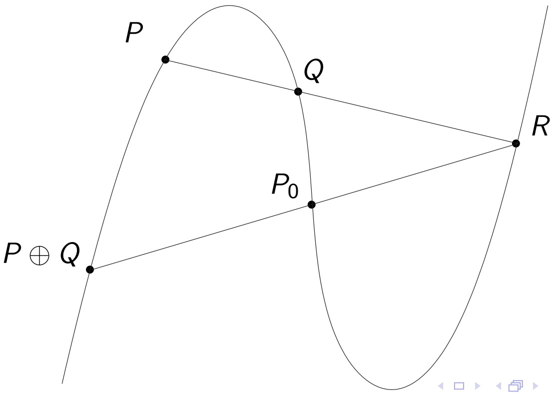
$(\Lambda_1, \Lambda_2, \Lambda_3)$:= dual 3-net of order n; $p > n$ or $p = 0$;

Proposition (Blokhuis, Korchmaros, Mazzocca, 2011). If Λ_3 is contained in a line then $(\Lambda_1, \Lambda_2, \Lambda_3)$ is either triangular or conic-line type. The same holds whenever $\Lambda_1 \cup \Lambda_2$ is contained in a conic.

Operation on Cubics

Proposition

A non-singular plane cubic $\mathcal{F}$ can be equipped with an additive group $(\mathcal{F}, +)$ on the set of all its points. If an inflection point P_0 of F is chosen to be the identity 0, then three distinct points $P, Q, R \in \mathcal{F}$ are collinear if and only if $P + Q + R = 0$.



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Proposition

Let $\mathcal{F}$ be an irreducible singular plane cubic with its unique singular point U, and define the operation $+$ on $\mathcal{F} \setminus \{U\}$ in exactly the same way as on a non–singular plane cubic. Then $(\mathcal{F}, +)$ is an abelian group isomorphic to the additive group of $\mathbb{K}$, or the multiplicative group of $\mathbb{K}$, according as U is a cusp or a node.

Examples of dual 3-nets from cubic curves

Theorem

Let G be the abelian group associated to a non-singular cubic curve $\mathcal{F}$. Take a finite subgroup H of G whose index is greater than two, with $0 \in H$, and choose three pairwise distinct cosets of H in G, say

$$A = a + H, \; B = b + H, \; C = c + H,$$

with $a, b, c \in G$ and collinear, i.e. $a + b + c = 0$. Then $A \cup B \cup C$ is a dual 3-net whose order is equal to the size of H.

A few results and conjectures

Can we realize non-abelian groups?

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What can we say about 3-nets realizing abelian groups?

A few results and conjectures

Can we realize non-abelian groups? YES

- Dihedral Group: $D_n = \langle x, y | x^2 = y^n = 1, y^x = y^{n-1} \rangle$, $n \geq 3$ (Pereira, Yuzvinsky, 2008; Stipins, 2007)
- Quaternions: $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, if $\mathrm{char}(\mathbb{K}) \neq 2$ (Urzua, 2007)

What can we say about 3-nets realizing abelian groups?

A few results and conjectures

Can we realize non-abelian groups? YES

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- Quaternions: $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, if $\mathrm{char}(\mathbb{K}) \neq 2$ (Urzua, 2007)

What can we say about 3-nets realizing abelian groups?

- A nice result: If an abelian group G contains an element of order ≥ 10 then every dual 3-net realizing G is algebraic. (Yuzvinsky, 2003)
- Conjecture (Yuzvinsky, 2003): Every 3-net realizing an abelian group is algebraic. (TRUE, Korchmaros, Nagy, Pace, 2012)

Subnets realizing subgroups

If $H \leq G$ and $\Gamma_1 = \alpha(H)$, $\Gamma_2 = \beta(H)$, $\Gamma_3 = \gamma(H)$,
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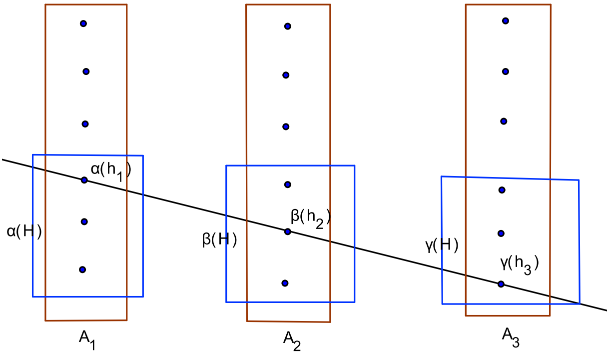

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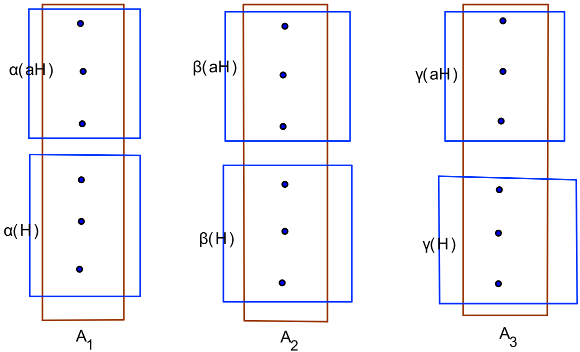

Lemma

Let (A_1, A_2, A_3) be a dual 3-net that realizes a group $(G, \cdot)$ of order kn containing a normal subgroup $(H, \cdot)$ of order n. For any two cosets g_1H and g_2H of H in G, let

$$\Gamma_1 = \alpha(g_1 H), \Gamma_2 = \beta(g_2 H) \text{ and } \Gamma_3 = \gamma((g_1 \cdot g_2)H).$$

Then $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a 3-subnet of (A_1, A_2, A_3) which realizes H.

Tetrahedron: Dihedral Group

The dual 3-net (A_1, A_2, A_3) is said to be tetrahedron-type if its components lie on the sides of a non-degenerate quadrangle such that $A_i = \Gamma_i \cup \Delta_i$, $|\Gamma_i| = |\Delta_i| = n$, and Γ_i and Δ_i are contained in opposite sides, for $i = 1, 2, 3$.

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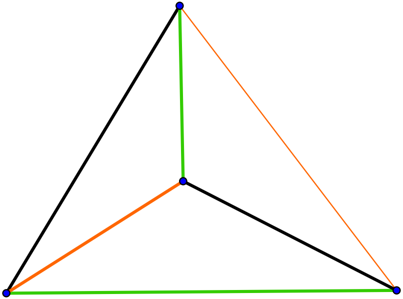

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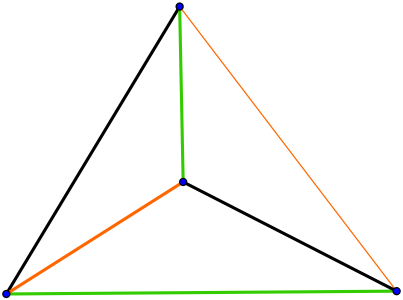

Theorem

Any tetrahedron-type dual 3-net realizes a dihedral group.

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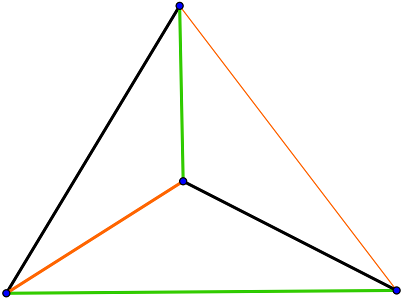

Theorem

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Theorem (Korchmaros, Nagy, Pace)

Any dual 3-net that realizes a dihedral group is of tetrahedron-type.

Classification of low order dual 3-nets

Proposition

Any dual 3–net realizing an abelian group of order ≤ 8 is algebraic.

Proposition

Any dual 3–net realizing an abelian group of order 9 is algebraic.

Proposition

If $p = 0$, no dual 3–net realizes Alt_4.

Reference:

G. Nagy, N. Pace, *On small 3-nets embedded in a projective plane over a field*, J. Combinatorial Theory, Series A, Volume 120, Issue 7, September 2013, Pages 1632–1641.

Dual 3-nets containing algebraic 3-subnets of order n with $n \geq 5$

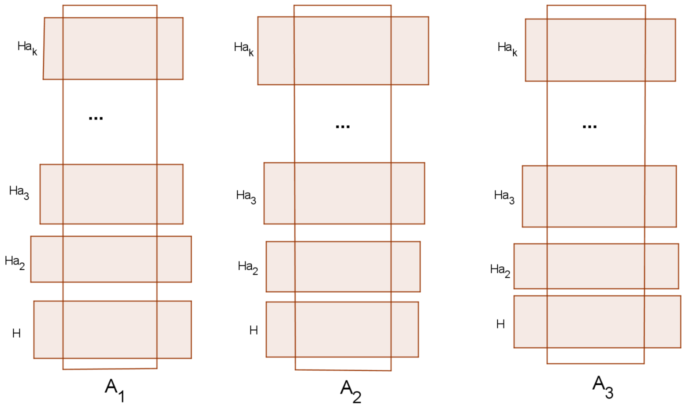
Proposition

Let $p = 0$ or $p > |G|$. Let G be a group containing a proper abelian normal subgroup H of order $n \geq 5$. If a dual 3-net $(\Lambda_1, \Lambda_2, \Lambda_3)$ realizes G such that all its dual 3-subnets realizing H as a subgroup of G are algebraic, then one of the following holds.

(i) *$(\Lambda_1, \Lambda_2, \Lambda_3)$ is algebraic, and G is either cyclic or the direct product of two cyclic groups.*

(ii) *$(\Lambda_1, \Lambda_2, \Lambda_3)$ is of tetrahedron type, and G is dihedral.*

Algebraic Subnets: Irreducible Cubic Case



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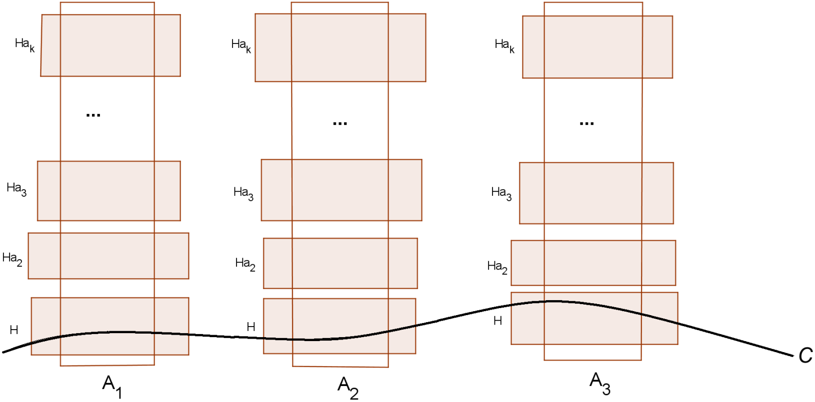

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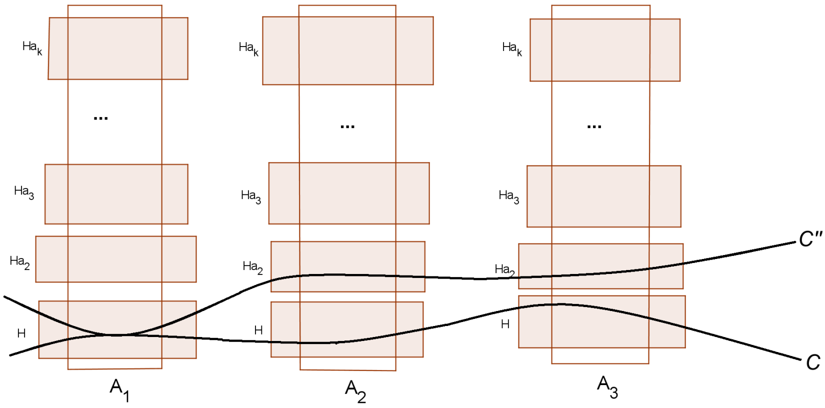

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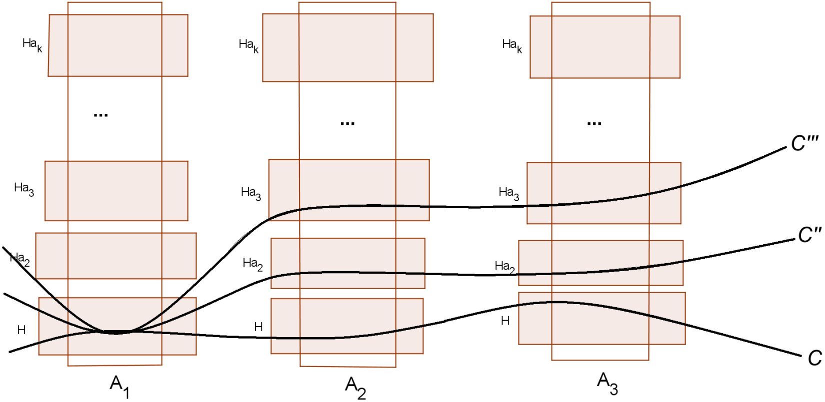

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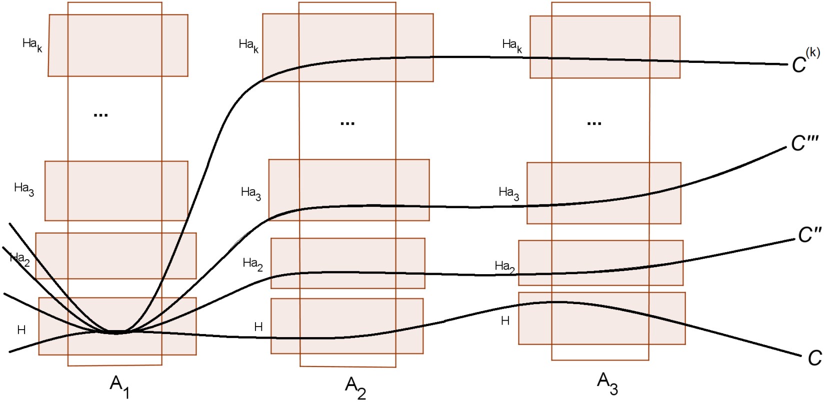

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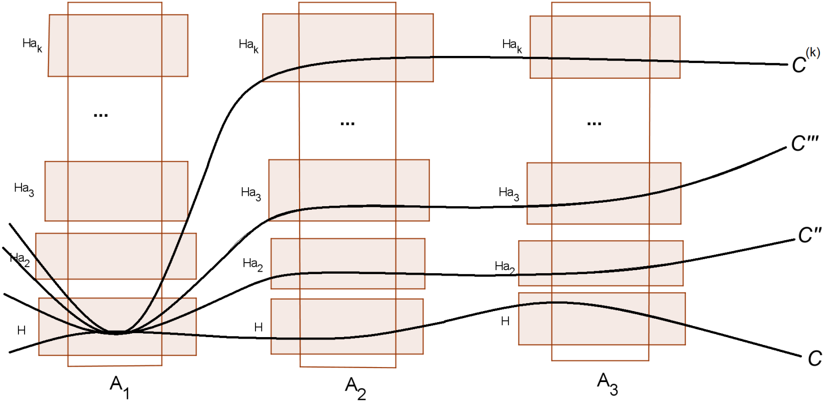

See also (G.Korchmaros, N.P., Coset Intersection of Irreducible Plane Cubics, to appear in Des. Codes and Cryptography, 2013).

Classification

Theorem (Korchmaros, Nagy, Pace)

In the projective plane $PG(2, \mathbb{K})$ defined over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$, let (A_1, A_2, A_3) be a dual 3-net of order $n \geq 4$ which realizes a group G. If either $p = 0$ or $p > n$ then one of the following holds:

Infinite families:

(I) *G is either cyclic or the direct product of two cyclic groups, and (A_1, A_2, A_3) is algebraic;*

(II) *G is dihedral and (A_1, A_2, A_3) is of tetrahedron type.*

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* If $p = 0$ then (IV) does not occur.

Open questions:

If $p > n$,

We couldn't find any example for the cases: Alt_4, Sym_4, Alt_5.

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Reference:
G.Korchmaros, G.Nagy, N.Pace, 3-nets realizing a group in a projective plane, to appear in J. Algebraic Combinatorics, 2013.

Definition

Let k be an integer, $k \geq 3$. A k-net in $PG(2, \mathbb{K})$ is a pair $(\mathcal{A}, \mathcal{X})$ where $\mathcal{A}$ is a finite set of lines partitioned into k subsets $\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{A}_i$ and $\mathcal{X}$ is a finite set of points subject to the following conditions:

1. *for every $i \neq j$ and every $\ell \in \mathcal{A}_i, \ell' \in \mathcal{A}_j$, we have $\ell \cap \ell' \in \mathcal{X}$*
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k-nets

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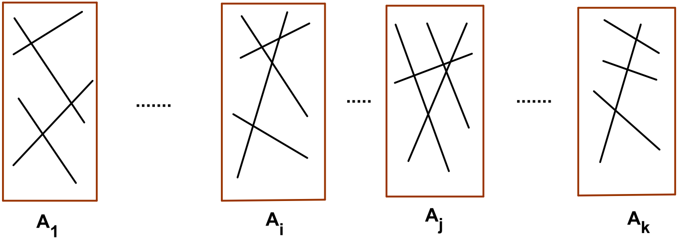
Note:
$|\mathcal{A}_1| = |\mathcal{A}_2| = \ldots = |\mathcal{A}_k| = n, \; |\mathcal{X}| = n^2$
(n is the *order* of the k-net)

k-nets



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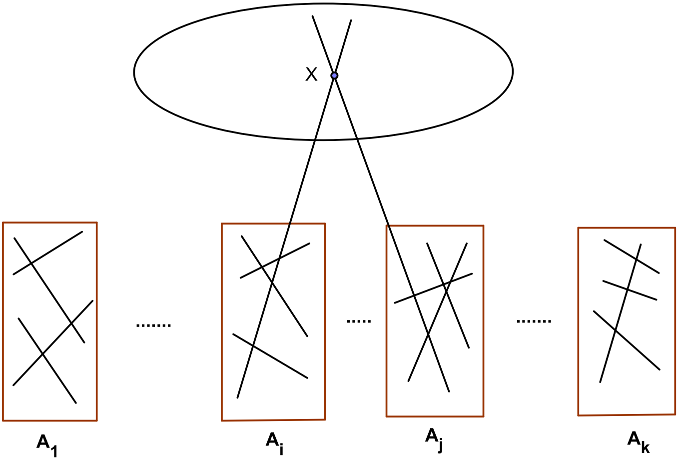

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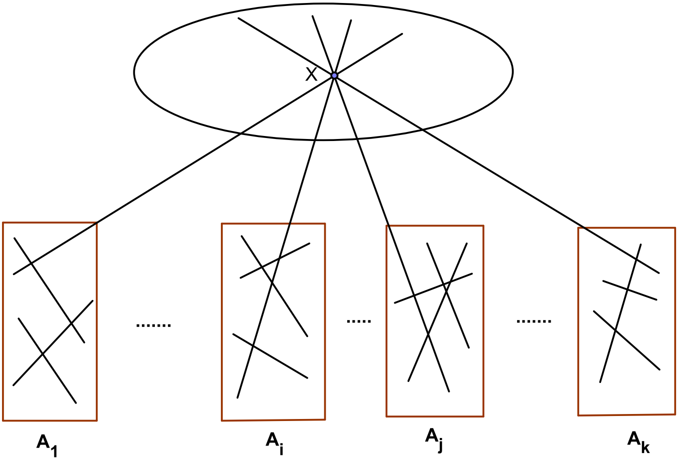

In the complex plane, we know only one 4-net up to projectivity.

k-nets in characteristic zero

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This 4-net, called the classical 4-net, has order 3 and it exists since $PG(2, \mathbb{C})$ contains an affine subplane $AG(2, \mathbb{F}_3)$ of order 3, unique up to projectivity, and the four parallel line classes of $AG(2, \mathbb{F}_3)$ are the components of a 4–net in $PG(2, \mathbb{C})$.

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By a result of Stipins, no k–net with $k \geq 5$ exists in $PG(2, \mathbb{C})$. Stipins' result holds true in $PG(2, \mathbb{K})$ provided that $\mathbb{K}$ has zero characteristic.

References:

J. Stipins, Old and new examples of k-nets in P2, math.AG/0701046.
S. Yuzvinsky, A new bound on the number of special fibers in a pencil of curves, Proc. Amer. Math. Soc. 137 (2009), 1641–1648.

k-nets in positive characteristic

Let $\mathbb{K}$ be a field of characteristic $p > 0$. In this case, $PG(2, \mathbb{K})$ contains an affine subplane $AG(2, F_p)$ of order p from which k–nets for $3 \leq k \leq p + 1$ arise taking k parallel line classes as components. Similarly, if $PG(2, \mathbb{K})$ contains an affine subplane $AG(2, F_{p^h})$, then k-nets of order p^h for $3 \leq k \leq p^h + 1$ exist in $PG(2, \mathbb{K})$.

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Theorem (Korchmaros, Nagy, Pace)

If $p > 3^{\varphi(n^2-n)}$, where φ is the classical Euler φ function, then $k \leq 4$. Moreover, This approach also works in zero characteristic and provides a new proof for Stipins' result.

Reference: G. Korchmaros, G. Nagy, N. Pace, k-nets embedded in a projective plane over a field (preprint arXiv:1306.5779)

Thank you!