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On Products of Shifts in Arbitrary Fields

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Abstract

We adapt the approach of Rudnev, Shakan, and Shkredov presented in [8] to prove that in an arbitrary field $\mathbb{F}$, for all $A \subset \mathbb{F}$ finite with $|A| < p^{1/4}$ if $p := Char(\mathbb{F})$ is positive, we have

$$|A(A+1)| \gg \frac{|A|^{11/9}}{(\log|A|)^{7/6}}, \qquad |AA| + |(A+1)(A+1)| \gg \frac{|A|^{11/9}}{(\log|A|)^{7/6}}.$$

This improves upon the exponent of 6/5 given by an incidence theorem of Stevens and de Zeeuw.

1 Introduction and Main Result

For finite $A \subseteq \mathbb{F}$, we define the *sumset* and *product set* of A as

$$A + A = \{a + b : a, b \in A\}, \quad AA = \{ab : a, b \in A\}.$$

It is an active area of research to show that one of these sets must be large relative to A. The central conjecture in this area is the following.

Conjecture 1 (Erdős - Szemerédi)**.** *For all $\epsilon > 0$, and for all $A \subseteq \mathbb{Z}$ finite, we have*

$$|AA| + |A + A| \gg |A|^{2-\epsilon}.$$

The notation $X \ll Y$ is used to hide absolute constants, i.e. $X \ll Y$ if and only if there exists an absolute constant $c > 0$ such that $X \ll cY$. If $X \ll Y$ and $Y \ll X$ we write $X \asymp Y$. We will let p denote the characteristic of $\mathbb{F}$ throughout (p may be zero). Due to the possible existence of finite subfields in $\mathbb{F}$, extra restrictions on $|A|$ relative to p must be imposed if p is positive; *all such conditions can be ignored if* $p = 0$.

Although Conjecture 1 is stated over the integers it can be considered over fields, the real numbers being of primary interest. Current progress over $\mathbb{R}$ places us at an exponent of $\frac{4}{3} + c$ for some small c, due to Shakan [10], building on works of Konyagin and Shkredov [4] and Solymosi [11]. Incidence geometry, and in particular the Szemerédi-Trotter Theorem, are tools often used to prove such results in the real numbers.

Conjecture 1 can also be considered over arbitrary fields $\mathbb{F}$. Over arbitrary fields we replace the Szemerédi-Trotter Theorem with a point-plane incidence theorem of Rudnev [7], which was used by Stevens and de Zeeuw to derive a point-line incidence theorem [12]. An exponent of 6/5 was proved in 2014 by Roche-Newton, Rudnev, and Shkredov [6]. An application of the Stevens - de Zeeuw Theorem also gives this exponent of 6/5 for Conjecture 1, so that 6/5 became a threshold to be broken.

The 6/5 threshold has recently been broken, see [9], [8], and [1]. The following theorem was proved in [8] by Rudnev, Shakan, and Shkredov, and is the current state of the art bound.

Theorem 1. [8] *Let $A \subset \mathbb{F}$ be a finite set. If $\mathbb{F}$ has positive characteristic p, assume $|A| < p^{18/35}$. Then we have*

$$|A + A| + |AA| \gg |A|^{\frac{11}{9} - o(1)}.$$

Another way of considering the sum-product phenomenon is to consider the set $A(A + 1)$, which we would expect to be quadratic in size. This encapsulates the idea that a translation of a multiplicatively structured set should destroy its structure, which is a main theme in sum-product questions. Study of growth of $|A(A + 1)|$ began in [2] by Garaev and Shen, see also [3], [13], and [5]. Current progress for $|A(A+1)|$ comes from an application of the Stevens - de Zeeuw Theorem, giving the same exponent of 6/5. In this paper we use the multiplicative analogue of ideas in [8] to prove the following theorem.

Theorem 2. *Let $A, B, C, D \subset \mathbb{F}$ be finite with the conditions*

$$|C(A + 1)||A| \leq |C|^3, \quad |C(A + 1)|^2 \leq |A||C|^3, \quad |B| \leq |D|, \quad |A|, |B|, |C|, |D| < p^{1/4}.$$

Then we have

$$|AB|^8|C(A + 1)|^2|D(B - 1)|^8 \gg \frac{|B|^{13}|A|^5|D|^3|C|}{(\log |A|)^{17}(\log |B|)^4}.$$

In our applications of this theorem we have $|A| = |B| = |C| = |D|$, so that the first three conditions are trivially satisfied. The conditions involving p could likely be improved, however for sake of exposition we do not attempt to optimise these. The main proof closely follows [8] (in the multiplicative setting), the central difference being a bound on multiplicative energies in terms of products of shifts. An application of Theorem 2 beats the threshold of 6/5, matching the 11/9 appearing in Theorem 1. Specifically, we have

Corollary 1. *Let $A \subseteq \mathbb{F}$ be finite, with $|A| < p^{1/4}$. Then*

$$|A(A+1)| \gg \frac{|A|^{11/9}}{(\log|A|)^{7/6}}, \qquad |AA| + |(A+1)(A+1)| \gg \frac{|A|^{11/9}}{(\log|A|)^{7/6}}.$$

Corollary 1 can be seen by applying Theorem 2 with $B = A+1$, $C = A$ and $D = A+1$ for the first result, and $B = -A$, $D = C = A+1$ for the second result.

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2 Preliminary Results

We require some preliminary theorems. The first is the point-line incidence theorem of Stevens and de Zeeuw.

Theorem 3 (Stevens - de Zeeuw, [12])**.** *Let A and B with $|B| \leq |A|$ be finite subsets of a field $\mathbb{F}$, and let L be a set of lines. Assuming $|L||B| \ll p^2$ and $|B||A|^2 \leq |L|^3$, we have*

$$I(A \times B, L) \ll |A|^{1/2}|B|^{3/4}|L|^{3/4} + |L|.$$

Note that as $|B| \leq |A|$, we have $|A|^{1/2}|B|^{3/4} \leq |A|^{3/4}|B|^{1/2}$, in particular with the same conditions we have the above result with the exponents of A and B swapped. Because of this, the condition $|B| \leq |A|$ is only needed to specify the second two conditions. We may therefore restate Theorem 3 as

Theorem 4. *Let A and B be finite subsets of a field $\mathbb{F}$, and let L be a set of lines. Assuming $|L|\min\{|A|,|B|\} \ll p^2$ and $|A||B|\max\{|A|,|B|\} \leq |L|^3$, we have*

$$I(A \times B, L) \ll \min\{|A|^{1/2}|B|^{3/4}, |A|^{3/4}|B|^{1/2}\}|L|^{3/4} + |L|.$$

This second formulation will be how we apply Theorem 3. Before stating the next two theorems we require some definitions. For $x \in \mathbb{F}$ we define the *representation function*

$$r_{A/D}(x) = \left|\left\{(a, d) \in A \times D : \frac{a}{d} = x\right\}\right|.$$

Note that for all x, $r_{A/D}(x) \leq \min\{|A|, |D|\}$. This is seen as fixing one of a, d in the equation $\frac{a}{d} = x$ necessarily determines the other. The set A/D in this definition can be changed to any other combination of sets, changing the fraction $\frac{a}{d}$ in the definition to match. For $n \in \mathbb{R}^+$, we define the n'th moment *multiplicative energy* of sets A, $D \subseteq \mathbb{F}$ as

$$E_n^*(A, D) = \sum_x r_{A/D}(x)^n.$$

When $n = 2$ we shall simply write $E^*(A, D)$, and when $A = D$ we write $E_n^*(A) := E_n^*(A, A)$. By considering that we have $\frac{a}{a} = 1$ for all $a \in A$, we have the trivial lower bound $E_n^*(A) \geq |A|^n$. when n is in fact a natural number, $E_n^*(A, D)$ can be considered as the number of solutions to

$$\frac{a_1}{d_1} = \frac{a_2}{d_2} = ... = \frac{a_n}{d_n} \quad a_i \in A, d_i \in D$$

giving the trivial upper bound $E_n^*(A, D) \leq |A|^n|D|$ by fixing a_1 to a_n and then choosing a single d_i, which necessarily determines all other d_i.

We use Theorem 4 to prove two further results. The first is a bound on the fourth order multiplicative energy relative to products of shifts.

Theorem 5. *For all finite non-empty $A, C, D \subset \mathbb{F}$ with $|A|^2|C(A+1)| \leq |D||C|^3$, $|A||C(A+1)|^2 \leq |D|^2|C|^3$, and $|A||C||D|^2 \ll p^2$, we have*

$$E_4^*(A, D) \ll \min\left\{\frac{|C(A+1)|^2|D|^3}{|C|}, \frac{|C(A+1)|^3|D|^2}{|C|}\right\} \log |A|.$$

The second result is similar, but for the second moment multiplicative energy.

Theorem 6. *For all finite and non-empty A, C, $D \subset \mathbb{F}$ with $|A|^2|C(A+1)| \leq |D||C|^3$, $|A||C(A+1)|^2 \leq |D|^2|C|^3$, and $|A||C||D| \min\{|C|, |D|\} \ll p^2$, we have*

$$E^*(A, D) \ll \frac{|C(A+1)|^{3/2}|D|^{3/2}}{|C|^{1/2}} \log |A|.$$

The set $A+1$ appearing in these theorems can be changed to any translate $A+\lambda$ for $\lambda \neq 0$, by noting that $|C(A+1)| = |C(\lambda A+\lambda)|$ and renaming $A' = \lambda A$. For our purposes, we will use $\lambda = \pm 1$.

Proof of Theorem 5. WLOG we can assume that $0 \notin A, C, D$. We begin by proving that

$$E_4^*(A, D) \ll \frac{|C(A+1)|^2|D|^3}{|C|} \log |A|.$$

Define the set

$$S_\tau := \{x \in A/D : \tau \leq r_{A/D}(x) < 2\tau\}.$$

By a dyadic decomposition, there is some τ with

$$|S_\tau|\tau^4 \ll E_4^*(A, D) \ll |S_\tau|\tau^4 \log |A|$$

Note that $\tau \leq \min\{|A|, |D|\}$. Take an element $t \in S_\tau$. It has τ representations in A/D, so there are τ ways to write $t = a/d$ with $a \in A$, $d \in D$. For all $c \in C$, we have

$$\begin{aligned} t &= \frac{a}{d} \\ &= \frac{1}{d}\left(\frac{ac + c - c}{c}\right) \\ &= \frac{1}{d}\left(\frac{\alpha}{c} - 1\right) \end{aligned}$$

where $\alpha = c(a+1) \in C(A+1)$. This shows that we have $|S_\tau|\tau|C|$ incidences between the lines

$$L = \{l_{d,c} : d \in D, c \in C\}, \quad l_{d,c} \text{ given by } y = \frac{1}{d}\left(\frac{x}{c} - 1\right)$$

and the point set $P = C(A+1) \times S_\tau$. Under the conditions $|D||C| \min\{|S_\tau|, |C(A+1)|\} \ll p^2$ and $|S_\tau||C(A+1)| \max\{|S_\tau|, |C(A+1)|\} \leq |D|^3|C|^3$, we have that

$$|S_\tau|\tau|C| \leq I(P, L) \ll |C(A+1)|^{1/2}|S_\tau|^{3/4}|C|^{3/4}|D|^{3/4} + |D||C|.$$

The conditions are satisfied under the assumptions $|D||A||C| \min\{|D|, |C|\} \ll p^2$, $|A|^2|C(A+1)| \leq |D||C|^3$, and $|A||C(A+1)|^2 \leq |D|^2|C|^3$. Assuming that the leading term is dominant, we have

$$|S_\tau|\tau^4|C| \ll |C(A+1)|^2|D|^3$$

so that as $\frac{E_4^*(A,D)}{\log |A|} \ll |S_\tau|\tau^4$, we have

$$E_4^*(A, D) \ll \frac{|C(A+1)|^2|D|^3}{|C|} \log |A|.$$

We therefore assume the leading term is not dominant. Suppose $|D||C|$ is dominant, so that

$$|C(A+1)|^{1/2}|S_\tau|^{3/4}|C|^{3/4}|D|^{3/4} \leq |D||C|. \tag{1}$$

Multiplying by τ^3 and simplifying, we have

$$|C(A+1)|^2 \frac{E_4^*(A,D)^3}{\log|A|^3} \ll |C(A+1)|^2 |S_\tau|^3 \tau^{12} \le |D||C|\tau^{12} \implies E_4^*(A,D) \ll \frac{|D|^{1/3}|C|^{1/3}\tau^4}{|C(A+1)|^{2/3}} \log|A|.$$

The result now follows if

$$\frac{|D|^{1/3}|C|^{1/3}\tau^4}{|C(A+1)|^{2/3}} \ll \frac{|C(A+1)|^2|D|^3}{|C|}.$$

We must therefore prove the result in the case that this is not true; we will prove the result under the assumption

$$\frac{|C(A+1)|^2|D|^3}{|C|} \le \frac{|D|^{1/3}|C|^{1/3}\tau^4}{|C(A+1)|^{2/3}}$$

which gives (using $\tau \le |A|$)

$$|D|^8|C|^4|A|^4 \le |D|^8|C(A+1)|^8 \le \tau^{12}|C|^4 \le |A|^{12}|C|^4$$

so that we have $|D| \le |A|$. We then have (using $|C(A+1)| \ge |C|^{1/2}|A|^{1/2}$)

$$|D||C| \ge |C(A+1)|^{1/2}|S_\tau|^{3/4}|C|^{3/4}|D|^{3/4} \ge |C(A+1)|^{1/2}|C|^{3/4}|D|^{3/4} \ge |A|^{1/4}|C||D|^{3/4} \ge |D||C|$$

so that the two terms are in fact balanced and the result follows.

Secondly, we prove that

$$E_4^*(A,D) \ll \frac{|C(A+1)|^3|D|^2}{|C|} \log|A|.$$

To do this, we swap the roles of D and S_τ from above. We define the line set and point set by

$$L = \{l_{t,c} : t \in S_\tau, c \in C\}, \qquad P = C(A+1) \times D.$$

Any incidence from the previous point and line set remains an incidence for the new ones, via $t = \frac{1}{d}\left(\frac{\alpha}{c} - 1\right) \iff d = \frac{1}{t}\left(\frac{\alpha}{c} - 1\right)$. Under the conditions

$$|S_\tau||C| \min\{|D|, |C(A+1)|\} \ll p^2, \qquad |D||C(A+1)| \max\{|D|, |C(A+1)|\} \le |S_\tau|^3|C|^3 \quad (2)$$

we have

$$|S_\tau|\tau|C| \le I(P,L) \ll |C(A+1)|^{3/4}|S_\tau|^{3/4}|C|^{3/4}|D|^{1/2} + |S_\tau||C|.$$

If the leading term dominates, the result follows from $|S_\tau|\tau^4 \gg \frac{E_4^*(A,D)}{\log|A|}$. Assume the leading term is not dominant, that is,

$$|C(A+1)|^3|D|^2 \le |S_\tau||C|.$$

Then by using $|S_\tau| \leq |A||D|$ and $|A|, |C| \leq |C(A+1)|$ we have

$$|A||C|^2|D|^2 \leq |C(A+1)|^3|D|^2 \leq |S_\tau||C| \leq |A||D||C|$$

so that $|C| = |D| = 1$ and the result is trivial by $E_4^*(A, D) \leq |A||D|^4 \leq |A|$.

We now check the conditions (2) for using Theorem 3. The first condition in (2) is satisfied if $|A||C||D|^2 \ll p^2$, which is true under our assumptions. The second condition depends on $\max\{|D|, |C(A+1)|\}$, which we assume is $|D|$ (if not the first term in Theorem 5 gives stronger information, which we have already proved). Assuming the second condition does not hold, we have

$$|S_\tau|^3|C|^3 < |D|^2|C(A+1)|.$$

Multiplying by τ^{12} and bounding $\tau \leq |A|$, we get

$$E_4^*(A, D) \ll \frac{|A|^4|D|^{2/3}|C(A+1)|^{1/3}}{|C|} \log |A|. \tag{3}$$

We may now assume the bound

$$\frac{|C(A+1)|^3|D|^2}{|C|} \leq \frac{|A|^4|D|^{2/3}|C(A+1)|^{1/3}}{|C|}. \tag{4}$$

Indeed, if we were to have

$$\frac{|A|^4|D|^{2/3}|C(A+1)|^{1/3}}{|C|} < \frac{|C(A+1)|^3|D|^2}{|C|}$$

then we may apply this bound in (3) and the result follows. Assuming (4), we have

$$|A|^8|D|^4 \leq |C(A+1)|^8|D|^4 \leq |A|^{12}.$$

So that $|D| \leq |A|$. In turn, this implies $|A| \geq |D| \geq |C(A+1)| \geq |A|$, so that $|A| = |C(A+1)| = |D|$. Returning to equation (3), this gives

$$E_4^*(A, D) \ll \frac{|A|^4|D|^{2/3}|C(A+1)|^{1/3}}{|C|} \log |A| = \frac{|C(A+1)|^3|D|^2}{|C|} \log |A|$$

and the result is proved. □

Proof of Theorem 5. The proof follows similarly to that of Theorem 5. We again define the lines and points

$$L = \{l_{d,c} : d \in D, c \in C\}, \quad l_{d,c} \text{ given by } y = \frac{1}{d}\left(\frac{x}{c} - 1\right), \quad P = C(A+1) \times S_\tau,$$

where in this case the set S_τ is rich with respect to $E^*(A,D)$, so that

$$|S_\tau|\tau^2 \ll E^*(A,D) \ll |S_\tau|\tau^2 \log|A|.$$

With the conditions $|A||C||D|\min\{|D|,|C|\} \ll p^2$ and $|S_\tau||C(A+1)|\max\{|S_\tau|,|C(A+1)|\} \leq |D|^3|C|^3$, (which are satisfied under our assumptions) we have by Theorem 4,

$$|S_\tau|\tau|C| \leq I(P,L) \ll |S_\tau|^{1/2}|C(A+1)|^{3/4}|D|^{3/4}|C|^{3/4} + |D||C|.$$

If the leading term dominates, we have

$$|S_\tau|\tau^2 \ll \frac{|C(A+1)|^{3/2}|D|^{3/2}}{|C|^{1/2}}$$

and the result follows from $\frac{E^*(A,D)}{\log|A|} \ll |S_\tau|\tau^2$. We therefore assume that the leading term does not dominate, that is,

$$|S_\tau|^{1/2}|C(A+1)|^{3/4}|D|^{3/4}|C|^{3/4} \leq |D||C|.$$

Multiplying through by τ and squaring, we get the bound

$$E^*(A,D) \ll \frac{|D|^{1/2}|C|^{1/2}\tau^2}{|C(A+1)|^{3/2}} \log|A|. \tag{5}$$

In a similar way to before, we may now assume the bound

$$\frac{|D|^{3/2}|C(A+1)|^{3/2}}{|C|^{1/2}} \leq \frac{|D|^{1/2}|C|^{1/2}\tau^2}{|C(A+1)|^{3/2}} \tag{6}$$

as assuming otherwise yields the result via (5). Bound (6) then gives

$$|D||C(A+1)|^3 \leq |C|\tau^2$$

Bounding $\tau \leq |A|$ and $|C||A|^2 \leq |C(A+1)|^3$ we have $|D| = 1$. Similarly, bounding $\tau^2 \leq |A||D|$ and $|C(A+1)|^3 \geq |C|^2|A|$, we find $|C| = 1$, so that the result is trivial. □

3 Proof of Theorem 2

We follow a multiplicative analogue of the argument in [8]. WLOG we may assume $A, B \subseteq \mathbb{F}^*$. For some $\delta > 0$, define a popular set of products as

$$P := \left\{ x \in AB : r_{AB}(x) \geq \frac{|A||B|}{|AB|\delta} \right\}.$$

Let $P^c := AB \setminus P$. Note that by writing

$$|\{(a,b) \in A \times B : ab \in P\}| + |\{(a,b) \in A \times B : ab \in P^c\}| = |A||B|$$

and noting that

$$|\{(a,b) \in A \times B : ab \in P^c\}| < |P^c|\frac{|A||B|}{|AB|\delta} \leq \frac{|A||B|}{\delta}$$

we have

$$|\{(a,b) \in A \times B : ab \in P\}| \geq \left(1 - \frac{1}{\delta}\right)|A||B|.$$

We also define a popular subset of A with respect to P, as

$$A' := \left\{a \in A : |\{b \in B : ab \in P\}| \geq \frac{2}{3}|B|\right\}.$$

We have

$$|\{(a,b) \in A \times B : ab \in P\}| = \sum_{a \in A'} |\{b : ab \in P\}| + \sum_{a \in A \setminus A'} |\{b : ab \in P\}| \geq \left(1 - \frac{1}{\delta}\right)|A||B| \qquad (7)$$

Suppose that $|A \setminus A'| = c|A|$ for some $c \geq 0$, so that $|A'| = (1-c)|A|$. Noting that

$$\sum_{a \in A'} |\{b : ab \in P\}| \leq (1-c)|A||B|, \qquad \sum_{a \in A \setminus A'} |\{b : ab \in P\}| \leq \frac{2c}{3}|A||B|,$$

we have by (7)

$$(1-c)|A||B| + \frac{2c}{3}|A||B| \geq (1 - \frac{1}{\delta})|A||B| \implies c \leq \frac{3}{\delta},$$

so that $|A'| \geq \left(1 - \frac{3}{\delta}\right)|A|$.

We use a multiplicative version of Lemma 8 in [8]. The proof we present is an expanded version of the proof present in [8].

Lemma 1. *For all finite $A \subset \mathbb{F}$, there exists $A_1 \subseteq A$ with $|A_1| \gg |A|$, such that*

$$E^*_{4/3}(A_1') \gg E^*_{4/3}(A_1)$$

Proof. We give an algorithm which shows such a subset exists, as otherwise we have a contradiction. We recursively define

$$A_i = A'_{i-1}, \qquad A_0 = A, \qquad i \leq \log|A|$$

where A_i' is defined relative to A_i. Using the same arguments as above, we have $|A_i'| \geq \left(1 - \frac{3}{\delta}\right)|A_i|$. We shall set $\delta = \log|A|$. We have the chain of inequalities

$$|A_i| = |A'_{i-1}| \geq \left(1 - \frac{3}{\log|A|}\right)|A_{i-1}| \geq ... \geq \left(1 - \frac{3}{\log|A|}\right)^i |A|.$$

Note that assuming $|A| \geq 16$ (if this is not true then the result is trivial), we have

$$\left(1 - \frac{3}{\log|A|}\right)^i \geq \left(1 - \frac{3}{\log|A|}\right)^{\log|A|} \geq \left(\frac{1}{4}\right)^4$$

since the function $\left(1 - \frac{3}{z}\right)^z$ is increasing for $z > 3$. We now have

$$|A_i| \geq \left(\frac{1}{4}\right)^4 |A| \gg |A|$$

at all steps i. We assume that at all steps, we have

$$E^*_{4/3}(A'_i) < \frac{E^*_{4/3}(A_i)}{4}$$

as otherwise we have $E^*_{4/3}(A'_i) \gg E^*_{4/3}(A_i)$ and we are done. After $\log|A|$ steps, we have a set A_k with

$$|A_k| \gg |A|, \qquad E^*_{4/3}(A'_k) < \frac{E^*_{4/3}(k)}{4} < \frac{E^*_{4/3}(A_{k-1})}{16} < \dots < \frac{E^*_{4/3}(A)}{4^{\log|A|}}.$$

But then we have

$$E^*_{4/3}(A) > E^*_{4/3}(A'_k)4^{\log|A|} \gg |A|^{4/3+2} = |A|^{10/3}$$

which is a contradiction. Therefore at some step we have an A_i satisfying the lemma. □

We now return to the proof of Theorem 2, with $\delta = \log|A|$ applied in the definition of P. We apply Lemma 1 to A to find a large subset $A_1 \subset A$ with $E^*_{4/3}(A'_1) \gg E^*_{4/3}(A_1)$, $|A_1| \gg |A|$. Noting that proving the result for A_1 implies it for A, we shall rename A_1 as A for simplicity.

We use a dyadic decomposition to find a set $Q \subset A'/A'$ such that

$$|Q|\Delta^{4/3} \ll E^*_{4/3}(A') \ll |Q|\Delta^{4/3}\log|A|$$

for some $\Delta > 0$.

We will bound the size of the set

$$N = \left\{(a, a', b, b') \in (A')^2 \times B^2 : \frac{a}{a'} \in Q, ab, ab', a'b, a'b' \in P\right\}.$$

By summing over all $a, a' \in A'$ with $\frac{a}{a'} \in Q$, we have

$$|N| = \sum_{\substack{a,a' \in A' \\ :a/a' \in Q}} |\{b \in B : ab, a'b \in P\}|^2$$

and we see that as for all $a \in A'$, $|\{b \in B : ab \in P\}| \geq \frac{2}{3}|B|$, by considering the intersection of $\{b \in B : ab \in P\}$ and $\{b \in B : a'b \in P\}$ we have that for all $a, a' \in A'$, $|\{b \in B : ab, a'b \in P\}| \geq \frac{1}{3}|B|$. Using that elements $q \in Q$ have at least Δ representations in A'/A', we have $|N| \geq \frac{1}{9}|B|^2|Q|\Delta$.

We now find an upper bound on $|N|$. Define an equivalence relation on $A^2 \times B^2$ via

$$(a, a', b, b') \sim (c, c', d, d') \iff \exists\, \lambda \text{ s.t. } a = \lambda c, a' = \lambda c', b = \frac{d}{\lambda}, b' = \frac{d'}{\lambda}.$$

Note that the conditions

$$\frac{a}{a'} \in Q, \quad ab, a'b, ab', a'b' \in P \tag{8}$$

are invariant in the class (i.e. if one class element satisfies these conditions, then they all do), as λ cancels in each condition. Let X denote the set of equivalence classes $[a, a', b, b']$ where the conditions (8) are satisfied. We can bound $|N|$ by the sum of the size of each equivalence class $[a, a', b, b']$ in X;

$$|N| \leq \sum_X |[a, a', b, b']| \,.$$

By the Cauchy-Schwarz inequality and completing the sum over all equivalence classes, we have

$$|Q|^2\Delta^2|B|^4 \ll |N|^2 \leq |X| \sum_{[a,a',b,b']} |[a, a', b, b']|^2 \tag{9}$$

We must now bound the two quantities on the right hand side of this equation. We first claim that

$$\sum_{[a,a',b,b']} |[a, a', b, b']|^2 \leq \sum_x r_{A/A}(x)^2 r_{B/B}(x)^2. \tag{10}$$

To see this, note that the left hand side of (10) counts pairs of elements of equivalence classes. Take any two elements (a, a', b, b'), $(c, c', d, d') \in A^2 \times B^2$ from the same equivalence class. By definition, we may write $(c, c', d, d') = (\lambda a, \lambda a', \frac{b}{\lambda}, \frac{b'}{\lambda})$. As $0 \notin A, B$, the 8-tuple $(a, a', b, b', c, c', d, d')$ satisfies

$$\lambda = \frac{c}{a} = \frac{c'}{a'} = \frac{b}{d} = \frac{b'}{d'}$$

for some $\lambda \in \mathbb{R}$, and thus corresponds to a contribution to the quantity $r_{A/A}(\lambda)^2 r_{B/B}(\lambda)^2$, and thus also corresponds to a contribution to the sum $\sum_x r_{A/A}(x)^2 r_{B/B}(x)^2$. We also see that different pairs from equivalence classes necessarily give different 8-tuples, and so the claim is proved. We use Cauchy-Schwarz on the right hand side of equation (10) to bound it by a product of fourth energies.

$$\sum_x r_{A/A}(x)^2 r_{B/B}(x)^2 \leq E_4^*(A)^{1/2} E_4^*(B)^{1/2}.$$

We use Theorem 5 to bound these energies. We bound via

$$E_4^*(A) \ll \frac{|C(A+1)|^2|A|^3}{|C|}\log|A|, \quad E_4^*(B) \ll \frac{|D(B-1)|^2|B|^3}{|D|}\log|B|$$

with conditions

$$|C(A+1)||A| \leq |C|^3,\ |C(A+1)|^2 \leq |A||C|^3,\ |A|^3|C| \ll p^2$$

$$|D(B-1)||B| \leq |D|^3,\ |D(B-1)|^2 \leq |B||D|^3,\ |B|^3|D| \ll p^2$$

which are all satisfied under our assumptions. Returning to equation (9), we now have

$$|Q|^2\Delta^2|B|^4 \ll |X|\frac{|C(A+1)||A|^{3/2}|D(B-1)||B|^{3/2}}{|C|^{1/2}|D|^{1/2}}(\log|A|\log|B|)^{1/2}. \tag{11}$$

We now bound $|X|$, the number of equivalence classes where the conditions (8) are satisfied. Note that any (a,a',b,b') belonging to an equivalence class in X maps to a solution of the equation

$$w = \frac{s}{t} = \frac{u}{v} \tag{12}$$

with $w \in Q$, $s,t,u,v \in P$, by taking $w = \frac{a}{a'}$, $s = ab, t = a'b, u = ab', v = a'b'$. Note that taking two solutions (a,a',b,b') and (c,c',d,d') that are *not* from the same equivalence class necessarily gives us two different solutions to equation (12) via the map above. Therefore we may bound $|X|$ by the number of solutions to (12).

$$\begin{aligned} |X| &\leq \left|\left\{(w,s,t,u,v) \in Q \times P^4 : w = \frac{s}{t} = \frac{u}{v}\right\}\right| \\ &= \left|\left\{(s,t,u,v) \in P^4 : \frac{s}{t} = \frac{u}{v} \in Q\right\}\right|. \end{aligned}$$

The popularity of P allows us to bound this by

$$|X| \leq \frac{|AB|^4(\log|A|)^4}{|A|^4|B|^4}\left|\left\{(a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4) \in A^4 \times B^4 : \frac{a_1b_1}{a_2b_2} = \frac{a_3b_3}{a_4b_4} \in Q\right\}\right|.$$

We dyadically pigeonhole the set $\frac{BA}{A}$ in relation to the number of solutions to $r/a = r'/a' \in Q$ with $r,r' \in \frac{BA}{A}$, $a,a' \in A$ to find popular subsets $R_1, R_2 \subseteq \frac{BA}{A}$ in terms of these solutions. We have

$$|X| \leq \frac{|AB|^4(\log|A|)^4}{|A|^4|B|^4}\sum_{i=1}^{2\log|A|}\sum_{\substack{x \in \frac{AB}{A} \\ 2^i \leq r_{\frac{AB}{A}}(x) < 2^{i+1}}} r_{\frac{AB}{A}}(x)\left|\left\{(a_3,a_4,b_1,b_3,b_4) \in A^2 \times B^3 : \frac{x}{b_1} = \frac{a_3b_3}{a_4a_4} \in Q\right\}\right|.$$

We use the pigeonhole principle to give us $\Delta_1 > 0$ and $R_1 \subseteq \frac{AB}{A}$ such that

$$|X| \ll \Delta_1\frac{|AB|^4(\log|A|)^5}{|A|^4|B|^4}\left|\left\{(r_1,a_3,a_4,b_2,b_3,b_4) \in R_1 \times A^2 \times B^3 : \frac{r_1}{b_2} = \frac{a_3b_3}{a_4b_4} \in Q\right\}\right|.$$

We perform a similar dyadic decomposition to get $\Delta_1' > 0$ and $R_2 \subseteq \frac{AB}{A}$ such that

$$|X| \ll \Delta_1 \Delta_1' \frac{|AB|^4 (\log|A|)^6}{|A|^4|B|^4} \left| \left\{ (r_1, r_2, b_2, b_4) \in R_1 \times R_2 \times B^2 : \frac{r_1}{b_2} = \frac{r_2}{b_4} \in Q \right\} \right|.$$

These decompositions now allow us to bound via fourth energies, as follows.

$$\begin{aligned} |X| &\ll \Delta_1 \Delta_1' \frac{|AB|^4 (\log|A|)^6}{|A|^4|B|^4} \left| \left\{ (r_1, r_2, b_2, b_4) \in R_1 \times R_2 \times B^2 : \frac{r_1}{b_2} = \frac{r_2}{b_4} \in Q \right\} \right| \\ &= \Delta_1 \Delta_1' \frac{|AB|^4 (\log|A|)^6}{|A|^4|B|^4} \sum_{q \in Q} r_{R_1/B}(q) r_{R_2/B}(q) \\ &\leq \Delta_1 \Delta_1' \frac{|AB|^4 (\log|A|)^6}{|A|^4|B|^4} \left(\sum_{q \in Q} r_{R_1/B}(q)^2 \right)^{1/2} \left(\sum_{q \in Q} r_{R_2/B}(q)^2 \right)^{1/2} \\ &\leq \Delta_1 \Delta_1' |Q|^{1/2} \frac{|AB|^4 (\log|A|)^6}{|A|^4|B|^4} E_4^*(B, R_1)^{1/4} E_4^*(B, R_2)^{1/4} \end{aligned} \tag{13}$$

where the third and fourth lines follow from applications of the Cauchy-Schwarz inequality. We will now show that given $|B||D||R_i|^2 \ll p^2$ and $|B| \leq |D|$ (which are true under our assumptions), we have

$$E_4^*(B, R_i) \ll \frac{|D(B-1)|^3 |R_i|^2}{|D|} \log|B|. \tag{14}$$

Firstly, with the additional conditions

$$|B|^2 |D(B-1)| \leq |R_i||D|^3, \quad |B||D(B-1)|^2 \leq |R_i|^2 |D|^3 \tag{15}$$

we may bound these fourth energies by Theorem 5 to get (14). We can therefore assume one of these conditions does not hold.

Firstly, suppose that $|B|^2|D(B-1)| > |R_i||D|^3$. We will use the trivial bound

$$E_4^*(B, R_i) \leq |R_i|^4 |B|.$$

Note that it would be enough to prove

$$E_4^*(B, R_i) \leq \frac{|D(B-1)|^3 |R_i|^2}{|D|}$$

which would follow from

$$|R_i|^4 |B| \leq \frac{|D(B-1)|^3 |R_i|^2}{|D|} \tag{16}$$

which is true if and only if $|R_i|^2|B||D| \leq |D(B-1)|^3$. Using our assumed bound $|B|^2|D(B-1)| > |R_i||D|^3$, we know that

$$|R_i|^2 |B||D| < \frac{|B|^5 |D(B-1)|^2}{|D|^5}.$$

By the assumption $|B| \leq |D|$, we have

$$|R_i|^2|B||D| < \frac{|B|^5|D(B-1)|^2}{|D|^5} \leq |D(B-1)|^3$$

and so by (16), the bound on the fourth energy holds.

Now assume the second condition from (15) does not hold, that is, $|B||D(B-1)|^2 > |R_i|^2|D|^3$. Again, we use the trivial bound

$$E_4^*(B, R_i) \leq |R_i|^4|B|.$$

We have

$$|R_i|^4|B| \leq \frac{|D(B-1)|^3|R_i|^2}{|D|} \iff |R_i|^2|B||D| \leq |D(B-1)|^3$$

so that it is enough to prove $|R_i|^2|B||D| \leq |D(B-1)|^3$, as before. Using the assumption $|B||D(B-1)|^2 > |R_i|^2|D|^3$, we have the information that

$$|R_i|^2|B||D| < \frac{|B|^2|D(B-1)|^2}{|D|^2}$$

and it follows from our assumption $|B| \leq |D|$ that

$$\frac{|B|^2|D(B-1)|^2}{|D|^2} \leq |D(B-1)|^3.$$

Therefore we have $|R_i|^2|B||D| < |D(B-1)|^3$ and so the bound on the fourth energy holds. Returning to equation (13), we use (14) to bound $|X|$ as

$$\begin{aligned} |X| &\ll \Delta_1\Delta_1'|Q|^{1/2}\frac{|AB|^4(\log|A|)^6}{|A|^4|B|^4}E_4^*(B,R_1)^{1/4}E_4^*(B,R_2)^{1/4} \\ &\ll \Delta_1\Delta_1'|R_1|^{1/2}|R_2|^{1/2}|Q|^{1/2}\frac{|AB|^4|D(B-1)|^{3/2}}{|A|^4|B|^4|D|^{1/2}}(\log|A|)^6(\log|B|)^{1/2} \end{aligned} \tag{17}$$

As $|R_i|\Delta_i \leq \sum_{x\in R_i} r_{BA/A}(x)$, the product $|R_1|^{1/2}|R_2|^{1/2}\Delta_1\Delta_1'$ can be bounded by

$$|R_1|^{1/2}|R_2|^{1/2}\Delta_1\Delta_1' \leq \left(\sum_{x\in R_1} r_{\frac{BA}{A}}(x)^2 \sum_{x\in R_2} r_{\frac{BA}{A}}(x)^2\right)^{1/2}$$

where it is important to note that $r_{\frac{BA}{A}}(x)$ gives a triple (b, a, a'). For $i = 1, 2$, we have

$$\sum_{x\in R_i} r_{\frac{BA}{A}}(x)^2 \leq \left|\left\{(a_1,a_2,a_3,a_4,b_1,b_2) \in A^4 \times B^2 : \frac{b_1a_1}{a_2} = \frac{b_2a_3}{a_4}\right\}\right|.$$

Following a similar dyadic decomposition as before, we find a pair of subsets $S_1, S_2 \subseteq A/A$ with respect to these solutions, and some $\Delta_2, \Delta_2' > 0$ with

$$\sum_{x \in R_i} r_{\frac{BA}{A}}(x)^2 \ll \Delta_2 \Delta_2' (\log |A|)^2 \left| \{(s_1, s_2, b_1, b_2) \in S_1 \times S_2 \times B^2 : s_1 b_1 = s_2 b_2\} \right|$$
$$\leq \Delta_2 \Delta_2' (\log |A|)^2 \sum_x r_{S_1 B}(x) r_{S_2 B}(x)$$
$$\leq \Delta_2 \Delta_2' (\log |A|)^2 E^*(B, S_1)^{1/2} E^*(B, S_2)^{1/2}.$$

Where the third inequality is given by the Cauchy-Schwarz inequality. We will use a similar argument as above to prove that with the two conditions $|B||D||S_i| \min\{|D|, |S_i|\} \ll p^2$ and $|B| \leq |D|$ (which are satisfied under our assumptions), we have

$$E^*(B, S_i) \ll \frac{|S_i|^{3/2} |D(B-1)|^{3/2}}{|D|^{1/2}} \log |B|. \tag{18}$$

Under the extra conditions

$$|B|^2 |D(B-1)| \leq |S_i||D|^3, \qquad |B||D(B-1)|^2 \leq |S_i|^2 |D|^3 \tag{19}$$

we can bound this energy by Theorem 5 to get (18). We therefore assume the first condition from (19) does not hold, that is, $|B|^2|D(B-1)| > |S_i||D|^3$. We bound the energy via the trivial estimate

$$E^*(B, S_i) \leq |B||S_i|^2.$$

It is now enough to show that

$$|B||S_i|^2 \leq \frac{|S_i|^{3/2}|D(B-1)|^{3/2}}{|D|^{1/2}} \quad \text{which is true iff} \quad |B||D|^{1/2}|S_i|^{1/2} \leq |D(B-1)|^{3/2}.$$

Using our assumption $|B|^2|D(B-1)| > |S_i||D|^3$, we have that

$$|B||D|^{1/2}|S_i|^{1/2} < \frac{|B|^2|D(B-1)|^{1/2}}{|D|}.$$

Our assumption that $|B| \leq |D|$ then gives

$$\frac{|B|^2|D(B-1)|^{1/2}}{|D|} \leq |B||D(B-1)|^{1/2} \leq |D(B-1)|^{3/2}$$

so that $|B||D|^{1/2}|S_i|^{1/2} < |D(B-1)|^{3/2}$, and the bound (18) holds. Next we assume that the second condition in (19) does not hold, that is, $|B||D(B-1)|^2 > |S_i|^2|D|^3$. We again use the trivial bound

$$E^*(B, S_i) \leq |B||S_i|^2.$$

Comparing this to our desired bound, we have

$$|B||S_i|^2 \leq \frac{|S_i|^{3/2}|D(B-1)|^{3/2}}{|D|^{1/2}} \iff |B||D|^{1/2}|S_i|^{1/2} \leq |D(B-1)|^{3/2}$$

so that the desired bound would follow from the second inequality above. Using our assumption $|B||D(B-1)|^2 > |S_i|^2|D|^3$, we know that

$$|B||D|^{1/2}|S_i|^{1/2} < \frac{|B|^{5/4}|D(B-1)|^{1/2}}{|D|^{1/4}}$$

and by our assumption that $|B| \leq |D|$, we have

$$\frac{|B|^{5/4}|D(B-1)|^{1/2}}{|D|^{1/4}} \leq |D(B-1)|^{3/2}$$

so that we have $|B||D|^{1/2}|S_i|^{1/2} < |D(B-1)|^{3/2}$ as needed.

In all cases the bound on $E^*(B, S_i)$ holds, so that we find

$$\begin{aligned}\left[|R_1|^{1/2}|R_2|^{1/2}\Delta_1\Delta_1'\right]^2 &\ll \Delta_2^2\Delta_2'^2 E^*(B,S_1)E^*(B,S_2)(\log|A|)^4 \\ &\ll \frac{\Delta_2^2\Delta_2'^2|S_1|^{3/2}|S_2|^{3/2}|D(B-1)|^3}{|D|}(\log|A|)^4(\log|B|)^2 \\ &\leq \frac{E^*_{4/3}(A)^3|D(B-1)|^3}{|D|}(\log|A|)^4(\log|B|)^2.\end{aligned}$$

Where the final inequality follows as Δ_2 and Δ_2' correspond to representations of elements of S_1 and S_2 in A/A, so that $|S_1|^{3/2}\Delta_2^2 = \left(|S_1|\Delta_2^{4/3}\right)^{3/2} \leq \left(\sum_x r_{A/A}(x)^{4/3}\right)^{3/2} \leq E^*_{4/3}(A)^{3/2}$, and similarly for S_2. Combining the bounds (11), (17), and the above, we have

$$|Q|^{3/2}\Delta^2|B|^{13/2}|A|^{5/2}|D|^{3/2}|C|^{1/2} \ll |AB|^4|C(A+1)||D(B-1)|^4E^*_{4/3}(A)^{3/2}(\log|A|)^{17/2}(\log|B|)^2$$

which simplifies to

$$E^*_{4/3}(A')^3|B|^{13}|A|^5|D|^3|C| \ll |AB|^8|C(A+1)|^2|D(B-1)|^8E^*_{4/3}(A)^3(\log|A|)^{17}(\log|B|)^4.$$

We know by Lemma 1 that $E_{4/3}(A') \gg E_{4/3}(A)$, so we have that

$$|B|^{13}|A|^5|D|^3|C| \ll |AB|^8|C(A+1)|^2|D(B-1)|^8(\log|A|)^{17}(\log|B|)^4$$

as needed. □

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