A new construction of strength-3 covering arrays using linear feedback shift register (LFSR) sequences

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joint work with **Sebastian Raaphorst** and **Brett Stevens**

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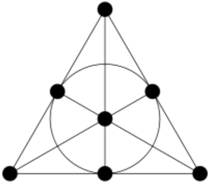
What are combinatorial designs?

Combinatorial designs are combinatorial objects such as arrays or set systems with some type of **“balance property”**.

- The construction in this talk relates many interesting combinatorial designs:
  block designs, Steiner triple systems, projective planes, orthogonal arrays, covering arrays.
- The construction uses LFSR sequences in finite fields to build **partial orthogonal arrays** that we transform into (complete) **covering arrays**.

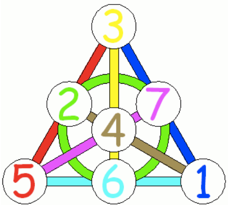


Steiner triple systems

Definition

A Steiner triple system of order n, $STS(n)$, is a set of 3-subsets (triples) of $X = \{1, 2, \ldots, n\}$ such that each unordered pair of elements of X appears in exactly 1 triple.

$STS(7)$:



$\{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{2,3,5\}, \{2,6,7\}, \{3,4,6\}, \{4,5,7\}$

Balanced incomplete block designs

A *balanced in complete block design*,

$\text{BIBD}(n, k, \lambda)$, λ k

Definition

A ~~Steiner triple system of order n~~, $STS(n)$, is a set of 3-subsets (~~triples~~) of $X = \{1, 2, \ldots, n\}$ such that each unordered pair of elements of X appears in exactly 1 triple.

~~$STS(7)$~~:

$\{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{2,3,5\}, \{2,6,7\}, \{3,4,6\}, \{4,5,7\}$

$$BIBD(n, 3, 1) = STS(n)$$

Ex: BIBD(13,4,1) =BIBD$(n^2+n+1, n+1, 1)$ for $n=3$

$\{\mathbf{0},\mathbf{1},\mathbf{3},\mathbf{9}\}$ ← difference set
$\{1,2,4,10\}$
$\{2,3,5,11\}$
$\{3,4,6,12\}$
$\{4,5,7,0\}$
$\{5,6,8,1\}$
$\{6,7,9,2\}$
$\{7,8,10,3\}$
$\{8,9,11,4\}$
$\{9,10,12,5\}$
$\{10,11,0,6\}$
$\{11,12,1,7\}$
$\{12,0,2,8\}$

all possible distances mod 13 appear exactly once as difference of two elements in $\{0,1,3,9\}$

Orthogonal arrays

Strength $t = 2$; $v = 3$ symbols; $k = 4$ columns; 2^3 rows

$$\begin{bmatrix} 0000 \\ 0122 \\ 1220 \\ 2202 \\ 2021 \\ 0211 \\ 2110 \\ 1101 \\ 1012 \end{bmatrix}$$

Definition: Orthogonal Array

An *orthogonal array* of strength t, k columns, v symbols and index λ denoted by $OA_\lambda(t, k, v)$, is an $\lambda v^t \times k$ array with symbols from $\{0, 1, \ldots, v-1\}$ such that in every $t \times N$ subarray, every t-tuple of $\{0, 1, \ldots, v-1\}^t$ appears in exactly λ rows.

Covering arrays

Strength $t = 3$; $v = 2$ symbols; $k = 10$ columns; $N = 13$ rows

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
1	1	1	0	1	0	0	0	0	1
1	0	1	1	0	1	0	1	0	0
1	0	0	0	1	1	1	0	0	0
0	1	1	0	0	1	0	0	1	0
0	0	1	0	1	0	1	1	1	0
1	1	0	1	0	0	1	0	1	0
0	0	0	1	1	1	0	0	1	1
0	0	1	1	0	0	1	0	0	1
0	1	0	1	1	0	0	1	0	0
1	0	0	0	0	0	0	1	1	1
0	1	0	0	0	1	1	1	0	1

Definition: Covering Array

A *covering array* of strength t, k factors, v symbols and size N, denoted by $CA(N; t, k, v)$, is an $N \times k$ array with symbols from $\{0, 1, \ldots, v-1\}$ such that in every $t \times N$ subarray, every t-tuple of $\{0, 1, \ldots, v-1\}^t$ is covered at least once.

Covering arrays

Strength $t = 3$; $v = 2$ symbols; $k = 10$ columns; $N = 13$ rows

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
1	1	1	0	1	0	0	0	0	1
1	0	1	1	0	1	0	1	0	0
1	0	0	0	1	1	1	0	0	0
0	1	1	0	0	1	0	0	1	0
0	0	1	0	1	0	1	1	1	0
1	1	0	1	0	0	1	0	1	0
0	0	0	1	1	1	0	0	1	1
0	0	1	1	0	0	1	0	0	1
0	1	0	1	1	0	0	1	0	0
1	0	0	0	0	0	0	1	1	1
0	1	0	0	0	1	1	1	0	1

Definition: Covering Array

A *covering array* of strength t, k factors, v symbols and size N, denoted by $CA(N; t, k, v)$, is an $N \times k$ array with symbols from $\{0, 1, \ldots, v-1\}$ such that in every $t \times N$ subarray, every t-tuple of $\{0, 1, \ldots, v-1\}^t$ is covered at least once.

Covering arrays generalize orthogonal arrays

$CAN(t,k,v) = \min\{N : CA(N;t,k,v) \text{ exists}\}$

An obvious lower bound: $CAN(t,k,v) \geq v^t$

An orthogonal array with index $\lambda = 1$: every every t-tuple of $\{0,1,\ldots,v-1\}^t$ appears exactly once in any t columns.
So, an $OA(t,k,v)$ is a $CA(v^t;t,k,v)$ that meets this lower bound.

For $t = 2$ and q a prime power, $\exists OA(2, k = q+1, q)$; $q-1$ MOLS.

For $t = 3$, the following orthogonal arrays exist:
$\exists OA(3,4,2), OA(3,4,3), OA(3,6,4), OA(3,6,5), OA(3,8,7)$, etc.
giving $CAN(3,4,2) = 2^3 = 8, \cdots, CAN(3,8,7) = 7^3 = 343$, etc.

Bose-Bush bound: $k \leq v+2$ is necessary for an $OA(3,k,v)$.

A Construction for Strength-3 Covering Arrays from Linear Feedback Shift Register Sequences

Work with Raaphorst, Stevens
Designs, Codes and Cryptography (September 2013).

- Use finite fields and linear feedback shift register sequences to build OA of strength 2 "almost" OA of strength 3.
- Build a CA of strength $t = 3$ by combining two of these "almost" OA of strength 3.
- We get a $CA(2q^3 - 1; 3, q^2 + q + 1; q)$.
- This improves upper bound for 512 parameter sets in Colbourn's covering array tables.

Our new construction of strength-3 covering arrays

The best CAs we can get from OAs are $CA(q^3; 3, q+2, q)$.
Our construction works for larger k up to $q^2 + q + 1$, guaranteeing an upper bound under a factor of 2 from the trivial lower bound.

Theorem (Construction for $t = 3$)

If q is a prime power then there exists a $CA(N = 2q^3 - 1; t = 3, k = q^2 + q + 1; v = q)$.

We will use liner feedback shift register sequences LFSR to build "partial" OAs (variable strength OA) that are concatenated vertically to create the CAs.

Example: our construction for $q = 2$

We get a $CA(2q^3 - 1; 3, q^2 + q + 1; q) = CA(15; 3, 7, 2)$

LFSR sequences of maximal period:

$$\underline{0011101}00111010011101\cdots$$

	0123456			
r_0:	0000000	uncovered triples		concatenate with reversals
r_1:	0011101	015	r_8:	1011100
r_2:	0111010	046	r_0:	0101110
r_3:	1110100	356	r_{10}:	0010111
r_4:	1101001	245	r_{11}:	1001011
r_5:	1010011	134	r_{12}:	1100101
r_6:	0100111	023	r_{13}:	1110010
r_7:	1001110	126	r_{14}:	0111001
		$BIBD(7, 3, 1)$		

Example: our construction for $q = 3$

We get a $CA(2q^3 - 1; 3, q^2 + q + 1; q) = CA(53; 3, 13, 3)$

$\underline{0121120111002}02122102220010121120111002021221022200 1\cdots$

	0123456789abc	$BIBD(q^2+q+1, 3, q-1)$		
r_0:	0000000000000	uncovered triples		conc. reversals
r_1:	0121120111002	3-sets of 06ab	r_{27} :	2001110211210
r_2:	1211201110020	3-sets of 59ac	r_{28} :	0200111021121
	$\cdots$	$\cdots$		$\cdots$
r_{12}:	0202122102220	3-sets of 028c	r_{38} :	0222012212020
r_{13}:	2021221022200	3-sets of 17bc	r_{39} :	0022201221202
r_{14}:	0212210222001	3-sets of 06ab	r_{40} :	1002220122120
r_{15}:	2122102220010	3-sets of 59ac	r_{41} :	0100222012212
	$\cdots$	$\cdots$		$\cdots$
r_{25}:	0101211201110	3-sets of 028c	r_{51} :	0111021121010
r_{26}:	1012112011100	3-sets of 17bc	r_{52} :	0011102112101
	matrix M	$BIBD(13, 3, 2)$		reversed M

A closer look at LFSRs

a linear feedback shift register sequence with primitive characteristic polynomial $f(x) = x^3 + 0x^2 + 2x + 1$ of degree $t = 3$ over $GF(q)$, $q = 3$ is defined by:
set arbitrary initial conditions (not all-zero): $a_0 = 0, a_1 = 1, a_2 = 2$
use f to define: $a_n = 0 \times a_{n-1} - 2 \times a_{n-2} - 1 \times a_{n-3}, \quad n \geq 3$
Because f is **primitive**, the sequence has **maximum period** $q^t - 1 = q^3 - 1 = 26$

$$\underline{0121120111002}02122102220010121120111002021221022200 1\cdots$$

properties:

- each nonzero 3-tuple of GF(q) appears once per period, starting at positions $i = 0, \ldots, q^3 - 2$
- the patterns of zeroes is the same at adjacent windows of size $q^3 - 1/(q-1) = q^2 + q + 1$
- there are exactly $q + 1$ such zeroes.

Variable strength orthogonal arrays (VOA)

Let f be a degree-t **primitive polynomial** over $GF(q)$ with root $\alpha \in GF(q^t)$. Then $\{1, \alpha, \alpha^2, \ldots, \alpha^{m-1}\}$ is a basis for $GF(q^t)$.
Consider the **LFSR sequence** with initial values $T = (a_0, \ldots, a_{t-1})$ not all zero and characteristic polynomial f.
Let $k = \frac{q^t-1}{q-1}$. Consider the following $q^t \times k$ array:

$$M = M(f,T) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ a_0 & a_1 & \ldots & a_{k-1} \\ a_1 & a_2 & \ldots & a_k \\ \vdots & \vdots & & \vdots \\ a_{q^t-2} & a_{q^t-1} & \ldots & a_{q^t-2+k-1} \end{bmatrix}$$

Every t consecutive columns have their q^t tuples covered.
Usually, M is not $OA(t,k,q)$: not all triples of columns **covered**.
Call Λ the hypergraph with hyper-edges the t-tuples of columns that are covered. We call M a $VOA(q^t; \Lambda, q)$.

For $t = 2$, M is the old construction for $OA(2, q+1, q)$

$t = 2$, $q = 3$, $k = \frac{q^2-1}{q-1} = q + 1$.
$T = (0, 1)$; $f(x) = x^2 + x + 2$, degree$(f) = t = 2$
LFSR: $01220211012202110122021 1 \cdots$

$$M = M(f,T) = \left[\begin{array}{c} 0000 \\ \hline 0122 \\ 1220 \\ 2202 \\ 2021 \\ 0211 \\ 2110 \\ 1101 \\ 1012 \end{array}\right]$$

is an orthogonal array of strength $t = 2$!!!

Our construction focus on $M = M(T, f)$ for $t = 3$

M is a $VOA(3, \Lambda, q)$ for $\Lambda = BIBD(q^2 + q + 1, 3, q^2)$

0121120111002021221022200101211201110020212210222001 $\cdots$

	0123456789abc	$BIBD(q^2 + q + 1, 3, q - 1)$
r_0:	0000000000000	uncovered triples
r_1:	0121120111002	3-sets of 06ab
r_2:	1211201110020	3-sets of 59ac
	$\cdots$	$\cdots$
r_{12}:	0202122102220	3-sets of 028c
r_{13}:	2021221022200	3-sets of 17bc
r_{14}:	0212210222001	3-sets of 06ab
r_{15}:	2122102220010	3-sets of 59ac
	$\cdots$	$\cdots$
r_{25}:	0101211201110	3-sets of 028c
r_{26}:	1012112011100	3-sets of 17bc

Triples of zeroes in rows of M relate to coverage

Theorem

Let q be a prime power, f be a primitive polynomial of degree 3 over $GF(q)$ with root $\alpha \in GF(q^3)$, and let $k = \frac{q^3-1}{q-1} = q^2 + q + 1$. Consider the $q^3 \times k$ array $M = M(f)$, the subinterval array of f. Then M is a $VOA(q^3; \Lambda, q)$, and for a set $\{i_0, i_1, i_2\}$, $0 \leq i_0 < i_1 < i_2 < q^2 + q + 1$, the following are equivalent:

1. $\{i_0, i_1, i_2\} \in \Lambda$ *(i.e. $\{i_0, i_1, i_2\}$ is "covered" in M).*
2. *There is no row r in M, $0 \leq r < q^3$, other than the all-zero row such that $M_{r,i_0} = M_{r,i_1} = M_{r,i_2} = 0$.*
3. $\{\alpha^{i_0}, \alpha^{i_1}, \alpha^{i_2}\}$ *is linearly independent over $GF(q)$.*

The structure of zeroes in rows of M

Theorem

Let f be a primitive polynomial f of degree 3 over $GF(q)$.

1. *Define $M = M(f)$ as before, the set $\mathcal{B} = \{\{a_1, \ldots, a_{q+1}\} : M_{i,a_1} = \ldots = M_{i,a_{q+1}} = 0$ for some $0 \leq i < q^3 - 1\}$ is the set of blocks of a projective plane of order q.*
2. *Consider Λ associated with $M = M(f)$. Then, $\binom{V}{3} \setminus \Lambda$ is a simple $BIBD(q^2 + q + 1, 3, q - 1)$, and Λ is a simple $BIBD(q^2 + q + 1, 3, q^2)$.*

Key properties: completing coverage with "reversal" of M

- Let $H = \{0 \leq i < k : a_i = 0\}$ (pos of 0's in 1st row of M).
  H is a $(q^2 + q + 1, q + 1, 1)$-difference set.
  Its translates are the blocks of the projective plane $\mathcal{B}$.
- If $\{a, b, c\} \subset H$ with $a < b < c$, then,
  $b - c + a \bmod q^2 + q + 1 \notin H$.
- Let $D = \{a, b, c\}$ with $0 \leq a < b < c < q^2 + q + 1$. If triple of columns D is uncovered in $M(f)$, then $D' = \{a, b, b - c + a\}$ is covered in $M(f)$.
- Let $\hat{f} = f(1/x)x^{\deg(f)}$, the reciprocal polynomial of f.
  If $D = \{a, b, c\}$ is not covered in $M(f)$, then D is covered in $M(\hat{f})$.
- $M(\hat{f})$ is obtained by reversal (mirror image) of $M(f)$.

For $t = 3$, there exists a $CA(2q^3 - 1; 3, q^2 + q + 1; q)$

$01211201110020212210222001\,01211201110020212210222001 \cdots$

	$M(f)$	$BIBD(q^2 + q + 1, 3, q - 1)$		$M(\hat{f})$
r_0:	0000000000000	uncovered triples		~~0000000000000~~
r_1:	0121120111002	3-sets of 06ab	r_{27} :	2001110211210
r_2:	1211201110020	3-sets of 59ac	r_{28} :	0200111021121
	...	...		...
r_{12}:	0202122102220	3-sets of 028c	r_{38} :	0222012212020
r_{13}:	2021221022200	3-sets of 17bc	r_{39} :	0022201221202
r_{14}:	0212210222001	3-sets of 06ab	r_{40} :	1002220122120
r_{15}:	2122102220010	3-sets of 59ac	r_{41} :	0100222012212
	...	...		...
r_{25}:	0101211201110	3-sets of 028c	r_{51} :	0111021121010
r_{26}:	1012112011100	3-sets of 17bc	r_{52} :	0011102112101
		$BIBD(13, 3, 2)$		

Improved CA bounds: $q \leq 25$, prime powers

q	k	new N	old N
2	7	15	12
3	13	53	50
4	21	**127**	152
5	31	**249**	365
7	57	**685**	1015
8	73	**1023**	1492
9	91	**1457**	2169
11	133	**2661**	3971
13	183	**4393**	6565
16	273	**8191**	12226
17	307	**9825**	15874
19	381	**13717**	24158
23	553	**24333**	38590
25	651	**31249**	49346

← improved upper bounds in Colbourn's CAs table for all $q \neq 2, 3$, $q \leq 25$

Improved bounds for $v \leq 25$, non prime powers

Non-prime-powers: "drop the symbols+fusion" for the next prime power.

$v \leq q$	k	new N	old N
6	57	684	624
10	133	**2659**	3794
12	183	**4391**	6350
14	273	**8187**	11996
15	273	**8189**	11998
18	381	**13715**	20191
20	553	**24327**	35941
21	553	**24329**	35943
22	553	**24331**	35945
24	651	**31247**	46196

← improved upper bounds in Colbourn's CAs table for all $v \neq 2, 3, 6$, $v \leq 25$

Improving upper bounds for higher k

- Using constructed CAs as ingredients in recursive constructions we improve many upper bounds.
- Before-and-after run of Colbourn tables of best bounds, gives upper bound improvements for 512 (ranges of) parameter sets. `http://www.public.asu.edu/~ccolbou/src/tabby/catable.html`

Open Problem: How this extends to $t \geq 4$?

For general t, q, $M(f)$ is a $q^t \times \frac{q^t-1}{q-1}$ array which is a $VOA(q^t, \Lambda, q)$ for some hypergraph Λ on $\frac{q^t-1}{q-1}$ vertices.

- Find s permutations of the columns of $M(f)$ such that the vertical concatenation of the s permuted $M(f)$ is a $CA(s(q^t - 1) + 1; t, \frac{q^t-1}{q-1}, q)$.
- Determine $s(t, q)$ the smallest such s.
  From our constructions, we know $s(2, q) = 1$, $s(3, q) = 2$.
  We experimentally determined $s(4, 2) \leq 4$, $s(4, 3) \leq 6$, $s(5, 2) \leq 9$; none of these cases improved best bounds.
- Determine a largest subset of the $\frac{q^t-1}{q-1}$ columns where it is enough to paste $s = 2$ matrices. For $t = 4$, this would lead to $CA(N = 2q^4 - 1; t = 4, k, q)$ where $k \leq \frac{q^t-1}{q-1}$.
- Study the structure of Λ (covered t-tuples) for $t \geq 4$, to get insight on constructions.