Conference Guide

Mini Special Semester on Inverse Problems 2010

WORKSHOP: Impact of smoothness on regularization

Organized by B. Hofmann and S. Pereverzyev

RICAM
JOHANN · RADON · INSTITUTE
FOR COMPUTATIONAL AND APPLIED MATHEMATICS

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Austrian Academy of Sciences

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June 29th - July 2nd, 2010 - RICAM, Linz, Austria

Organizers

Bernd Hofmann, Chemnitz University of Technology, Germany

Sergei Pereverzyev, Johann Radon Institute, Austria

Invited Speakers

Cara Brooks, Rose-Hulman Institute of Technology, USA

Markus Grasmair, University of Vienna, Austria

Dinh Nho Hào, Vietnam Academy of Science and Technology, Vietnam

Markus Hegland, Australian National University, Australia

Bernd Hofmann, Chemnitz University of Technology, Germany

Patricia Lamm, Michigan State University, USA

Christine de Mol, Universite Libre de Bruxelles, Belgium

Thamban Nair, Indian Institute of Technology Madras, India

Zuhair Nashed, University of Central Florida, USA

Teresa Regińska, Polish Academy of Sciences, Poland

Ulrich Tautenhahn, Hochschule Zittau/Görlitz, Germany

Yuesheng Xu, Syracuse University, USA

Location

The workshop takes place at the Johann Radon Institute (RICAM), Hochschulfondsgebäude (HF). The talks are held in the Room HF 136 (first floor Hochschulfondgebäude).

Time Table

Time	Tuesday	Wednesday	Thursday	Friday
8:45 – 9:00	Opening			
9:00 – 10:00	Z. Nashed	P. Lamm	D. N. Hào	M. Grasmair
10:00 – 10:30	*Coffee Break*			
10:30 – 11:30	M. Hegland	M. T. Nair	T. Regińska	S. Kindermann
11:30 – 14:00	*Lunch Break + Discussion*			
14:00 – 15:00	Y. Xu	C. de Mol	C. Brooks	S. Lu
15:00 – 16:00	S. Pereverzyev	B. Hofmann	R. Ramlau	
18:00 –	*Conference Dinner*			

Abstracts

Error bounds of regularisation techniques based on range inclusions and variable Hilbert scales

MARKUS HEGLAND, Australian National University Canberra

In my talk I will discuss error bounds based on range inclusions using variable Hilbert scales for general index functions. I will also present some related formulae for the modulus of continuity of linear operators with non-closed range. These new results are similar to earlier results obtained by several authors. The earlier results were derived from the work by Mathe and Pereverzyev, who used variable Hilbert scales based on rate functions. An important observation which helps to understand the connections between the new and the earlier work uses characterisations of concavity and in particular the concavity preserving property of a special involution.

As an application of this theory I will consider spectral sharpening where explicit range inclusions are known.

The talk is based on joint research with B. Hofmann, Chemnitz.

References

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2. M. HEGLAND: Error Bounds For Spectral Enhancement Which Are Based On Variable Hilbert Scale Inequalities, *Journal of Integral Equations and Applications* (in print) (2010). Preprint as ArXiv e-print arXiv:0911.2695, 2009, http://arxiv.org/abs/0911.2695.

Regularization of natuarlly linearized parameter identification problems and the application of the balancing principle

SERGEI PEREVERZEV, RICAM, Linz

Natural linearization was introduced as an alternative to output least-squares approach to nonlinear parameter identification problems for partial differential equations. The purpose of the talk is to present a kind of survey of the recent research on natural linearization as well as numerical experiments illustrating theoretical results

Operator smoothing and source conditions for the method of local regularization

Patricia K. Lamm, Department of Mathematics, Michigan State University

In contrast to classical methods of regularization, the method of local regularization for linear ill-posed problems is not based on spectral theory for self-adjoint linear operators. Thus, the usual source conditions seen in classical methods are not relevant in the theory for local regularization. In this talk we will survey what is currently known about the link between operator smoothness and source conditions in the theory for local regularization for first-kind integral equations of Volterra type.

Error Estimates for Tikhonov Regularization Using a General Stabilizing Operator

M. Thamban Nair, Department of Mathematics, I.I.T. Madras

For obtaining stable approximate solutions for an ill-posed operator equation $Tx = y$, we consider Tikhonov regularization with a general stabilizing operator L, where $T : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y and $L : D(L) \subset X \to X$ is a (possibly unbounded) closed densely defined operator in X. We assume that the operators T and L are related by the condition

$$\|Tx\|^2 + \|Lx\|^2 \geq \gamma \|x\|^2 \quad \forall x \in D(L)$$

for some constant $\gamma > 0$. Then, the regularized solution x_α^δ, using inexact data y_δ with $\|y - y_\delta\| \leq \delta$ for a known noise level $\delta > 0$ and for a positive parameter α, is defined as the solution of the well-posed equation

$$(T^*T + \alpha L^*L)x_\alpha^\delta = T^*y_\delta.$$

The regularization parameter $\alpha_\delta := \alpha(\delta, y_\delta)$ is chosen according to the Morozov's discrepancy principle

$$c_1\delta \leq \|Tx_\alpha^\delta - y^\delta\| \leq c_2\delta$$

for constants $c_1, c_2 \geq 1$. It is known that under mild conditions, the regularized solution $x_{\alpha_\delta}^\delta$converges to $\hat{x}$, the unique LRN-solution which minimizes the map $x \to \|Lx\|$. However, for obtaining estimates for the error $\|\hat{x} - x_{\alpha_\delta}^\delta\|$, one has to impose additional smoothness assumptions on $\hat{x}$. This aspect has been considered extensively in the literature in recent years by assuming that the operator L^*L is of the form B^s for some strictly positive operator B which generates a Hilbert scale $\{X_t\}_{t>0}$ and requiring $\hat{x}$ to lie in X_p for some $p > 0$.

Our aim in this talk is to derive a convergence rate under a source condition which does not have to rely on a Hilbert scale. The derived estimate also includes the known order optimal rate for ordinary Tikhonov regularization, i.e., $L = I$. We shall also discuss a special situation where the operators T and L are related to a Hilbert scale in certain manner and derive an improved error estimate.

Fast convergence rates or excessive smoothness?

Christine De Mol, Department of Mathematics and ECARES Universitè Libre de Bruxelles

It is well known that stability in ill-posed inverse problems is restored through regularizing constraints or penalties imposing some regularity conditions on the solution. The stronger are the constraints and the regularity, the better are the rates at which the regularized solution converges to the true generalized solution. In particular, so-called "source conditions" have been widely advocated in the literature but may appear in some inverse problems as very restrictive smoothness assumptions on the solution. The talk will review results of this type, both for classical quadratic regularization and for sparsity-enforcing regularization. It also aims at investigating, in relationship with several practical applications, to what extent good rates are or not obtained through smoothness conditions which may appear as unrealistic for the application under study. Hopefully, this question will provoke some discussions among the participants to the workshop.

On the variational inequality approach for nonlinear ill-posed problems

Bernd Hofmann, Department of Mathematics, Chemnitz University of Technology

Twenty years ago Engl, Kunisch and Neubauer presented in a seminal paper the fundamentals of a systematic Hilbert space theory for convergence rates in *nonlinear* Tikhonov regularization of the form $\|F(x) - y^\delta\|^2 +$

$\alpha\|x-x^*\|^2 \to \min$ applied to ill-posed inverse problems $F(x) = y$ with nonlinear forward operator F, solutions $x^\dagger$ and noisy data y^δ, where $\|y - y^\dagger\| \le \delta$ and $\delta > 0$ denotes the noise level. In general, convergence rates results for nonlinear ill-posed problems are based on two ingredients: 1. structural conditions concerning the nonlinearity of F, mostly characterized by the behaviour of the Fréchet derivative $F'(x)$ in a neighbourhood of $x^\dagger$. 2. source conditions expressing the solution smoothness with respect to the forward operator F, for example of the form $x^\dagger - x^* = F'(x^\dagger)^* w$ for some source element w with sufficiently small norm. In this presentation, following the lines of [2,3] and [5,7], we show that both nonlinearity and smoothness conditions can be expressed by variational inequalities in a unified manner emphasizing the capability of those inequalities for yielding convergence rates. In this context, as in [4,6] we also extend and modify the idea of using approximate source conditions presented in [1] for linear ill-posed problems to the nonlinear case. To formulate the results in a Banach space setting we use Bregman distances for measuring the regularization error.

The research is supported by DFG under Grant HO1454/7-2.

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Convergence Rates for Tikhonov regularization of coefficient identification problems in elliptic equations

DINH NHO HÀO, Hanoi Institute of Mathmatics and TRAN NHAN TAM QUYEN, Da Nang University of Education, Vietnam

We investigate the convergence rates for Tikhonov regularization of the problem of identifying

1. the coefficient q in the Dirichlet problem $-\mathrm{div}(qu) = f$ in Ω, $u = 0$ on $\partial\Omega$,
2. the coefficient a in the Dirichlet problem $-\Delta u + au = f$ in Ω, $u = 0$ on $\partial\Omega$,

when u is imprecisely given in the whole domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$.

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Regularization of ill-posed differential problems related to multiplication operators in frequency space

TERESA REGIŃSKA, Institute of Mathematics, Polish Academy of Sciences, Warsaw

Let us consider Cauchy problems for elliptic or parabolic differential equations on domains $\Omega = \{(\rho, z) \in \mathbb{R}^{n-1} \times (0, d)\}$ or $\Omega = \{(\rho, z, t) \in \mathbb{R}^{n-1} \times (0, d) \times \mathbb{R}\}$, respectively. Let Cauchy data be given on one part of the boundary of the ininite strip, say at $z = d$. The problem of reconstructing a solution for a fixed point z can be formulated as an operator equation in frequency space with an operator $\hat{A}(z)$ depending on z. It is easy to see that sometimes these operators are multiplication operators in $L^2(\mathbb{R}^{n-1})$ or $L^2(\mathbb{R}^{n-1} \times \mathbb{R})$, respectively and their spectra contain 0, i.e.,$\hat{A}^{-1}(z)$ are unbounded operators. Moreover, $\hat{A}(z)$ itself may be unbounded for some $z \in (0, d)$, as in the case of the Helmholtz equation with sufficiently large wave number.

The talk will concern an analysis of regularized solutions. Among others, the question what is the best possible accuracy for reconstructing an exact solution from noisy data and the question what regularization methods guarantee an optimal order of convergence, will be considered.

References
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A Modified Discrepancy Principle for Local Regularization of Linear and Nonlinear Inverse Problems

Cara Brooks, Rose-Hulman Institute of Technology, USA

Over the past 15 years, the work of P. K. Lamm and others has led to the development of local regularization methods for solving linear and nonlinear inverse problems. The convergence theory associated with *a priori* parameter selection now includes finitely smoothing linear Volterra problems, nonlinear Hammerstein and autoconvolution problems, as well as linear non-Volterra integral equations. Recent advancements in the theory involve the construction of a modified discrepancy principle for the *a posteriori* selection of the local regularization parameter. The theoretical justification for use of this principle is well-established for finitely smoothing linear Volterra convolution equations. We will present the convergence theory for its use as an effective parameter selection strategy for generalized local regularization of linear inverse problems and for local regularization of a class of nonlinear Volterra Hammerstein equations.

An SVD based Wavefront Reconstruction for Adaptive Optics

Ronny Ramlau, Industrial Mathematics Institute, University Linz, Austria

Most of the large earthbound astronomical telescopes use Adaptive Optics technology (AO) in order to enhance the image quality. The degradation of the measured images is caused by atmospheric turbulences. The correction is achieved by the use of deformable mirrors, where the deformation is obtained from the measurements of the light of a bright star. In order to measure the incoming wavefront, different types of sensors are used. We consider the so called Shack-Hartmann sensor which measures an average of the gradient of the wavefront. The Inverse Problem is now the reconstruction of the wavefront from the (noisy) sensor measurements. Besides the reconstruction quality the reconstruction time is most important, as the reconstructions have to be carried out within 2ms. Our reconstruction algorithm is based on the singular value decomposition of the underlying operator. Our analysis allows also for a characterization of the reconstructable wavefronts and gives results for the degree of ill-posedness of the operator that describes the measurements. The analytical results are illustrated by numerical experiments.

This research is joint work with A. Neubauer and S. Kindermann, Industrial Mathematics Institute, Johannes Kepler University Linz.

Variational Inequalities and Convergence Rates for Non-convex Regularization

Markus Grasmair, Computational Science Center, University of Vienna

One strategy for deriving convergence rates for Tikhonov regularization is the method of variational inequalities, introduced by Hofmann et al. in 2007. The method uses the minimality properties of the solution of Tikhonov regularization combined with an inequality, which, in the linear case, is equivalent to standard range conditions. In this talk, we will show that the same ansatz can also be used for the derivation of convergence rates for Tikhonov regularization with a *non-convex* penalty term. To that end, it is necessary to generalize the notion of Bregman distances in such a way that it also makes sense for non-convex functionals. This generalization is achieved usings concepts from abstract convexity.

We demonstrate the results of the generalized theory by means of two examples. First, we consider non-convex regularization of linear, ill-posed equations on Hilbert spaces. In this case the variational inequalities can be reduced to a condition of range type. Moreover, in the convex case, the standard range condition is recovered. The second example treats the case of non-convex, sparse regularization on sequence spaces, where convergence rates in the norm are derived under the assumption of a range condition. In addition, it is shown that the rate of convergence only depends on the growth of the regularization functional near zero.

Regions of Stability and Regularization of the Cauchy Problem for the Helmholtz equation

STEFAN KINDERMANN, Industrial Mathematics Institute, University Linz, Austria

The Cauchy Problem for the Helmholtz equation is a well-known example of an ill-posed problem. Recently, it was observed in [2–3] that associated conditional stability estimates improve with higher wavenumber. In this work, we investigate the degree of ill-posedness of this problem using an appropriate operator formulation. Similar to the conditional stability estimates the stability increases with the wavenumber but in a particular fashion: there are subspaces on which the Cauchy Problem can be solved in a stable way and these regions of stability increase with the wavenumber, such that in the limit the problem is well-posed. We investigate the effect of such stability regions on the regularization and the parameter choice, taking into account the dependencies on the wavenumber.

An explicit calculation of the singular values is only possible for simple geometries, for general geometries we look at the degree of ill-posedness using numerical calculations. They indicate that independent of the geometry there is always a region of stability increasing with the wavenumber. This leads us to the conjecture that this phenomenon occurs (at least in the plane) independent of the geometry.

This is joint work with Victor Isakov, Dept. of Math and Stat., Wichita State University, KS.

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Regularized total least squares: computational aspects and error bounds

Shuai Lu, RICAM, Linz

For solving linear ill-posed problems, regularization methods are required when the right hand side and/or the operator are corrupted by some noise. In the present talk, regularized solutions are constructed using regularized total least squares and dual regularized total least squares. We discuss computational aspects and provide order optimal error bounds that characterize the accuracy of the regularized solutions. The results extend earlier results where the operator is exactly given. We also present some numerical experiments, which shed light on the relationship between RTLS, dual RTLS and the standard Tikhonov regularization.

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