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Abstract. We show that the unresolved examples of Christol's conjecture $_3F_2\left([2/9, 5/9, 8/9], [2/3, 1], x\right)$ and $_3F_2\left([1/9, 4/9, 7/9], [1/3, 1], x\right)$, are indeed diagonals of rational functions. We also show that other $_3F_2$ and $_4F_3$ unresolved examples of Christol's conjecture are diagonals of rational functions. Finally we give two arguments that show that it is likely that the $_3F_2([1/9, 4/9, 5/9], [1/3, 1], 27 \cdot x)$ function is a diagonal of a rational function.

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1. Introduction

There is a plethora of multiple integrals in physics: Feynman integrals, lattice Green functions, the summands of the magnetic susceptibility of the 2D Ising model [1, 2], that have very specific mathematical properties. These functions are D-finite, i.e., solutions of linear differential operators with polynomial coefficients, and have series expansions with *integer coefficients*. Furthermore, it was also shown that the linear differential operators annihilating the summands of the magnetic susceptibility of the Ising model $\tilde{\chi}(n)$, verify the specific property of being Fuchsian† operators: the critical exponents of all their singularities are given by rational numbers, and their Wronskians are N-th roots of rational functions [1, 2]. It was also shown that the $\tilde{\chi}(n)$ functions are solutions of globally nilpotent operators [11], and that they "come from geometry" being G-operators [3].

The unifying scheme behind these seemingly sparse properties is that these functions are *diagonals of rational functions* [4, 5]. It was shown for example in [5], that if summands of the magnetic susceptibility $\tilde{\chi}(n)$ for any n have an integer coefficient series expansion reducing to algebraic series modulo any prime, it is because they are diagonals of rational functions for any integer n. In fact many problems

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† Denoting by θ the homogeneous derivative $x \cdot \frac{d}{dx}$, the degrees of all the polynomial terms of the Fuchsian linear differential operator $\sum_n P_n(x) \cdot \theta^n$ are equal.

in mathematical physics involving n-fold integrals, could be interpreted in terms of diagonals of algebraic or rational¶ functions†.

Gilles Christol has shown in [7] that for every rational function, its diagonal $f(x)$ has the following properties:

- it is globally bounded: there exist integers c and d in $\mathbb{N}^*$, such that $d\,f(c\,x) \in \mathbb{Z}[[x]]$ and $f(x)$ has a radius of convergence that is non-zero in $\mathbb{C}$.
- D-finite: there exists a differential operator $L \in \mathbb{Z}[x][\frac{d}{dx}]$, with $L \neq 0$, such that $L(f) = 0$.

Christol conjectured in [8] that every series that verifies these two properties *is the diagonal of a rational function*. In this paper [8] Christol came up with an unresolved example to his conjecture [8], and a longer list was generated by Christol and his co-authors in 2012 in [5]. In this paper we show that two of the unresolved examples of the conjecture given in [5] on page 58, namely the $_3F_2\left([2/9, 5/9, 8/9], [2/3, 1], 3^6 \cdot x\right)$ and $_3F_2\left([1/9, 4/9, 7/9], [1/3, 1], 3^6 \cdot x\right)$ are indeed diagonals of rational functions and provide a generalization of this result.

2. Recalls on diagonals of rational functions and on Christol's conjecture

2.1. Definition of the diagonal of a rational function

The diagonal of a rational function in n variables $\mathcal{R}(x_1, \ldots, x_n) = \mathcal{P}(x_1, \ldots, x_n)/\mathcal{Q}(x_1, \ldots, x_n)$, where $\mathcal{P}, \mathcal{Q} \in \mathbb{Q}[x_1, \ldots, x_n]$ such that $\mathcal{Q}(0, \ldots, 0) \neq 0$, is defined through its multi-Taylor expansion around $(0, \ldots, 0)$:

$$\mathcal{R}\left(x_1, \ldots, x_n\right) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} R_{m_1, \ldots, m_n} \cdot x_1^{m_1} \ldots x_n^{m_n}, \qquad (1)$$

as the series in *one variable* x:

$$Diag\left(\mathcal{R}\left(x_1, \ldots, x_n\right)\right) = \sum_{m=0}^{\infty} R_{m, m, \ldots, m} \cdot x^m. \qquad (2)$$

2.2. Hadamard product of algebraic functions and Christol's conjecture

Recall that the Hadamard product of two series $f(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n$ and $g(x) = \sum_{n=0}^{\infty} \beta_n \cdot x^n$ is given by:

$$f(x) \star g(x) = \sum_{n=0}^{\infty} \alpha_n \cdot \beta_n \cdot x^n. \qquad (3)$$

Hypergeometric series of the form $_pF_{p-1}([a_1, \ldots, a_p, b_1 \ldots, b_{p-1}], [1, \ldots, 1], x)$ of height $h = h(a_1, \ldots, a_p, b_1 \ldots, b_{p-1})$, where the height h is given by:

$$h = \#\{1 \leq j \leq p \mid b_j \in \mathbb{Z}\} - \#\{1 \leq j \leq p \mid a_j \in \mathbb{Z}\} \qquad (4)$$

with $b_p = 1$, '#' meaning "the number of", can be written‡ as the Hadamard product of h globally bounded♯ series of height 1, were shown to verify Christol's conjecture.

¶ Any diagonal of an algebraic function in n variables can be rewritten as the diagonal of a rational function in $2n$ variables: see [6].
† See [4, 5] p.26 and p.58.
‡ See [8] p.15.
♯ Globally bounded series can be recast into series with *integer* coefficients [4, 5].

For example, the globally bounded hypergeometric series $_3F_2([1/3,\ 1/3,\ 1/3],\ [1,\ 1],x)$ has height 3, and it can be written as the Hadamard product of three algebraic functions†:

$$_3F_2\Big([\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}],\ [1,\ 1],\ x\Big)\ =\ (1-x)^{-1/3}\star(1-x)^{-1/3}\star(1-x)^{-1/3}, \tag{5}$$

and can thus be written as the diagonal of the algebraic function in three variables:

$$(1-x)^{-1/3}\cdot(1-y)^{-1/3}\cdot(1-z)^{-1/3}. \tag{6}$$

Unlike the case of $_3F_2([1/3,\ 1/3,\ 1/3],\ [1,\ 1],x)$, the hypergeometric functions $_3F_2([2/9,5/9,8/9],\ [2/3,1],\ x)$ and $_3F_2([1/9,4/9,7/9],\ [1/3,1],\ x)$, while being *globally bounded* functions [9], were constructed in a way that prevents them from being written as "simple" Hadamard products of algebraic functions.

2.3. Unresolved examples to the conjecture

Generalized hypergeometric functions with regular singularities $_pF_{p-1}$ are a simple and natural testing ground for Christol's conjecture.

All $_2F_1([a,b],[c],\ x)$ hypergeometric series that are globally bounded are diagonals of rational functions. There are two cases that fall into this category:

- Either the parameter c is an integer, and the $_2F_1$ function can be written as the Hadamard product of two $_1F_0$ functions, which are algebraic functions, and thus are diagonals of rational functions by Furstenberg's [10] theorem¶.

- The parameter c is not an integer, and in this case the $_2F_1$ function is a diagonal of a rational function if and only if it is an algebraic function†† (and consequently its series is globally bounded).

Moving on to $_3F_2$ hypergeometric functions, one can ask when is a $_3F_2$ hypergeometric function a diagonal of a rational function?

- When the parameters d and e in $_3F_2([a,b,c],\ [d,e],\ x)$ are integers, because, in this case, the $_3F_2$ can be written as the Hadamard product of three $_1F_0$ algebraic functions, and is thus the diagonal of a rational function, by the closure of diagonals under the Hadamard product.

- When the parameters d and e in $_3F_2([a,b,c],\ [d,e],\ x)$ are rational numbers but not integers, because in this case the $_3F_2$ is algebraic, and is thus a diagonal by Furstenberg's theorem.

Hence the interesting case occurs when *only one* of the two parameters d or e is rational, and the other is an integer. But even in this case, a lot of the $_3F_2$ functions are easily seen to be diagonals of a rational function. Suppose that a $_3F_2([a,b,c],\ [1,e],\ x)$ is globally bounded, with the parameters $a,\ b,\ c,\ e\in\mathbb{Q}\setminus\mathbb{Z}$, then there are six ways to write the $_3F_2([a,b,c],[1,e],x)$ function as the diagonal of a rational function. This corresponds to the six ways to write the $_3F_2([a,b,c],[1,e],x)$ as a Hadamard product of hypergeometric functions:

† Diagonals are closed under the Hadamard product: if two series are diagonals of rational functions, their Hadamard product is also a diagonal of a rational function.
¶ Furstenberg's theorem states that any algebraic function is the diagonal of a rational function in two variables.
†† The only $_2F_1$ hypergeometric functions that are globally bounded with $c\in\mathbb{Q}$ are the algebraic ones: they are the ones appearing in the list of Schwarz [12].

- $_2F_1([a,b],\,[e],\,x) \star {}_1F_0([c],\,x)$
- $_2F_1([a,c],\,[e],\,x) \star {}_1F_0([b],\,x)$
- $_2F_1([b,c],\,[e],\,x) \star {}_1F_0([a],\,x)$
- $_2F_1([a,b],\,[1],\,x) \star {}_2F_1([c,1],\,[e],\,x)$
- $_2F_1([a,c],\,[1],\,x) \star {}_2F_1([b,1],\,[e],\,x)$
- $_2F_1([b,c],\,[1],\,x) \star {}_2F_1([a,1],\,[e],\,x)$

Now $_1F_0([c],\,x)$ and $_2F_1([a,b],[1],x)$ are diagonals of rational functions by what we have said above. Then $_3F_2([a,b,c],\,[1,e],\,x)$ is a diagonal of rational functions if $_2F_1([c,1],\,[e],\,x)$ or $_2F_1([a,b],[e],x)$††, are diagonals of rational functions, i.e. if and only if they are algebraic functions, since $e \in \mathbb{Q} \setminus \mathbb{Z}$. Now $_2F_1([c,1],[e],x)$ cannot be an algebraic functions for $e \in \mathbb{Q}$ by Goursat [13]. Hence if one of $_2F_1([a,b],\,[e],\,x)$, $_2F_1([b,c],\,[e],\,x)$, or $_2F_1([a,c],\,[e],\,x)$ is an algebraic function, then $_3F_2([a,b,c],[1,e],\,x)$ is the diagonal of a rational function. Now taking the two examples given in [5] or by G. Christol in [14]♯ that we are looking at here, we see that neither $H_1 = {}_3F_2\left([2/9,5/9,8/9],[2/3,1],\,x\right)$, nor $H_2 = {}_3F_2\left([1/9,4/9,7/9],[2/3,1],\,x\right)$, can be obtained as diagonals of rational functions through Hadamard products† since the three $_2F_1$ hypergeometric series are *not* globally bounded¶:

$$_2F_1\left([\tfrac{2}{9},\tfrac{5}{9}],\,[\tfrac{2}{3}],\,x\right), \qquad {}_2F_1\left([\tfrac{2}{9},\tfrac{8}{9}],\,[\tfrac{2}{3}],\,x\right), \qquad {}_2F_1\left([\tfrac{5}{9},\tfrac{8}{9}],\,[\tfrac{2}{3}],\,x\right), \qquad (7)$$

and nor are the $_2F_1$ hypergeometric series:

$$_2F_1\left([\tfrac{1}{9},\tfrac{4}{9}],\,[\tfrac{1}{3}],\,x\right), \qquad {}_2F_1\left([\tfrac{4}{9},\tfrac{7}{9}],\,[\tfrac{1}{3}],\,x\right), \qquad {}_2F_1\left([\tfrac{1}{9},\tfrac{7}{9}],\,[\tfrac{1}{3}],\,x\right). \qquad (8)$$

3. The main results

The globally bounded $_3F_2$ hypergeometric series

$$_3F_2\left([\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9}],\,[\tfrac{2}{3},1],\,27\cdot x\right), \qquad {}_3F_2\left([\tfrac{1}{9},\tfrac{4}{9},\tfrac{7}{9}],\,[\tfrac{1}{3},1],\,27\cdot x\right) \qquad (9)$$

are‡ respectively *the diagonals of the two algebraic functions*

$$_3F_2\left([\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9}],\,[\tfrac{2}{3},1],\,27\cdot x\right) \;=\; Diag\left(\frac{(1-x-y)^{1/3}}{1-x-y-z}\right), \qquad (10)$$

and

$$_3F_2\left([\tfrac{1}{9},\tfrac{4}{9},\tfrac{7}{9}],\,[\tfrac{1}{3},1],\,27\cdot x\right) \;=\; Diag\left(\frac{(1-x-y)^{2/3}}{1-x-y-z}\right). \qquad (11)$$

These two hypergeometric series†† (9) can be recast into series with *integer* coefficients

$$_3F_2\left([\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9}],\,[\tfrac{2}{3},1],\,3^6\cdot x\right) \;=\; 1 + 120x + 47124x^2 + 23483460x^3 + \cdots, \qquad (12)$$

†† Instead of $_2F_1([c,1],[e],x)$, or one could take any one of the three permuted versions: $_2F_1([b,1],\,[e],\,x)$, etc.
♯ Appendix F p.58 of [5], also see [14] p.19.
† See [23] for a proof that H_2 cannot be written as a Hadamard product.
¶ One can see this experimentally by taking the series expansion of any of the Gauss hypergeometric functions: the prime numbers in the denominators of the coefficients grow continuously.
‡ The operators annihilating the two hypergeometric functions (9) are adjoint of each other.
†† The hypergeometric function $_3F_2([2/9,5/9,8/9],[2/3,1],27x)$ can be rewritten as the Hadamard product $_2F_1\left([\tfrac{2}{9},\tfrac{5}{9}],[\tfrac{2}{3}],\,27x\right) \star (1-x)^{-8/9}$ with $_2F_1\left([\tfrac{2}{9},\tfrac{5}{9}],[\tfrac{2}{3}],\,27x\right)$ being associated with a *Shimura curve* [15]. For more details please refer to Appendix A.

and

$$_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{1}{3}, 1], \ 3^6 \cdot x\Big) \ = \ 1 \ + 84x \ + 32760x^2 \ + 16302000x^3 \ + \cdots \qquad (13)$$

Now Denef and Lipshitz in [6] show that any power series in $\mathbb{Q}[[x_1, \ldots, x_n]]$, algebraic over $\mathbb{Q}(x_1, \ldots, x_n)$, is the diagonal of a rational function in $2n$ variables, and they give an algorithm to build this rational function. This means that we can construct the rational functions, whose corresponding diagonals are the $_3F_2([2/9, 5/9, 8/9], [2/3, 1], 27 \cdot x)$ and the $_3F_2([1/9, 4/9, 7/9], [1/3, 1], 27 \cdot x)$ functions. We recall the algorithm of Denef and Lipshitz and apply it to the algebraic function $(1-x-y)^{1/3}/(1-x-y-z)$ in the first subsection below, and then we give the rational function and a generalization of the result in the second subsection. Finally, we give a second proof of the general result using binomial sums.

3.1. From diagonals of algebraic functions to diagonals of rational functions: Denef and Lipshitz

We explain a method which, for a given algebraic power series in n variables, constructs a rational function in $2n$ variables whose diagonal equals the diagonal of the given algebraic series. Moreover, the partial diagonal of that $2n$-variable rational function, with respect to the pairs of variables $(x_1, x_{n+1}), \ldots, (x_{n-1}, x_{2n})$, yields the original n-variable algebraic power series. The method is described in the paper by Denef and Lipshitz [6] in the proof of their Theorem 6.2. As a running example we use the three-variable algebraic function

$$f(x, y, z) \ = \ \frac{(1-x-y)^{1/3}}{1-x-y-z}, \qquad (14)$$

whose multi-Taylor series expansion at 0 is actually a power series in the three variables x, y, z:

$$f(x, y, z) \ = \ 1 \ + \frac{2}{3}x \ + \frac{2}{3}y \ + z \ + \frac{10}{9}xy \ + \frac{5}{3}xz \ + \frac{5}{3}yz \ + \frac{40}{9}xyz \ + \ldots \qquad (15)$$

Note that the minimal polynomial of f is given by

$$p(x, y, z, f) \ = \ \Big((x+y+z-1) \cdot f\Big)^3 \ + 1 \ - x - y. \qquad (16)$$

Denef and Lipshitz's theorem is formulated for étale extensions, which basically means that the partial derivative (w.r.t. f) of the minimal polynomial has a nonzero constant coefficient at 0. Clearly, the above polynomial $p(x, y, z, f)$ does not meet this criterion. However, by considering $\tilde{f} = f - 1$, i.e. by removing the constant term of f, we can achieve an étale extension. The minimal polynomial then reads

$$\tilde{p}(x, y, z, f) \ = \ \Big((x+y+z-1) \cdot (f+1)\Big)^3 \ + 1 \ - x - y. \qquad (17)$$

Indeed, $\frac{\partial \tilde{p}}{\partial f}(0,0,0,0) = -3 \neq 0$. According to the proof of Theorem 6.2 (i) in [6], the rational function

$$\tilde{r}(x, y, z, f) \ = \ f^2 \cdot \frac{\frac{\partial \tilde{p}}{\partial f}(xf, yf, zf, f)}{\tilde{p}(xf, yf, zf, f)} \qquad (18)$$

has the property that $\mathcal{D}\big(\tilde{r}(x, y, z, f)\big) = \tilde{f}(x,y,z)$, and hence $\mathcal{D}\big(r(x,y,z,f)\big) = f(x,y,z)$ for $r(x, y, z, f) = \tilde{r}(x, y, z, f) + 1$. Here the operator $\mathcal{D}$ denotes a special kind of "diagonalization" with respect to the last variable: for

$$f(x_1, \ldots, x_n, y) \ = \ \sum a_{i_1, \ldots, i_n, j} \cdot x_1^{i_1} \cdots x_n^{i_n} y^j, \qquad (19)$$

one defines

$$\mathcal{D}\big(f(x_1, \ldots, x_n, y)\big) \;=\; \sum_{j=i_1+\cdots+i_n} a_{i_1,\ldots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n}. \tag{20}$$

In our running example we obtain:

$$r(x, y, z, f) \;=\; \frac{3\,f^2 \cdot (f+1)^2 \cdot (xf + yf + zf\; -1)^3}{(f+1)^3 \cdot (xf + yf + zf\; -1)^3\; -xf\; -yf\; +1} \;+\; 1. \tag{21}$$

In the second step, which is explained in the proof of Theorem 6.2(ii) of [6], one has to transform the rational function r (that has $n+1$ variables) into another rational function (having $2n$ variables) such that its "true" (partial) diagonal gives the n-variable algebraic series f. It consists of a sequence of $n-1$ elementary steps, each of which is adding one more variable. In our example, we have to do the following

$$r_1(x, y, z, u_1, v_1) \;=\; \frac{u_1 \cdot r(x, y, z, u_1) - v_1 \cdot r(x, y, z, v_1)}{u_1 - v_1}, \tag{22}$$

$$r_2(x, y, z, u_1, u_2, v_2) \;=\; \frac{u_2 \cdot r_1(x, y, z, u_1, u_2) - v_2 \cdot r_1(x, y, z, u_1, v_2)}{u_2 - v_2},$$

and obtain with r_2 the desired rational function in *six* variables.

3.2. Generalization of the previous result

By the algorithm of Denef and Lipshitz given in the previous section, it is possible to show that the algebraic function

$$\frac{(1\; -x\; -y)^{a/b}}{1\; -x\; -y\; -z}, \tag{23}$$

corresponds to the following rational function in *six* variables, by taking the diagonal with respect to (x, u), (y, v) and (z, w):

$$\begin{aligned}
&\frac{a \cdot u^3 v \cdot (1 - ux - uy - uz) \cdot (1+u)^{a-1} \cdot (1 - ux - uy - uz)^{a-1}}{(1+u)^a \cdot (1 - ux - uy - uz)^a - (1 - ux - uy)^b \cdot (u-v) \cdot (v-w)} \\
&\quad - \frac{a \cdot v^4 \cdot (1 - vx - vy - vz) \cdot ((1+v) \cdot (1 - vx - vy - vz))^{a-1}}{(1+v)^a \cdot (1 - vx - vy - vz)^a - (1 - vx - vy)^b \cdot (u-v)(v-w)} \\
&\quad - \frac{a \cdot u^3 w \cdot (1 - ux - uy - uz)((1+u) \cdot (1 - ux - uy - uz))^{a-1}}{(1+u)^a \cdot (1 - ux - uy - uz)^a - (1 - ux - uy)^b \cdot (u-w) \cdot (v-w)} \\
&\quad - \frac{a w^4 \cdot (1 - wx - wy - wz) \cdot (1+w)^{a-1} \cdot (1 - wx - wy - wz)^{a-1}}{(1+w)^a \cdot (1 - wx - wy - wz)^a - (1 - wx - wy)^b \cdot (u-w) \cdot (v-w)} + 1.
\end{aligned} \tag{24}$$

The diagonal of the rational function (24) is annihilated by the linear differential operator of order three:

$$\begin{aligned}
&b^3\,x^2\,(1 - 27\,x) \cdot D_x^3 + b^2\,x\,((27\,a - 135\,b) \cdot x - a + 3\,b) \cdot D_x^2 \\
&\quad - b \cdot ((9\,a^2 - 63\,a\,b + 114\,b^2) \cdot x + a\,b - b^2) \cdot D_x + (a - 3\,b) \cdot (a - 2\,b) \cdot (a - b),
\end{aligned} \tag{25}$$

and can be expressed as the $_3F_2$ hypergeometric function

$$_3F_2\Big([\frac{3\,a - b}{3\,a}, \frac{2\,a - b}{3\,a}, \frac{a - b}{3\,a}], \ [\frac{a - b}{a}, 1], \ 27 \cdot x\Big). \tag{26}$$

In particular, the two hypergeometric functions $_3F_2([2/9, 5/9, 8/9], [2/3, 1], 27 \cdot x)$ and $_3F_2([1/9, 4/9, 7/9], [1/3, 1], 27 \cdot x)$ appearing in (9), correspond respectively to the parameters $(a, b) = (3, 1)$, and $(a, b) = (3, 2)$ in the algebraic function (23). Other

values of the parameters (a, b) are not necessarily unresolved examples of Christol's conjecture.

For example if we consider the parameter values $a = 7$ and $b = 1$, we see that the diagonal of (24) is given by the *globally bounded*† series (27)

$$_3F_2\Big([\frac{2}{7},\ \frac{13}{21},\ \frac{20}{21}],\ [\frac{6}{7},\ 1],\ 27\,x\Big)\ =\ 1\ +\ \frac{260}{49}\,x\ +\ \frac{188190}{2401}\,x^2\ +\ \cdots \tag{27}$$

with the $_2F_1$ series

$$_2F_1\Big([\frac{13}{21},\ \frac{20}{21}],\ [\frac{6}{7}],\ 27\,x\Big),\quad _2F_1\Big([\frac{2}{7},\ \frac{20}{21}],\ [\frac{6}{7}],\ 27\,x\Big),\quad _2F_1\Big([\frac{2}{7},\ \frac{13}{21}],\ [\frac{6}{7}],\ 27\,x\Big),$$

being series that are *not* globally bounded. Hence the hypergeometric series (27) cannot be easily written as a Hadamard product, as explained in Section 2.3.

In contrast, for $a = 4$ and $b = 3$ the diagonal of (24) which is given by the *globally bounded*‡ series (28)

$$_3F_2\Big([\frac{3}{4},\ \frac{5}{12},\ \frac{1}{12}],\ [\frac{1}{4},\ 1],\ 27\,x\Big)\ =\ 1\ +\ \frac{45}{16}\,x\ +\ \frac{41769}{1024}\,x^2\ +\ \cdots \tag{28}$$

with the $_2F_1$ series

$$_2F_1\Big([\frac{5}{12},\ \frac{1}{12}],\ [\frac{1}{4}],\ 27\,x\Big), \tag{29}$$

being a *globally bounded series*, which means that it can be written as a diagonal using one of the procedures given in Section 2.3. We note that algebraic functions close to the algebraic functions appearing in (10) and (11), also give $_3F_2$ or $_4F_3$ hypergeometric functions as their diagonals that are unresolved examples to Christol's conjecture:

$$Diag\Big(\frac{(1-x-2\,y)^{2/3}}{1-x-y-z}\Big)\ =\ _3F_2\Big([\frac{1}{9},\ \frac{4}{9},\ \frac{7}{9}],\ [\frac{2}{3},\ 1],\ 27\cdot x\Big), \tag{30}$$

$$Diag\Big(\frac{(1-x-2\,y)^{1/3}}{1-x-y-z}\Big)\ =\ _3F_2\Big([\frac{2}{9},\ \frac{5}{9},\ \frac{8}{9}],\ [\frac{5}{6},\ 1],\ 27\cdot x\Big), \tag{31}$$

$$Diag\Big(\frac{(1-x)^{1/3}}{1-x-y-z}\Big)\ =\ _4F_3\Big([\frac{2}{9},\ \frac{5}{9},\ \frac{8}{9},\ \frac{1}{2}],\ [\frac{1}{3},\ \frac{5}{6},\ 1],\ 27\cdot x\Big), \tag{32}$$

$$Diag\Big(\frac{(1-x-y)^{1/3}}{1-x-z}\Big)\ =\ _4F_3\Big([\frac{2}{9},\ \frac{5}{9},\ \frac{8}{9},\ \frac{-1}{3}],\ [\frac{1}{3},\ \frac{5}{6},\ 1],\ 27\cdot x\Big). \tag{33}$$

3.3. Proof

A computer algebra proof of this result can easily be obtained using the creative telescoping program [16]: one computes the operator (25) using the program [16], and verifies that this operator does annihilate the diagonal of (23)††. Another longer way to do it which we provide below, is through binomial sums.

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† $_3F_2\Big([\frac{2}{7},\ \frac{13}{21},\ \frac{20}{21}],\ [\frac{6}{7},\ 1],\ 27\cdot 7^3\cdot x\Big)$ is a series with *integer* coefficients.

‡ $_3F_2\Big([\frac{3}{4},\ \frac{5}{12},\ \frac{1}{12}],\ [\frac{1}{4},\ 1],\ 1728\cdot x\Big)$ is a series with *integer* coefficients.

†† One also needs to note that initial conditions have to be compared.

The denominator of the algebraic function $(1 - x - y)^{a/b}/(1 - x - y - z)$ can be expanded as a geometric series:

$$
\begin{aligned}
(1 - x - y - z)^{-1} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n}{m} \cdot (x + y)^m \, z^{n-m} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \binom{n}{m} \binom{m}{l} \cdot x^l \, y^{m-l} \, z^{n-m},
\end{aligned}
\tag{34}
$$

while the numerator can be written as the sum:

$$
(1 - (x + y))^{a/b} = \sum_{k=0}^{\infty} \frac{(-a/b)_k}{k!} \cdot (x + y)^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-a/b)_k}{k!} \cdot \binom{k}{j} x^j y^{k-j}.
\tag{35}
$$

Multiplying these two sums (34) and (35) and re-indexing, we obtain:

$$
\sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{u=0}^{\infty} x^s \, y^t \, z^u \cdot \sum_{j=0}^{s} \sum_{k=0}^{\infty} \frac{(-a/b)_k}{k!} \cdot \binom{k}{j} \binom{s+t+u-k}{s+t-k} \binom{s+t-k}{s-j}.
\tag{36}
$$

Now taking the coefficients corresponding to the diagonal in (36), i.e. such that $s = t = u = n$, we get:

$$
\begin{aligned}
\sum_{j=0}^{n} \sum_{k=0}^{\infty} \frac{(-a/b)_k}{k!} &\cdot \binom{k}{j} \binom{3n-k}{2n-k} \binom{2n-k}{n-j} \\
&= \sum_{k=0}^{2n} \frac{(-a/b)_k}{k!} \cdot \binom{3n-k}{2n-k} \cdot \sum_{j=0}^{n} \binom{k}{j} \binom{2n-k}{n-j}.
\end{aligned}
\tag{37}
$$

Now recalling the Chu-Vandermonde identity which says that $\binom{2n}{n} = \sum_{j=0}^{n} \binom{k}{j} \binom{2n-k}{n-j}$, we find that (37) can be written as

$$
S(n) = \binom{2n}{n} \cdot \sum_{k=0}^{2n} \frac{(-a/b)_k}{k!} \cdot \binom{3n-k}{2n-k},
\tag{38}
$$

and by using a computer algebra tool like *Mathematica* or *Maple* to simplify this sum into a closed form, from which we can read off the hypergeometric function representation of the diagonal. More precisely, we used creative telescoping (Zeilberger's algorithm) to prove that (38) satisfies the first-order recurrence:

$$
\begin{aligned}
(a - 3b - 3bn) \cdot (a - 2b - 3bn) \cdot (a - b - 3bn) \cdot S(n) \\
= b^2 \cdot (n+1)^2 \cdot (a - b - bn) \cdot S(n+1).
\end{aligned}
\tag{39}
$$

Together with the initial condition $S(0) = 1$, we obtain the closed form

$$
S(n) = \frac{3^{3n} \cdot \left((b-a)/3b\right)_n \cdot \left((2b-a)/3b\right)_n \cdot \left((3b-a)/3b\right)_n}{\left((b-a)/b\right)_n \cdot \left(n!\right)^2}.
\tag{40}
$$

4. Telescopers of algebraic functions versus diagonals of algebraic functions

The diagonal of a rational function and a solution of a telescoper† of a rational function are close, yet distinct notions. A telescoper annihilates an n-fold integral of a rational

---

† By "telescoper" of a rational function $R(x, y, z)$ we denote the output of the creative telescoping program [16], applied to the transformed rational function $R(x/y, y/z, z)/(yz)$, which is a differential operator that annihilates the diagonal of R.

function over *all* integration cycles††. For example the $_3F_2([a, b, c], [d, 1], x)$ can be written through the well-known integral representation as:

$$(1 - y)^{-1-b+d} \cdot y^b \cdot (1 - x \cdot y^2)^{-a} \cdot (1 - z)^{-c}, \tag{41}$$

with $a, b, c, d \in \mathbb{Q}$. Hence if one takes the parameters a, b, c, d to have the values $a = 1/9$, $b = 4/9$, $c = 7/9$, $d = 1/3$, one immediately obtains that the telescoper of the algebraic function

$$\frac{y^{4/9}}{(1 - y)^{10/9} \cdot (1 - x\, y^2)^{1/9} \cdot (1 - z)^{7/9}}, \tag{42}$$

admits as a solution the hypergeometric function $_3F_2([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{1}{3}, 1], x)$. Yet the diagonal of the algebraic function (42) is equal to zero, and it is through the algebraic function (11) that we were able to obtain it as the diagonal of an algebraic function. Other $_3F_2$ unresolved examples to Christol's conjecture like [8, 14]

$$_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], 27 \cdot x\Big), \tag{43}$$

were not obtained here as diagonals of a rational function, yet they are solution of the telescoper of an algebraic function and can thus be seen as a *period of an algebraic variety over a non-evanescent cycle*‡, but not necessarily as a diagonal of an algebraic function (i.e. a period over an evanescent cycle). We give two arguments in favour of the fact that the $_3F_2$ hypergeometric function (43) is most probably a diagonal of an algebraic function.

4.1. Diagonal: algebraic mod p

If one expects $_3F_2$ hypergeometric functions like (43) to be diagonals of an algebraic function, one should find [4, 5] that the corresponding series expansion reduces to an algebraic series modulo any prime number p, or power of a prime number p^r. In order to verify this fact on (43) we look at the series expansion of

$$_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], 27^2 \cdot x\Big) = 1 + 60\,x + 20475\,x^2 + 9373650\,x^3 \tag{44}$$
$$+ 4881796920\,x^4 + 2734407111744\,x^5 + 1605040007778900\,x^6 + \cdots$$

which becomes modulo 2:

$$_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], 27^2 \cdot x\Big) = 1 + x^2 + x^{128} + x^{130}$$
$$+ x^{8192} + x^{8194} + x^{8320} + x^{8322}$$
$$+ x^{524288} + x^{524290} + x^{524416} + x^{524418} + x^{532480} + x^{532482} + x^{532608} + x^{532610}$$
$$+ O(x^{600000})$$
$$= (1 + x^2) \cdot (1 + x^{128}) \cdot (1 + x^{8192}) \cdot (1 + x^{524288}) + O\left(x^{600000}\right). \tag{45}$$

Straightforward guessing gives the infinite product formula

$$F(x) = (1 + x^2) \cdot (1 + x^{2^7}) \cdot (1 + x^{2^{13}}) \cdot (1 + x^{2^{19}}) \cdots (1 + x^{2^{6\,n\,+1}}) \cdots \tag{46}$$

which is solution of

$$F(x) = (1 + x^2) \cdot F(x^{64}) \qquad mod. \quad 2, \tag{47}$$

††Diagonals correspond *only* to *evanescent integration cycles* over rational functions.
‡ To be totally rigorous, one has to consider the two certificates of the telescoping equation see if that the integral of the derivatives of these two certificates over that cycle are actually zero.

i.e. $_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], 27^2 \cdot x\Big)$ is an algebraic function modulo 2 satisfying:

$$F(x) = (1 + x^2) \cdot F(x)^{64} \qquad mod. \quad 2 \tag{48}$$

or:

$$(1 + x^2) \cdot F(x)^{63} = 1 \qquad mod. \quad 2. \tag{49}$$

Modulo 3 we have the following expansion

$$\frac{_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], \ 27^2 \cdot x\Big) - 1}{3} = 2 \cdot F(x) \qquad mod. \quad 3, \tag{50}$$

where:

$$F(x) = x + x^3 + x^9 + x^{27} + x^{81} + x^{243} + x^{729} + x^{2187} + x^{6561}$$
$$+ x^{19683} + x^{59049} + O\left(x^{60000}\right) \tag{51}$$

which is solution of

$$x + F(x^3) = F(x) \qquad mod. \quad 3, \tag{52}$$

i.e. $F(x)$ is an algebraic function modulo 3:

$$x + F(x)^3 = F(x) \qquad mod. \quad 3. \tag{53}$$

Unlike the situation on the $_3F_2\big([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{1}{3}, 1], \ 27^2 \cdot x\big)$ hypergeometric series, it is less obvious how to obtain the $_3F_2\big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], \ 27^2 \cdot x\big)$ as the diagonal of a rational function. It is however possible to obtain the solution of $_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], \ 27^2 \cdot x\Big)$, as a telescoper of an algebraic function, and this solution is an algebraic function modulo p.

4.2. A relation between $_3F_2([1/9, 4/9, 5/9], [1/3, 1], 27 \cdot x)$ and a $_4F_3$ diagonal of an algebraic function

The diagonal of the product of algebraic functions

$$\frac{(1 - x - y)^{2/3}}{(1 - x - y - z)} \cdot (1 - w)^{-5/9}, \tag{54}$$

is given by the $_4F_3$ hypergeometric function $\mathcal{H}$ which is the Hadamard product of $_3F_2([1/9, 4/9, 7/9], [1/3, 1], 27 \cdot x)$ and $(1 - x)^{-5/9}$:

$$\mathcal{H} = {}_4F_3\Big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}], [\frac{1}{3}, 1, 1], \ 27 \cdot x\Big)$$
$$= (1 - x)^{-5/9} \star {}_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{1}{3}, 1], \ 27 \cdot x\Big)$$
$$= Diag\Big(\frac{(1 - x - y)^{2/3}}{(1 - x - y - z)} \cdot (1 - w)^{-5/9}\Big). \tag{55}$$

This $_4F_3$ hypergeometric series (55) *can also* be written as the Hadamard product:

$$\mathcal{H} = (1 - x)^{-7/9} \star {}_3F_2([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], \ 27 \cdot x). \tag{56}$$

So even though we did not find a rational (or algebraic) function whose diagonal is given by (43), knowing that $_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{1}{3}, 1], \ 27 \cdot x\Big)$ is the diagonal of a rational function, we see that the Hadamard product of (43) with a simple algebraic function $(1 - x)^{-7/9}$ is actually a diagonal of an algebraic (or rational) function. This suggests but does not prove, that $_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{5}{9}], [\frac{1}{3}, 1], \ 27 \cdot x\Big)$ could also be a diagonal of a rational function.

5. Conclusion

Because of the crucial role played by diagonals of rational functions in physics [4, 5], Christol's conjecture is an important open problem. We have shown that the hypergeometric series $_3F_2\left([2/9, 5/9, 8/9], [2/3, 1], x\right)$ and $_3F_2\left([1/9, 4/9, 7/9], [1/3, 1], x\right)$ appearing in [5, 14] are diagonals of rational functions. We did so by first finding two algebraic functions whose diagonals were given by these two hypergeometric functions, and through an algorithm outlined in the paper [6], we were able to recover the rational functions whose diagonals are given by these two $_3F_2$ hypergeometric functions.

We were also able to give a generalization of this result, and obtain other unresolved examples of Christol's conjecture as diagonals of rational functions. Furthermore, even though we were not able to write the $_3F_2([1/9, 4/9, 5/9], [1/3, 1], 27 \cdot x)$ given by Christol in [8], as a diagonal of a rational function, we gave two arguments that suggested that it was likely to be so. More generally, we believe after writing the $_3F_2\left([2/9, 5/9, 8/9], [2/3, 1], x\right)$ and $_3F_2\left([1/9, 4/9, 7/9], [1/3, 1], x\right)$ as diagonal of rational functions, that it is likely that the other $_3F_2$ unresolved examples of Christol's conjecture are diagonals of rational functions.

Appendix A. Counterexamples and links with Shimura curves

The Gauss hypergeometric function appearing on the left in (9)

$$_3F_2\left([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}], [\frac{2}{3}, 1], \quad 27\,x\right) \tag{A.1}$$

can be seen as the Hadamard product of a Gauss hypergeometric function and an algebraic function given by:

$$_2F_1\left([\frac{2}{9}, \frac{5}{9}], [\frac{2}{3}], \quad 27\,x\right) \, \star \, (1-x)^{-8/9}. \tag{A.2}$$

Now the Gauss hypergeometric function $_2F_1\left([\frac{2}{9}, \frac{5}{9}], [\frac{2}{3}], \quad 27\,x\right)$ happens to be a hypergeometric function corresponding to an automorphic form associated with a

Shimura curve [15, 18, 19]. One has the identity:

$$_2F_1\Big([\frac{2}{9}, \frac{5}{9}], [\frac{2}{3}], \ 27\,x\Big) \ = \ (1 - 27\,x)^{-1/9} \cdot (1 - 36\,x + 216\,x^2)^{-1/18}$$

$$\times \ _2F_1\Big([\frac{1}{36}, \frac{19}{36}], [\frac{8}{9}], \ -1728 \cdot \frac{x^3 \cdot (1 - 27\,x)}{(1 - 36\,x + 216\,x^2)^2}\Big). \tag{A.3}$$

The Gauss hypergeometric function $_2F_1\Big([\frac{1}{36}, \frac{19}{36}], [\frac{8}{9}], \ x\Big)$ can be also expressed as:

$$_2F_1\Big([\frac{1}{36}, \frac{19}{36}], [\frac{8}{9}], \ x\Big) \ = \ (1 - x)^{-1/36} \cdot \ _2F_1\Big([\frac{1}{36}, \frac{13}{36}], [\frac{8}{9}], \ -\frac{x}{1 - x}\Big). \tag{A.4}$$

Now the Gauss hypergeometric function $_2F_1([\frac{1}{36}, \frac{13}{36}], [\frac{8}{9}], \ x)$ which occurs in p.14 of [20], corresponds to a hypergeometric function related to a *Shimura curve* since it has exponent differences† $(1/9, 1/2, 1/3)$, and these exponent differences are listed in the exhaustive list of hypergeometric functions that are associated with Shimura curves appearing in Table 1 of [22]. Other $_3F_2$ functions that are unresolved examples to Christol's conjecture that we found to be related to $_2F_1$ hypergeometric functions related to *Shimura curves* are given by:

$$_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{4}{3}, 1], \ 3^6\,x\Big) \ = \ (1 - x)^{-1/9} \ \star \ _2F_1\Big([\frac{4}{9}, \frac{7}{9}], [\frac{4}{3}], \ 3^6\,x\Big), \tag{A.5}$$

$$_3F_2\Big([\frac{2}{9}, \frac{5}{9}, \frac{7}{9}], [\frac{2}{3}, 1], \ 3^6\,x\Big) \ = \ (1 - x)^{-7/9} \ \star \ _2F_1\Big([\frac{2}{9}, \frac{5}{9}], [\frac{2}{3}], \ 3^6\,x\Big), \tag{A.6}$$

$$_3F_2\Big([\frac{4}{9}, \frac{5}{9}, \frac{8}{9}], [\frac{2}{3}, 1], \ 3^3\,x\Big) \ = \ (1 - x)^{-8/9} \ \star \ _2F_1\Big([\frac{4}{9}, \frac{5}{9}], [\frac{2}{3}], \ 3^3\,x\Big), \tag{A.7}$$

$$_3F_2\Big([\frac{1}{7}, \frac{2}{7}, \frac{4}{7}], [\frac{1}{2}, 1], \ 7^4\,x\Big) \ = \ (1 - x)^{-4/7} \ \star \ _2F_1\Big([\frac{1}{7}, \frac{2}{7}], [\frac{1}{2}], \ 7^4\,x\Big), \tag{A.8}$$

Besides two hypergeometric functions, the $_3F_2([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}], [\frac{2}{3}, 1], \ 27\,x)$ and the $_3F_2$ hypergeometric $_3F_2([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{4}{3}, 1], \ 27\,x)$, and the three globally bounded $_3F_2$ hypergeometric series (A.6), (A.7) and (A.8), we were not able to write the other examples given in this section as a Hadamard product involving a $_2F_1$ hypergeometric function associated to a *Shimura curve*. In any case, since the class of potential counterexamples formulated by Christol is *infinite*, while the list of Shimura in Table 1 of [22] is *finite*, a list of $_3F_2$ functions both related to *Shimura curves* and to Christol's conjecture is bound to be finite.

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† See [21] p.10 for a definition of exponent difference.

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