Blocking Sets of the Hermitian Unital

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Overview

1. Blocking sets on Hermitian curves
2. A lower bound
3. Background: Unitals via difference sets
4. A geometric construction
5. Explicit examples

The talk is based on joint work with A. Blokhuis, A. Brouwer, V. Krcadinac, S. Rottey, L. Storme, T. Szőnyi and P. Vandendriessche.

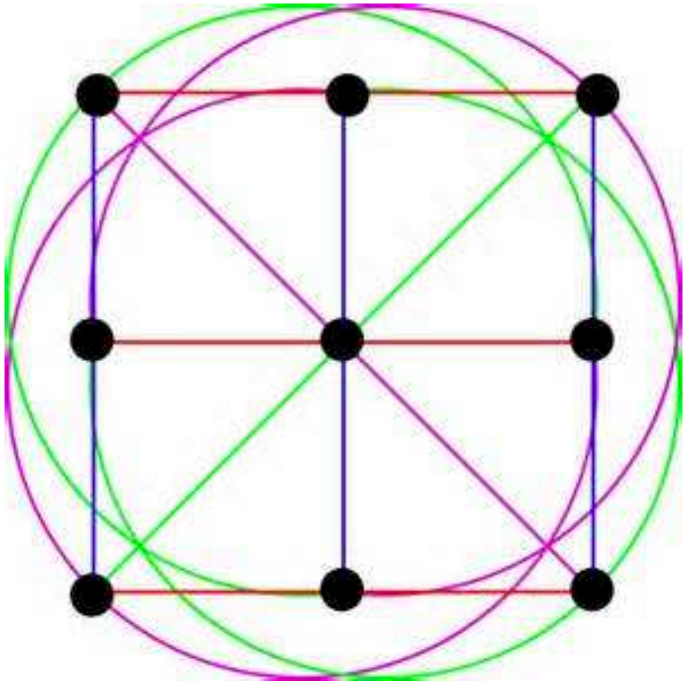
Hermitian units

- **Hermitian curve** $\mathcal{H}_{2,q^2}$ in PG$(2,q^2)$:

$$\mathcal{H}_{2,q^2} : \begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x^q \\ y^q \\ z^q \end{pmatrix} = 0,$$

  with $\det(A) \neq 0$, $A = (a_{ij})$, and $a_{ij}^q = a_{ji}$.
- Any line of PG$(2,q^2)$ intersects $\mathcal{H}_{2,q^2}$ in 1 point (tangent) or in $q+1$ points (secant).
- A secant intersects $\mathcal{H}_{2,q^2}$ in a Baer subline PG$(1,q)$ (**block**).
- Classical $(q^3+1, q+1, 1)$-design (**Hermitian unital**).

An example



$\mathcal{H}_{2,4}$ *yields* $AG(2,3)$ *embedded in PG(2,4)*

Blocking sets

Definition.

1. *Blocking set* B on $\mathcal{H}_{2,q^2}$: a set of points intersecting every block, but not containing any block completely.
2. *Minimal* blocking set B: no proper subset of B still is a blocking set.

Computer Results (A. Al-Azemi, A. Betten and D. Betten, *Unital designs with blocking sets*):

- 68806 different 2-$(28, 4, 1)$ unital designs have blocking sets.
- $\mathcal{H}_{2,9}$: no blocking sets.

A lower bound

Theorem. Let B be a blocking set of a Hermitian unital $\mathcal{U}$ in $\mathrm{PG}(2, q^2)$, $q = p^h$, p prime. Then

$$|B| \geq \frac{3q^2 - 2q - 1}{2} = q^2 - q + 1 + \frac{q^2 - 3}{2}.$$

The setup:

- Points of $\mathcal{U}$: $(x : y : z)$ with $(x : y : z)\, I\, [z^q : y^q : x^q]$,

  so $xz^q + y^{q+1} + zx^q = 0$.
- Tangents of $\mathcal{U}$: the lines $[t : u : v]$ with $tv^q + u^{q+1} + vt^q = 0$.
- Line at infinity: $z = 0$, the tangent in $(1 : 0 : 0)$.

- $B = S \cup \{(1:0:0)\}$
- $S := \{(a,b) \mid (a:b:1) \in B\}$
- Line $[1:u:v] : X + uY + v = 0$
- Tangent line $[1:u:v] : v^q + v + u^{q+1} = 0$

- A unital point outside B is on q^2 unital lines: $|S| \geq q^2$
- $|B| \; = \; |S| + 1 \; =: \; q^2 - q + 1 + k$
- *Claim:* $k \geq \frac{1}{2}(q^2 - 3)$
- W.l.o.g. $|S| < 2q^2 - q - 1$
- B minimal $\implies b \neq 0$ for all $(a,b) \in S$

The polynomial method

$$H(U,V) = C(U,V)R(U,V)$$
$$:= (V^q + V + U^{q+1}) \prod_{(a,b)\in S} (V + a + bU)$$

$H(U,V)$ vanishes identically on $\mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$!

$$H(U,V) = (V^{q^2} - V)f(U,V) + (U^{q^2} - U)g(U,V)$$

with

- $\deg(f), \deg(g) \leq k+1$
- $\deg(f) = k+1, \quad \deg_V f = k$

- Common linear factor $V + a_i + Ub_i$ of $f(U,V)$ and $g(U,V)$
  $\hat{=}$ non-necessary point for B.
- $C(U,V)$ divides $f(U,V)$ and $g(U,V)$
  $\hat{=}$ B blocking set of $\mathrm{PG}(2,q^2)$, so $B = \mathcal{H}_{2,q^2}$
- f and g are coprime.
- If $f(u,v) = 0$, then also $g(u,v) = 0$.
- $f(u,V)$ is fully reducible over $\mathbb{F}_{q^2}$ for all $u \in \mathbb{F}_{q^2}$.
- Let $f = f_0 \cdots f_m$ be the factorization of f into irreducible components.

Case 1

There is an irreducible factor f_0 of f with $\partial_V f_0 \not\equiv 0$.

- Put $m := \deg f_0$, so that $1 \leq m \leq \deg f = k+1$.

  Then $\deg_V(f_0) = m - \epsilon$, with $\epsilon \in \{0,1\}$, and $\epsilon = 0$ for $m = 1$.

- Let N be the number of zeros of f_0 in $\mathbb{F}_{q^2}^2$.
- By Bézout's theorem, $N \leq \deg f_0 \deg g \leq m(k+1)$.

- As $f(u,V)$ is fully reducible for all u, the number M of zeros *counted with multiplicity* is $q^2(m-\epsilon)$.
- Now $N \geq M - m(m-1)$.

- Hence $q^2(m-\epsilon) - m(m-1) \leq m(k+1)$.
- By case analysis, $k \geq \frac{1}{2}(q^2-3)$.

Case 2

$\partial_V f_i \equiv 0$ for all irreducible factors f_i of f.

- $f(u, V)$ is a p-th power.
- The multiplicity of v as a root of $H(u, V) = (V^{q^2} - V)f(u, V)$ is $1 \pmod p$.
- All (non-horizontal) secants intersect B in $1 \pmod p$ points.

- Summing over a parallel class of $\mathcal{U}$:

$$|B| \equiv (q^2 - q + 1) \cdot 1 \equiv 1 \pmod p.$$

- Summing over the q^2 lines through a point $p \notin B$:

$$|B| \equiv q^2 \cdot 1 \equiv 0 \pmod p.$$

Unitals via difference sets

- Represent $\mathrm{PG}(2, q^2)$ via a planar difference set D in the cyclic group G of order $q^4 + q^2 + 1$.
- Let D be fixed by every multiplier.
- $G = A \times B$, where $|A| = q^2 - q + 1$ and $|B| = q^2 + q + 1$.
- The cosets of A are arcs, the cosets of B Baer subplanes.
- Elements of G: pairs (i, j) with $0 \leq i \leq q^2 - q$ and $0 \leq j \leq q^2 + q$.
- The multiplier q^3 maps (i, j) to $(-i, j)$.
- $g \mapsto D - q^3 g$ defines a Hermitian polarity.
- The absolute points give the Hermitian unital $\mathcal{U} = \{a + \beta \,|\, a \in A, 2\beta \in B \cap D\}$.
- $\mathcal{U}$ is the union of $q + 1$ cosets of A.

A geometric construction

Theorem. $\mathcal{H}_{2,q^2}$, with $q \geq 7$, has blocking sets of size

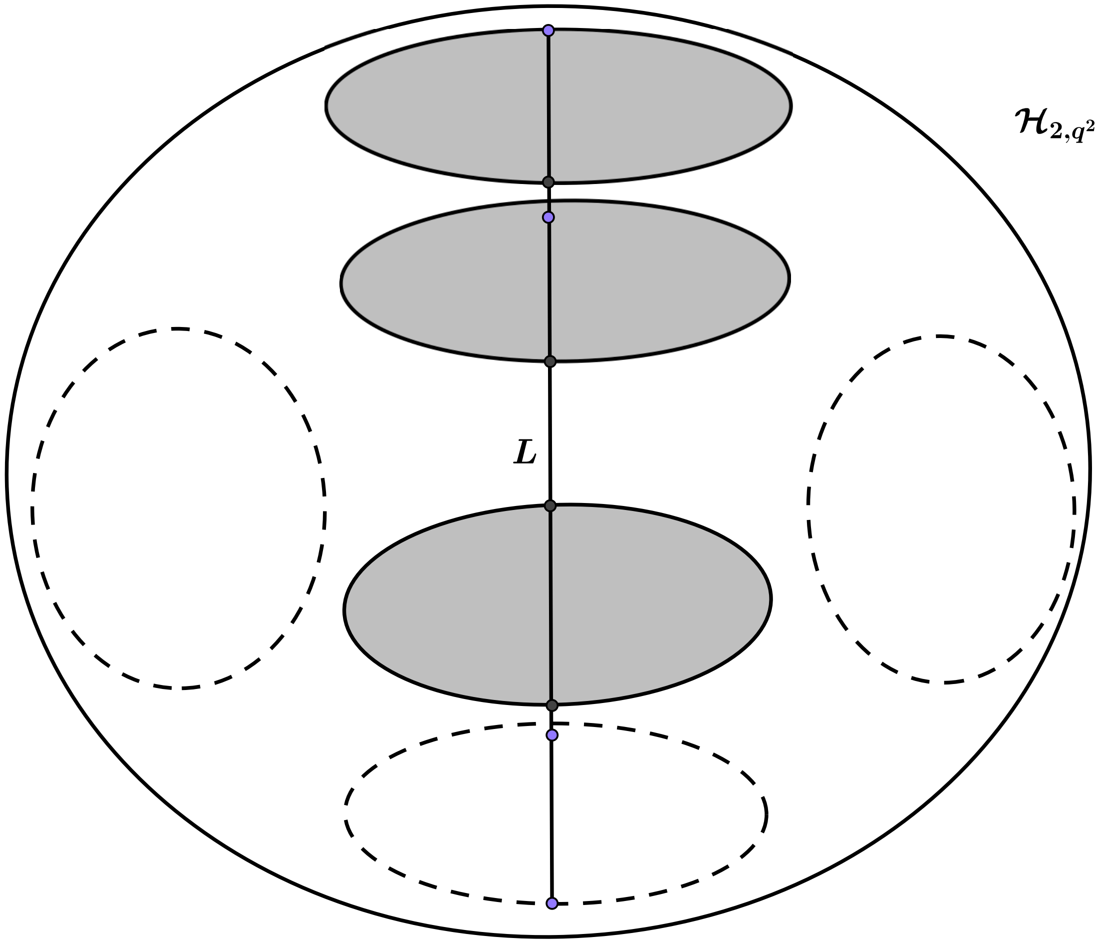
$$\frac{q^3+1}{2} \quad \text{if } q \text{ is odd,}$$

$$\frac{q^3-q^2+q}{2} \quad \text{if } q \text{ is even.}$$

Idea of proof:

- Let q be odd. Partition the $q+1$ cosets of A into two sets of size $(q+1)/2$ such that the union of each is a blocking set of $\mathcal{U}$.
- If q is even, partition $\mathcal{U}$ into collections of $q/2$ and $q/2+1$ cosets of A forming blocking sets.

Hermitian curve partitioned into arcs



Proof for q odd

The point set of $\mathcal{U}$ is $A + \frac{1}{2}(B \cap D)$, and $\frac{1}{2}(B \cap D)$ is an oval O in the Baer subplane B.

Lines have three types of intersection pattern with the $\mathcal{U}$-cosets of A:

- A tangent of O is also a tangent of the unital $\mathcal{U}$.
- A secant of O intersects two $\mathcal{U}$-cosets in a single point, and the remaining ones in 0 or 2 points. Both cases occur $(q-1)/2$ times.
- An external line of O intersects all $\mathcal{U}$-cosets of A in 0 or 2 points. Both cases occur $(q+1)/2$ times.

The $(q^2 - q)/2$ external lines give partitions of the set of $\mathcal{U}$-cosets *not* leading to blocking sets of $\mathcal{U}$.

As $\frac{1}{2}\binom{q+1}{(q+1)/2} > \frac{1}{2}(q^2 - q)$ for $q \geq 7$, the desired partition of the $\mathcal{U}$-cosets exists.

The main result

- $q = 2$: Non-existence (well-known)
- $q = 3$: Non-existence by computer search
- $q = 4$: Method works!
- $q = 5$: Method fails, but a random greedy computer search gives blocking sets of all sizes from 45 to 81.

Main Theorem.

The Hermitian unital in PG$(2, q^2)$ contains a blocking set if and only if $q > 3$.

An alternative construction

Theorem.

Let $r|(q-1)$, where $r > 1$ and $4r^2 + 1 < q$.

Then the Hermitian unital in $\mathrm{PG}(2, q^2)$ contains a blocking set B of size $k + q(q-1)^2/r$ for some k with $1 \leq k \leq q^2 - q + 1$.

For $r \sim \sqrt{q}/2$, this result leads to proper blocking sets of size approximately $2q^2\sqrt{q}$.

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Sketch of proof for q odd

- We again use the Hermitian curve $\mathcal{H}$ with affine equation $X^q + X + Y^{q+1} = 0$.
- Choose a non-square $k \in \mathbb{F}_q$ and $i \in \mathbb{F}_{q^2}$ with $i^2 = k$.
  Now the elements of $\mathbb{F}_{q^2}$ are $x = x_1 + ix_2$, with $x_1, x_2 \in \mathbb{F}_q$.
- Put $B := \{(x, y) \in \mathcal{H} \mid y = u^r + iv,\ \text{with } u, v \in \mathbb{F}_q\} \cup \{(1 : 0 : 0)\}$.
  So B contains $(1 : 0 : 0)$ and the points of $\mathcal{U}$ on the horizontal lines $Y = u^r + iv$, $u, v \in \mathbb{F}_q$.
- Trivially, B meets every horizontal line.
- B meets every non-horizontal line of $\mathcal{H}$ in z points, where $(q - 2 - (2r - 2)\sqrt{q})/r \leq z \leq (q + 1 + (2r - 2)\sqrt{q})/r$.

- Consider a cyclic $(q^2 - q + 1)$-arc A in $\mathcal{H}$ and passing through $(1:0:0)$.
- The $q+1$ lines through $(1:0:0)$ tangent to A form a dual Baer subline at $(1:0:0)$.
- One of these lines is the tangent line $Z = 0$ to $\mathcal{H}$ in $(1:0:0)$, and the remaining q are secant lines to $\mathcal{H}$.

- Delete all points $\neq (1:0:0)$ of the arc $A \cap B$ from B.
- Delete all points $\neq (1:0:0)$ lying on the above q secants of $\mathcal{H}$ through $(1:0:0)$ from B.
- This gives the desired blocking set.

Thanks for your attention.