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Abstract

Lattice rules are a powerful and popular form of quasi-Monte Carlo rules that are based on integration lattices. The study of the theory and application of lattice rules is intimately connected with the name Ian H. Sloan. We take the opportunity of Ian's 80th birthday to give an overview of his wide-ranging and fruitful contributions to this topic.

1 Introduction and Background

Ian Sloan is a major force in the area of numerical integration, and one of his main contributions to this subject is the theory of lattice rules. Therefore we find it very appropriate to devote an article appreciating his work on lattice rules to this anniversary volume.

Lattice rules belong to the family of quasi-Monte Carlo methods for numerical integration. The emphasis of these methods is on the multidimensional case. It is well known that quasi-Monte Carlo methods are effective even for integration problems of very high dimensions as they appear, for instance, in computational finance. We refer to the books of Dick and Pillichshammer [9] and Leobacher and Pillichshammer [43] as well as to the extensive survey article by Dick, Kuo, and Sloan [7] for a general background on quasi-Monte Carlo methods. An older monograph on quasi-Monte Carlo methods is [47].

Quasi-Monte Carlo methods are, in a nutshell, deterministic versions of Monte Carlo methods. We normalize the integration domain to be the s-dimensional unit cube $I^s := [0, 1]^s$ for some $s \geq 1$. For a Lebesgue-integrable function f on I^s, the Monte Carlo method uses the estimate

$$\int_{I^s} f(\boldsymbol{u})\mathrm{d}\boldsymbol{u} \approx \frac{1}{N} \sum_{n=1}^{N} f(\boldsymbol{x}_n), \tag{1}$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are independent and uniformly distributed random samples from I^s. If $f \in L^2(I^s)$, then the expected error in (1) has the order of magnitude $N^{-1/2}$ (note that the convergence rate is independent of the dimension s). The quasi-Monte Carlo method for the same integral uses again the approximation (1), but now $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \in I^s$

are deterministic points that are chosen to obtain a smaller error bound than the Monte Carlo error bound. Since $O\left(N^{-1/2}\right)$ is the mean-square error in (1) averaged over all samples of N points in I^s, such deterministic points must exist. One of the main issues in the theory of quasi-Monte Carlo methods is the explicit construction of deterministic point sets that yield better numerical integration schemes than the Monte Carlo method for large families of integrands. There are three principal approaches: (i) via Halton sequences and their variants (see [9, Section 3.4]); (ii) via the theory of nets (see most parts of [9]); (iii) via the theory of lattice rules. In this article, we focus on the third approach since it is here where Ian has made most of his contributions.

A lattice rule is a generalization of the classical one-dimensional numerical integration rule

$$\int_0^1 f(u)\mathrm{d}u \approx \frac{1}{N}\sum_{n=1}^N f\left(\frac{n-1}{N}\right) \tag{2}$$

called a rectangle rule which, in the case of an integrand f with $f(0) = f(1)$, agrees with the N-point trapezoidal rule for the interval $[0, 1]$. Lattice rules can be introduced from a group-theoretic and a geometric perspective.

We start with the first point of view. For a given dimension $s \geq 1$, the Euclidean space $\mathbb{R}^s$ is an abelian group under addition which has $\mathbb{Z}^s$ as a subgroup. Thus, we can form the factor group $\mathbb{R}^s/\mathbb{Z}^s$, also called the s-dimensional *torus group*. Now we consider, for the moment, the nodes $0, \frac{1}{N}, \ldots, \frac{N-1}{N}$ in (2) and their corresponding cosets $0 + \mathbb{Z}, \frac{1}{N} + \mathbb{Z}, \ldots, \frac{N-1}{N} + \mathbb{Z}$ in the one-dimensional torus group $\mathbb{R}/\mathbb{Z}$. Clearly, these cosets form a finite subgroup of $\mathbb{R}/\mathbb{Z}$; in fact, it is the cyclic group generated by $\frac{1}{N} + \mathbb{Z}$. The generalization is now obvious. For an arbitrary s, let $L/\mathbb{Z}^s$ be any finite subgroup of $\mathbb{R}^s/\mathbb{Z}^s$ and let $\boldsymbol{y}_n + \mathbb{Z}^s$ with $\boldsymbol{y}_n \in [0, 1)^s$ for $n = 1, \ldots, N$ be the distinct cosets making up the group $L/\mathbb{Z}^s$. The point set consisting of the points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N$ is called a *lattice point set* and the corresponding quasi-Monte Carlo approximation

$$\int_{I^s} f(\boldsymbol{u})\mathrm{d}\boldsymbol{u} \approx \frac{1}{N}\sum_{n=1}^N f(\boldsymbol{y}_n) \tag{3}$$

is called a *lattice rule*.

Why do we speak of a "lattice rule" and not, for instance, of a "finite-group rule"? The reason is a nice geometric interpretation of lattice rules. Recall that an s-dimensional *lattice* is defined to be a discrete subgroup of $\mathbb{R}^s$ that is not contained in any proper linear subspace of $\mathbb{R}^s$. Equivalently, an s-dimensional lattice is obtained by taking a basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_s$ of the vector space $\mathbb{R}^s$ and forming the set

$$L = \left\{ \sum_{i=1}^s k_i\boldsymbol{b}_i : k_i \in \mathbb{Z} \text{ for } 1 \leq i \leq s \right\}$$

of all linear combinations of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_s$ with integer coefficients. The lattices corresponding to lattice rules must have an additional property expressed in the following definition.

Definition 1 *An s-dimensional lattice is called an s-dimensional* integration lattice *if it contains $\mathbb{Z}^s$ as a subset.*

If we take an s-dimensional integration lattice L as the starting point, then the intersection $L \cap [0,1)^s$ is a finite set since L is discrete, and this finite set of points in $[0,1)^s$ forms a lattice point set. Furthermore, all lattice point sets can be obtained in this way.

An important concept for the analysis of lattice rules is that of the dual lattice of a given integration lattice.

Definition 2 *The* dual lattice $L^\perp$ *of the s-dimensional integration lattice L is defined by*

$$L^\perp = \big\{ \boldsymbol{h} \in \mathbb{R}^s : \boldsymbol{h} \cdot \boldsymbol{y} \in \mathbb{Z} \ \text{for all} \ \boldsymbol{y} \in L \big\},$$

where $\cdot$ denotes the standard inner product on $\mathbb{R}^s$.

It is easy to see that the dual lattice of an s-dimensional integration lattice is always a subgroup of $\mathbb{Z}^s$.

An interesting special case arises if the finite subgroup $L/\mathbb{Z}^s$ of $\mathbb{R}^s/\mathbb{Z}^s$ is cyclic. Let N be the order of the group $L/\mathbb{Z}^s$ and let $\boldsymbol{y} + \mathbb{Z}^s$ be a generator of $L/\mathbb{Z}^s$. Then $N\boldsymbol{y} \in \mathbb{Z}^s$, and so $\boldsymbol{y} = \frac{1}{N}\boldsymbol{g}$ for some $\boldsymbol{g} \in \mathbb{Z}^s$. The lattice rule (3) then attains the form

$$\int_{I^s} f(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} \approx \frac{1}{N} \sum_{n=1}^{N} f\Big(\Big\{\frac{n-1}{N}\boldsymbol{g}\Big\}\Big), \tag{4}$$

where the curly brackets denote fractional parts, that is, the points $\frac{n-1}{N}\boldsymbol{g}$, $n = 1, \ldots, N$, are considered modulo 1 in each coordinate. The numerical integration schemes in (4) were historically the first lattice rules and they are collectively known as the *method of good lattice points*. In this context, we call $\boldsymbol{g}$ the *generating vector* of the lattice rule.

We conclude this section with some comments on the history of lattice rules. The special case of the method of good lattice points was introduced by Korobov [28] in 1959 and a few years later independently by Hlawka [23]. The Soviet school quickly developed a quite satisfactory theory of the method of good lattice points, which is summarized in the book of Korobov [30] from 1963.

The first steps in the direction of general lattice rules were taken by Frolov [14] in 1977, but it was Ian who developed a systematic approach to the subject. His general perspective was first presented at a workshop in Canberra in December 1983 (see [68]). His fundamental paper [69] with Kachoyan on general lattice rules was submitted in August 1984, but unfortunately it took until 1987 to get published. In the meantime, his paper [63] had also advertised general lattice rules. Specialists in numerical integration got a first-hand exposure to general lattice rules through Ian's talk at an Oberwolfach workshop in November 1987 (see [75]). The second author first met Ian on a local train from Freudenstadt to Wolfach on their way to this workshop, but it was only after some curiosity-driven small talk that they were able to identify each other. Since then, Ian has had very close contacts with the Austrian community in the area of quasi-Monte Carlo methods, and we take this opportunity to thank him wholeheartedly for all his help and support.

An excellent summary of Ian's contributions to the theory of lattice rules as of 1994 can be found in his book [67] written jointly with Stephen Joe.

2   The Structure of Lattice Rules

In Section 1, we have defined lattice rules as quasi-Monte Carlo methods using the points of an integration lattice in I^s as the integration nodes. An obvious question is how such lattice rules can be represented in general. From (4), it can be seen that at least some lattice rules $Q_{N,s}$ for approximating the integral of a function f defined on I^s by N lattice points can be written as

$$Q_{N,s}(f) = \frac{1}{N} \sum_{j=1}^{N} f\left(\left\{\frac{j-1}{N}\boldsymbol{g}\right\}\right).$$

One may observe that the above representation is not unique. Indeed, choosing an integer $m \geq 2$ and replacing N and $\boldsymbol{g}$ by mN and $m\boldsymbol{g}$, respectively, yields an integration rule with mN points, each of them occurring with multiplicity m, such that the newly obtained rule is effectively equivalent to the one we started with. Furthermore, it is easy to construct examples where for the same N two distinct generating vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ yield rules with identical sets of integration nodes. Another question is of course whether there are maybe other lattice rules that can be represented in a different way than the one above. Hence, it is natural to ask for a canonical way of denoting lattice rules that makes it easier to structure and classify these, and to detect equivalences. Regarding this problem, it was Ian Sloan who, together with James Lyness, made substantial progress in providing a systematic way of representing lattice rules. We state a crucial theorem from [72] here.

Theorem 1 (Sloan and Lyness) *Let $Q_{N,s}$ be an s-dimensional lattice rule with $N \geq 2$ points. Then there exist uniquely determined integers*

- *r, with $r \in \{1, \ldots, s\}$,*

- *and $n_1, \ldots, n_r > 1$ with $n_{k+1}|n_k$ for $1 \leq k \leq r-1$, and $N = n_1 \cdots n_r$,*

such that $Q_{N,s}$ applied to a function f can be represented as

$$Q_{N,s}(f) = \frac{1}{N} \sum_{j_r=1}^{n_r} \cdots \sum_{j_1=1}^{n_1} f\left(\left\{\frac{j_1-1}{n_1}\boldsymbol{z}_1 + \cdots + \frac{j_r-1}{n_r}\boldsymbol{z}_r\right\}\right),$$

where $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ are linearly independent (over $\mathbb{Q}$) integer vectors in $\mathbb{Z}^s$.

The proof of Theorem 1 is based on the group structure of lattice points, and can be found in [72], see also [67] and [51, Section 4.3.2].

Remark 1 *The integer r in Theorem 1 is called* rank *of the lattice rule $Q_{N,s}$, and $n_1, \ldots, n_r$ are the* invariants.

Theorem 1 implies that the rank and the invariants of any given lattice rule are determined uniquely. What about the generating vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ of a rank-r lattice rule? In general, these cannot be determined uniquely. However, Sloan and Lyness [73] identified a class of lattice rules for which even the generating vectors can, in a certain sense, be identified unambiguously. This class is known as projection-regular lattice rules, which shall be described briefly here (we follow [67] and [73] in our terminology). Given an s-dimensional lattice rule $Q_{N,s}$ and $d \in \{1, \ldots, s\}$, we speak of the d-dimensional principal

projection of $Q_{N,s}$ when we consider the d-dimensional lattice rule obtained by omitting the last $s - d$ components of the integration nodes.

Note now that we can modify the representation of lattice rules outlined in Theorem 1 to a so-called extended canonical form by setting $n_{r+1} = \cdots = n_s = 1$, and by choosing arbitrary integer vectors $\boldsymbol{z}_{r+1}, \ldots, \boldsymbol{z}_s$. Then we can represent a lattice rule $Q_{N,s}$ as

$$Q_{N,s}(f) = \frac{1}{N} \sum_{j_s=1}^{n_s} \cdots \sum_{j_1=1}^{n_1} f\left(\left\{\frac{j_1 - 1}{n_1}\boldsymbol{z}_1 + \cdots + \frac{j_s - 1}{n_s}\boldsymbol{z}_s\right\}\right),$$

where we now trivially have $N = n_1 \cdots n_s$. Furthermore the rank r is in this case the maximal index such that $n_r > 1$. Using the latter representation of lattice rules, projection regularity is defined as follows (cf. [73]).

Definition 3 *Let $Q_{N,s}$ be an s-dimensional lattice rule with invariants $n_1, \ldots, n_s$ in its extended canonical form. The rule $Q_{N,s}$ is called* projection regular *if for any choice of $d \in \{1, \ldots, s\}$ the d-dimensional principal projection of $Q_{N,s}$ has invariants $n_1, \ldots, n_d$.*

The benefit of projection-regular lattice rules is that, by demanding some structure of the generating vectors, their choice is unique. Indeed, note that, given a lattice rule $Q_{N,s}$ in its extended canonical form, we can define an $s \times s$-matrix Z with the generating vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_s$ as its rows, i.e. $Z = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_s)^\top$. The matrix Z corresponding to a lattice rule is simply called Z-matrix in [73]. We say that Z is unit upper triangular if Z is upper triangular and has only 1s as the entries on the main diagonal. Using this terminology, we can state the following theorem, which is the main result of [73].

Theorem 2 (Sloan and Lyness) *A lattice rule $Q_{N,s}$ is projection-regular if and only if the rule $Q_{N,s}$ can be represented in an extended canonical form such that the corresponding Z-matrix is unit upper triangular.*

A question that remains is whether a projection-regular lattice rule can be represented in two different ways in an extended canonical form with unit upper triangular Z-matrices? The answer to this question is, as shown in [73], no, so it makes sense to speak of *the* standard form of a projection-regular lattice with unit upper triangular Z-matrix. For further details, we refer to the monograph [67] and the original paper [73]. We also remark that, for the special case of rank-2 lattices, Ian, again together with James Lyness, showed alternative representations, which subsequently made computer searches more effective (see, e.g., [44]). An overview of results related to the ones outlined in this section can also be found in [64].

A further important topic regarding the structure of lattice rules that was studied extensively by Ian and his collaborators is that of copy rules. Even though, due to their simple structure, rank-1 lattice rules have many convenient properties and have frequently been studied in many papers, Ian has pointed out on various occasions that also lattice rules of higher rank should be considered. To be more precise, as stated, e.g., in [13], lattice rules of higher or even maximal rank may perform better than rules of rank 1 with respect to certain criteria, such as the quantity P_α used in the classical literature on lattice rules. A prominent way of obtaining higher-rank lattice rules are copy rules. The basic idea of copy rules is elegant and simple: given a lattice rule $Q_{N,s}$ based on an integration lattice L, we obtain an integration rule consisting of m^s scaled "copies"

of $Q_{N,s}$ by considering the lattice rule based on the integration lattice $m^{-1}L$ for some positive integer m. In other words, a copy rule is obtained by scaling the original rule $Q_{N,s}$ and copying it to each of the cubes obtained by partitioning I^s into cubes of volume m^{-s}. Using the canonical representation form, let us suppose we start with a rank-1 rule $Q_{N,s}$ with generating vector $\boldsymbol{z}$, i.e.,

$$Q_{N,s}(f) = \frac{1}{N} \sum_{j=1}^{N} f\left(\left\{\frac{j-1}{N}\boldsymbol{z}\right\}\right).$$

The m^s copy rule $Q_{m,N,s}$ is then given by

$$Q_{m,N,s}(f) = \frac{1}{m^s N} \sum_{k_s=1}^{m} \cdots \sum_{k_1=1}^{m} \sum_{j=1}^{N} f\left(\left\{\frac{j-1}{mN}\boldsymbol{z} + \frac{(k_1-1,\ldots,k_s-1)}{m}\right\}\right).$$

It was shown by Sloan and Lyness in [72] that an m^s copy rule has rank s; to be more precise, an s-dimensional lattice rule is of rank s if and only if it is an m^s copy rule obtained from a rule of lower rank. Furthermore, in the paper [13], Disney and Sloan gave numerical results indicating that copy rules perform better with respect to P_α than rank-1 rules of comparable size. We refer to [13] and [67] for further details.

The rank of lattice rules plays a role in yet another variant of lattice rules, namely so-called embedded (sometimes also imbedded) lattice rules. The crucial feature of embedded lattice rules is that, at least up to a certain point, these can be extended by adding additional points to the rule without having to discard previous ones. This property can be a desirable advantage in practical implementations. In this context one then more precisely speaks of sequences of embedded integration rules, a concept that not only occurs with lattice rules but also other types of quadratures. Sequences of embedded lattice rules, as they shall be presented here, were originally described by Ian together with Stephen Joe (see [25] and [67]); they have the additional property that the more points we add, the higher the rank of the lattice rule gets. In [67, p. 164], the following is remarked:

> *"Embedded sequences of quadrature rules open up the possibility of obtaining an error estimate with little extra cost. The hope is that such an error estimate, along with the full set of approximate integrals obtained from the sequence of embedded rules, can give valuable information about the reliability of the final (and hopefully best) approximation in the sequence."*

Using a representation form similar to the canonical representation form introduced above, a sequence of embedded lattice rules is defined as follows. For a fixed positive integer m that is relatively prime to N, let for $r \in \{0, 1, \ldots, s\}$ the rule $Q_{r,m,N,s}$ be defined by

$$Q_{r,m,N,s}(f) = \frac{1}{m^r N} \sum_{k_r=1}^{m} \cdots \sum_{k_1=1}^{m} \sum_{j=1}^{N} f\left(\left\{\frac{j-1}{m}\boldsymbol{z} + \frac{(k_1-1,\ldots,k_r-1,0,\ldots,0)}{m}\right\}\right),$$

where $\boldsymbol{z}$ is a generating vector with components that are relatively prime to m. As pointed out in [25], the $Q_{r,m,N,s}$ have the property that the integration nodes in $Q_{r,m,N,s}$ also occur in $Q_{r+1,m,N,s}$ for $0 \le r \le s - 1$, and that $Q_{r,m,N,s}$ is a lattice rule with $m^r N$ points and

rank r for $0 \leq r \leq s$, the only exception being the rank of $Q_{0,m,N,s}$, which is 1. Hence, we see that $Q_{s,m,N,s}$, which is the rule based on the largest number of integration nodes and, thus, intuitively the "most precise", has maximal rank. From the representation of the $Q_{r,m,N,s}$ we see much similarity to copy rules, and indeed it is shown in [25] that $Q_{s,m,N,s}$ is nothing but an m^s copy rule obtained from $Q_{0,m,N,s}$. The papers [25, 27], and again the book [67] contain further remarks, results, and numerical experiments related to this subject. Results on component-by-component constructions (see Section 4) of embedded lattice rules are outlined in [35]. We also remark that, though sometimes using slightly different techniques, the question of how to extend the number of points of lattice rules has been addressed in numerous papers from theoretical and practical viewpoints, see, for example, [3, 10, 21, 22, 48].

3   Error Bounds for Lattice Rules

Integration rules have to be accompanied by an error analysis in order to be useful in practice. There are, in principle, two ways of establishing an upper bound on the integration error for the lattice rule (3). One method is based on the general Koksma-Hlawka inequality for an integrand f of bounded variation $V(f)$ in the sense of Hardy and Krause (see [31, Chapter 2]). This inequality yields the bound

$$\left| \int_{I^s} f(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} - \frac{1}{N} \sum_{n=1}^{N} f(\boldsymbol{y}_n) \right| \leq V(f) D_N^*(L),$$

where $D_N^*(L)$ is the *star discrepancy* of the integration nodes $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N$ corresponding to the integration lattice L. Recall that

$$D_N^*(L) = \sup_J \left| \frac{A(J;L)}{N} - \lambda_s(J) \right|,$$

where J runs through all half-open subintervals of I^s with one vertex at the origin, $A(J;L)$ is the number of integers n with $1 \leq n \leq N$ such that $\boldsymbol{y}_n \in J$, and λ_s denotes the s-dimensional Lebesgue measure.

Thus, one arrives at the problem of bounding the star discrepancy $D_N^*(L)$ that was treated by Niederreiter and Sloan [49]. We need some notation in order to state their results. Let $C_s(N)$ be the set of all nonzero $\boldsymbol{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s$ with $-N/2 < h_i \leq N/2$ for $1 \leq i \leq s$. For integers $h \in (-N/2, N/2]$, we put

$$r(h, N) = \begin{cases} N \sin(\pi |h|/N) & \text{if } h \neq 0, \\ 1 & \text{if } h = 0. \end{cases}$$

For $\boldsymbol{h} = (h_1, \ldots, h_s) \in C_s(N)$, we write

$$r(\boldsymbol{h}, N) = \prod_{i=1}^{s} r(h_i, N).$$

Then for an arbitrary s-dimensional integration lattice L with exactly N points in $[0,1)^s$, we define

$$R(L) = \sum_{\boldsymbol{h} \in C_s(N) \cap L^\perp} \frac{1}{r(\boldsymbol{h}, N)}, \tag{5}$$

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where $L^\perp$ is the dual lattice in Definition 2. According to [49, Proposition 3], the set $C_s(N) \cap L^\perp$ is nonempty for $s \geq 2$ and $N \geq 2$. The following discrepancy bound was shown in [49].

Theorem 3 (Niederreiter and Sloan) *Let L be an s-dimensional integration lattice with exactly N points in $[0,1)^s$, where $s \geq 2$ and $N \geq 2$. Then*

$$D_N^*(L) \leq \frac{s}{N} + R(L).$$

An important quantity related to $R(L)$ is the *figure of merit* $\rho(L)$, sometimes also called the *Zaremba index* of L. For $h \in \mathbb{Z}$ put $r(h) = \max(1, |h|)$, and for $\boldsymbol{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s$ put

$$r(\boldsymbol{h}) = \prod_{i=1}^s r(h_i).$$

Then for L as in Theorem 3, $\rho(L)$ is defined by

$$\rho(L) = \min_{\substack{\boldsymbol{h} \in L^\perp \\ \boldsymbol{h} \neq \boldsymbol{0}}} r(\boldsymbol{h}).$$

Now we can state the following result from [49].

Theorem 4 (Niederreiter and Sloan) *For L as in Theorem 3, we have*

$$R(L) < \frac{1}{\rho(L)} \Big(\frac{2}{\log 2} \Big)^{s-1} \Big((\log N)^s + \frac{3}{2}(\log N)^{s-1} \Big).$$

Lower bounds for $R(L)$ have been proved in [42, 59]. Theorem 4 suggests that the figure of merit $\rho(L)$ should be large for a useful integration lattice L. This is also demonstrated by the following lower bound on the star discrepancy from [49].

Theorem 5 (Niederreiter and Sloan) *For L as in Theorem 3, we have*

$$D_N^*(L) \geq \frac{c_s}{\rho(L)},$$

where c_s is an explicitly known positive constant depending only on s.

An interesting practical problem arises in connection with $R(L)$ in (5), namely how to compute this quantity efficiently. Joe and Sloan [26] use an asymptotic series in order to obtain a very good approximation to $R(L)$ which can be computed in $O(N)$ operations. This should be compared with the definition of $R(L)$ in (5) which requires the summation of $N^{s-1} - 1$ terms.

Now we turn to the second method of bounding the integration error in (3). Here we assume that the integrand f is a sufficiently regular periodic function on $\mathbb{R}^s$ of period 1 in each variable such that f is represented by its absolutely convergent Fourier series

$$f(\boldsymbol{u}) = \sum_{\boldsymbol{h} \in \mathbb{Z}^s} \widehat{f}(\boldsymbol{h}) \exp(2\pi \mathtt{i} \boldsymbol{h} \cdot \boldsymbol{u}),$$

where $\widehat{f}(\boldsymbol{h}) = \int_{I^s} f(\boldsymbol{u}) \exp(-2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{u}) \mathrm{d}\boldsymbol{u}$ is the $\boldsymbol{h}^{\mathrm{th}}$ Fourier coefficient of f. For a real number $\alpha > 1$, we say that $f \in \mathcal{E}^\alpha$ if there exists a constant $c(f) \geq 0$ such that

$$|\widehat{f}(\boldsymbol{h})| \leq c(f) r(\boldsymbol{h})^{-\alpha} \qquad \text{for all } \boldsymbol{h} \in \mathbb{Z}^s.$$

Let $\boldsymbol{g} \in \mathbb{Z}^s$ be a generating vector of a lattice rule of rank 1. Then for $f \in \mathcal{E}^\alpha$ we have the error bound

$$\Big| \int_{I^s} f(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} - \frac{1}{N} \sum_{n=1}^N f\Big(\frac{n-1}{N}\boldsymbol{g}\Big) \Big| \leq c(f) P_\alpha(\boldsymbol{g}, N), \tag{6}$$

where

$$P_\alpha(\boldsymbol{g}, N) = \sum_{\boldsymbol{h}} r(\boldsymbol{h})^{-\alpha}$$

with the summation running over all nonzero $\boldsymbol{h} \in \mathbb{Z}^s$ with $\boldsymbol{h} \cdot \boldsymbol{g} \equiv 0 \pmod{N}$.

The bound (6) raises the question of how small we can make the quantity $P_\alpha(\boldsymbol{g}, N)$. Disney and Sloan [12] have shown that for every $s \geq 3$ and every $\alpha > 1$ we have

$$\min_{\boldsymbol{g} \in \mathbb{Z}^s} P_\alpha(\boldsymbol{g}, N) \leq ((2\mathrm{e}/s)^{\alpha s} + o(1)) \frac{(\log N)^{\alpha s}}{N^\alpha} \quad \text{as } N \to \infty. \tag{7}$$

This implies that with the method of good lattice points we can obtain a convergence rate $O(N^{-\alpha})$ for $f \in \mathcal{E}^\alpha$, up to logarithmic factors.

4   The Search for Good Lattice Rules

So far there exists no general construction principle for lattice rules with excellent properties with respect to a given quality measure such as the star discrepancy, the criteria P_α, R, or the figure of merit ρ. (The only exception is in dimension $s = 2$ where one can use a relation to Diophantine approximation to find explicit constructions, see [2, 47, 67] or [51, Example 4.3.15]. In this context we would like to particularly mention so-called Fibonacci lattice rules.)

For dimensions s larger than two there are mainly averaging arguments which guarantee the existence of lattice rules which behave excellently with respect to a given quality criterion. The simple idea behind this principle is that there must exist a lattice point set for which a given quality criterion is at least as small as the average of the same criterion extended over all possible lattice point sets. However, such results give no information about how to obtain a good lattice point set.

For a long time, good lattice point sets were found by a brute force computer search, see, for example, [18, 19, 45, 46, 61]. But this is infeasible if N and s are not very small. For example for the method of good lattice points for given N one has to search for $\boldsymbol{g}$ within the set $\{0, 1, \dots, N-1\}^s$ which means that in the worst case one has to check N^s integer vectors to ensure success.

Since the 1980s Ian Sloan has contributed important results to the search for lattice rules. For example, we mention his work together with Linda Walsh [75, 76] (see also [67, Eq. (5.8)]) concerning the effective search for rank-2 lattice rules with small values of the quality criterion P_α. In their work Ian and Walsh restricted their attention to

relatively coprime invariants m and n, which allows to write the rank-2 lattice rule in a very symmetric three-sum form. This representation makes it easier to eliminate "geometrically equivalent" rules from the search. Since P_α is invariant for geometrically equivalent rules it suffices to calculate P_α only for one member of a geometrically equivalent family. This is an important observation since there may be a huge number of rules which are distinct but geometrically equivalent.

Back to the nowadays most important case of rank-1 rules: One way to reduce the search space for good lattice points is a method that goes back to Korobov [29]. He suggested considering only lattice points of the form

$$\boldsymbol{g}(g) = (1, g, g^2, \ldots, g^{s-1}) \quad \text{where } g \in \{1, \ldots, N-1\}.$$

This restriction reduces the size of the search space from N^s to $N-1$, since for given N one only has to search for g in the set $\{1, 2, \ldots, N-1\}$. The limitation to good lattice points of so-called "Korobov form" is in many cases justified by averaging results that are basically of the same quality as the averaging results over all lattice points from $\{0, 1, \ldots, N-1\}^s$ (see, for example, [29, 47, 51]). The method is very effective and in fact some of the good lattice points in the tables in [19, 46, 61] are of the specific Korobov form.

In [84] Ian, together with Xiaoqun Wang and Josef Dick, studied Korobov lattice rules in weighted function spaces. As the quality criterion they chose the worst-case integration error in weighted Korobov spaces which is more general than the classical quality measure P_α (in fact, in the unweighted case, P_α of a lattice rule is exactly the squared worst-case error of the same rule). For simplicity we restrict ourselves to the unweighted case and state the following special case of [84, Algorithm 1].

In the following let $Z_N := \{1, 2, \ldots, N-1\}$ and let $\zeta(\beta) = \sum_{j \geq 1} j^{-\beta}$ be the Riemann zeta function.

Algorithm 1 (Korobov lattice rule) *Let N be a prime number and let $s \in \mathbb{N}$. The optimal Korobov generator is found by minimizing the measure*

$$P_\alpha(\boldsymbol{g}(g), N) = -1 + \frac{1}{N} \sum_{k=0}^{N-1} \sum_{h_1, \ldots, h_s \in \mathbb{Z}} \prod_{j=1}^{s} \frac{1}{r(h_j)^\alpha} \exp\left(2\pi i k \frac{h_j g^{j-1}}{N}\right),$$

with respect to $g \in Z_N$.

It is pointed out in [84] that the number of operations needed to find the optimal Korobov lattice rule for even α (in this case one can get rid of the infinite sums in the definition of P_α) and a single dimension s is $O(sN^2)$. This is a remarkable improvement compared to the order $O(N^s)$ for a full search. Now the surprise is that the so found lattice rule is also of good quality. The following result is a specific case of the more general [84, Theorem 4]:

Theorem 6 (Wang, Sloan, and Dick) *Let N be a prime number and assume that g_* was found by Algorithm 1. Then for arbitrary $\tau \in [1, \alpha)$ we have*

$$P_\alpha(\boldsymbol{g}(g_*), N) \leq C_s(\alpha, \tau) \left(\frac{s}{N-1}\right)^\tau,$$

where $C_s(\alpha, \tau) = \exp(2s\tau \zeta(\frac{\alpha}{\tau}))$.

Ian's most important contribution to effective search routines for lattice rules, however, is the so-called component-by-component (CBC) construction of good lattice points. Although already described in the year 1963 by Korobov [30], this method fell into oblivion and it was Ian who re-invented it in a joint paper with Andrew V. Reztsov [74] in the year 2002. With this method good lattice points can be constructed one component at a time. At this time, this was a big surprise. We quote Ian and Reztsov, who wrote: *"The results may be thought surprising, since it is generally accepted that knowledge of a good lattice rule in s dimensions does not help in finding a good rule in s + 1 dimensions."* (cf. [74, p. 263])

To be more precise, given N and a quality measure, say, e.g., P_α, one starts with a sufficiently good one-dimensional generator (g_1). To this generator one appends a second component g_2 which is chosen as to minimize the quality criterion, in our case P_α. In a next step, one appends to the now two-dimensional generator (g_1, g_2) a third component g_3 which is again chosen as to minimize P_α. This procedure is repeated until one obtains an s-dimensional generating vector $\boldsymbol{g} = (g_1, g_2, \ldots, g_s)$. In each of the s steps the search space Z_N has cardinality $N - 1$ and hence the overall search space for the CBC method is reduced to a size of order $O(sN)$. Hence this provides a feasible way of finding a generating vector.

The use of the CBC method is justified by the following result which guarantees that the obtained generating vector is of excellent quality. In order to stress the dependence of P_α on the dimension s we will in the following write $P_\alpha^{(s)}$ for the s-dimensional case. We state [74, Theorem 2.1]:

Theorem 7 (Sloan and Reztsov)   *i) For arbitrary $\beta > 1$ and prime N, with $N \geq 2\zeta(\beta) + 1$, there exists a sequence $(g_j)_{j=1}^\infty$, with $g_j \in Z_N$, such that for all $s \geq 1$ and all $\alpha \geq \beta$*

$$P_\alpha^{(s)}((g_1, \ldots, g_s), N) \leq \frac{(1 + 2\zeta(\beta))^{s\alpha/\beta}}{N^{\alpha/\beta}}. \tag{8}$$

ii) The members g_j of a sequence $(g_j)_{j=1}^\infty$ satisfying (8) can be determined recursively, by setting $g_1 = 1$ and taking $g_{s+1} \in Z_N$ to be the least value of $g \in Z_N$ that minimizes $P_\beta^{(s+1)}((g_1, \ldots, g_s, g), N)$.

From this result Ian and Reztsov deduced the following:

Theorem 8 (Sloan and Reztsov)   *i) Let $\alpha > 1$ and let $s_{\max} \geq 2$ be a fixed positive integer, and let N be a prime number satisfying*

$$N > e^{s_{\max} \frac{\alpha}{\alpha - 1}}.$$

There exists a finite sequence $(g_j)_{j=1}^{s_{\max}}$ such that for any s satisfying $1 \leq s \leq s_{\max}$

$$P_\alpha^{(s)}((g_1, \ldots, g_s), N) \leq D(s, \alpha) \frac{(\log N)^{s\alpha}}{N^\alpha},$$

where

$$D(s, \alpha) := \left(\frac{3}{s_{\max}}\right)^{s\alpha} e^{s_{\max}\alpha}.$$

ii) The sequence $(g_j)_{j=1}^{s_{\max}}$ can be constructed as in part ii) of Theorem 7, with β given by

$$\beta := \frac{\log N}{\log N - s_{\max}}.$$

The bound on P_α from Theorem 8 should be compared to that in (7).

The search procedure from Theorem 7 is nowadays called "CBC algorithm" and can be summarized in concise form as follows:

Algorithm 2 (CBC algorithm) *Let $s, N \in \mathbb{N}$.*

1. *Choose $g_1 = 1$.*

2. *For $s = 1, 2, \ldots, s_{\max} - 1$, assume that $g_1, \ldots, g_s$ are already found. Choose $g_{s+1} \in Z_N$ to minimize $P_\alpha^{(s+1)}((g_1, \ldots, g_s, g), N)$ as a function of $g \in Z_N$.*

It is pointed out in [74] that the total cost of the CBC construction for P_α is of order of magnitude $O(s^2 N^2)$. This construction cost is typical for a straightforward implementation of the CBC algorithm also for other quality measures. Sometimes this cost can easily be reduced to $O(sN^2)$ under the assumption of a memory capacity of order $O(N)$. This is comparable to the search for generators of Korobov form and makes the CBC algorithm applicable for moderately large N. However, if one is interested in lattice rules with really large N, one has to further reduce the factor N^2 in the construction cost to get a feasible construction method. A breakthrough with respect to this problem was obtained by Dirk Nuyens and Ronald Cools [57, 58] in 2006 using fast Fourier transform (FFT) methods for the construction of lattice point sets. This way, it is possible to construct, for a given prime number N, an s-dimensional generating vector $\boldsymbol{g}$ in $O(sN \log N)$ operations, compared to $O(sN^2)$ operations for the usual CBC algorithm. The modified CBC construction is commonly known as "fast CBC construction".

The CBC construction is even more suited for the construction of good lattice rules in weighted spaces where the successive coordinates of the integrands have decreasing influence on the integral. This was already indicated in [74] although this paper only deals with P_α, i.e., the unweighted case. We quote from [74, p. 264]:

> *"Lattice rules obtained by the present component-by-component algorithm may be particularly valuable if the user knows that for the integrand under consideration the first coordinate is more important than the second, the second than the third, and so on. This is because the error bounds in Theorems 2.1 and 4.1 [here Theorems 7 and 8] hold simultaneously for all s up to the dimension of the particular integral. Thus the error bounds hold for all the principal projections obtained by omitting one or more of the later coordinates of the integrand."*

The different importance of the coordinate projections is usually modeled by a sequence of weights. For weighted spaces the CBC construction automatically adapts the constructed lattice point to the given sequence of weights. This means that a good lattice point constructed by CBC for a given sequence of weights need not be good for another weight sequence.

Nowadays the fast CBC construction is used with great success also for other quality measures such as the criterion R (see for example [43, Chapter 4]) or the (mean square) worst-case error of (shifted) lattice rules in weighted Sobolev spaces and Korobov spaces. This will be discussed further in Section 5.

5 Randomizations and Generalizations of Lattice Rules

Lattice rules are perfectly configured for the numerical integration of smooth and in each variable one-periodic functions. It is well known that they can achieve the optimal convergence rate in the class $\mathcal{E}^\alpha$ (see Section 3) or, in a more modern setting, in weighted Korobov spaces of smoothness α.

Let $\alpha > 1$ and let $\boldsymbol{\gamma} = (\gamma_j)_{j \in \mathbb{N}}$ be a sequence of positive weights[1]. Then the $\boldsymbol{\gamma}$-weighted Korobov space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ of smoothness α is a reproducing kernel Hilbert space (see [1] for general information about reproducing kernel Hilbert spaces) of one-periodic, complex valued functions with reproducing kernel

$$K_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h} \in \mathbb{Z}^s} \frac{\exp(2\pi\mathtt{i}\boldsymbol{h} \cdot (\boldsymbol{x} - \boldsymbol{y}))}{r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})},$$

where for $\boldsymbol{h} = (h_1, h_2, \ldots, h_s)$, $r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) = \prod_{j=1}^s r_{\alpha,\gamma_j}(h_j)$, and for $h \in \mathbb{Z}$ and $\gamma > 0$, $r_{\alpha,\gamma}(h) = \gamma^{-1}|h|^\alpha$ if $h \neq 0$ and $r_{\alpha,\gamma}(0) = 1$. The corresponding inner product is

$$\langle f, g \rangle_{s,\alpha,\boldsymbol{\gamma}} = \sum_{\boldsymbol{h} \in \mathbb{Z}^s} r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) \widehat{f}(\boldsymbol{h}) \overline{\widehat{g}(\boldsymbol{h})},$$

where $\widehat{f}(\boldsymbol{h}) = \int_{I^s} f(\boldsymbol{x}) \exp(-2\pi\mathtt{i}\boldsymbol{h} \cdot \boldsymbol{x}) \mathrm{d}\boldsymbol{x}$ is the $\boldsymbol{h}^{\text{th}}$ Fourier coefficient of f. The norm in $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ is defined as $\|\cdot\|_{s,\alpha,\boldsymbol{\gamma}} = \langle \cdot, \cdot \rangle_{s,\alpha,\boldsymbol{\gamma}}^{1/2}$.

The worst-case error of a linear integration rule

$$Q_{\mathcal{P},\boldsymbol{a}}(f) = \sum_{j=1}^N a_j f(\boldsymbol{x}_j) \quad \text{for } f \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$$

based on a point set $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ in I^s and coefficients $\boldsymbol{a} = (a_1, \ldots, a_N) \in \mathbb{R}^N$ is defined as

$$e(\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}, \mathcal{P}, \boldsymbol{a}) = \sup_f \left| \int_{I^s} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - Q_{\mathcal{P},\boldsymbol{a}}(f) \right|,$$

where the supremum is extended over the unit ball of $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$, i.e., over all $f \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ with $\|f\|_{s,\alpha,\boldsymbol{\gamma}} \leq 1$. If $\boldsymbol{a} = (1/N, \ldots, 1/N)$, then the integration rule is a quasi-Monte Carlo rule $Q_{\mathcal{P},\boldsymbol{a}}$ and we omit the parameter $\boldsymbol{a}$ in the notation. There is a convenient formula for the worst-case error in $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ of rank-1 lattice rules,

$$e^2(\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}, \mathcal{P}(\boldsymbol{g}, N)) = -1 + \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^s \\ \boldsymbol{g} \cdot \boldsymbol{h} \equiv 0 \pmod{N}}} \frac{1}{r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})},$$

see [79, Eq. (15)]. This formula should be compared to the definition of P_α in Section 3.

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[1]For simplicity here we only consider weights of product form, i.e., so-called *product weights*.

Let us first consider the unweighted case, i.e., the weights satisfy $\gamma_j = 1$ for all $j \in \mathbb{N}$ (in this case we also omit $\boldsymbol{\gamma}$ in the notation). Then for prime N one can construct by a CBC algorithm a lattice point $\boldsymbol{g} \in Z_N^s$ such that

$$e^2(\mathcal{H}_{s,\alpha}, \mathcal{P}(\boldsymbol{g}, N)) \leq (1 + 2\zeta(\alpha))^s \frac{1 + 2^{\alpha(s+1)}(1 + \log N)^{\alpha s}}{N^\alpha}.$$

Note that the order of magnitude in N is, up to the $\log N$-factor, best possible since it is known that for arbitrary N-element point sets $\mathcal{P}$ in I^s, $\boldsymbol{a} \in \mathbb{R}^N$, and for every $\alpha > 1$ we have

$$e^2(\mathcal{H}_{s,\alpha}, \mathcal{P}, \boldsymbol{a}) \geq C(s, \alpha) \frac{(\log N)^{s-1}}{N^\alpha},$$

where $C(s, \alpha) > 0$ depends only on α and s, but not on N.

This means that asymptotically, for N tending to infinity, lattice rules can achieve the optimal rate of convergence for the worst-case integration error in (unweighted) Korobov spaces. However, the question is how long one has to wait to see this excellent asymptotic behavior especially when the dimension s is large. This issue is the subject of tractability, a further topic to which Ian made significant contributions during the last two decades. We now switch again to the weighted setting.

The Nth *minimal error* in $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ is given by

$$e(N, s) := \inf_{\boldsymbol{a}, \mathcal{P}} e(\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}, \mathcal{P}, \boldsymbol{a}),$$

where the infimum is taken over all N-element point sets $\mathcal{P}$ in I^s and coefficients $\boldsymbol{a}$.

Furthermore, for $\varepsilon \in (0, 1]$ and $s \in \mathbb{N}$ the *information complexity* is the minimal number of information evaluations needed to achieve a minimal error of at most ε, to be more precise

$$N(\varepsilon, s) = \min\{N \in \mathbb{N} \,:\, e(N, s) \leq \varepsilon\}.$$

The subject of tractability deals with the question in which way the information complexity depends on ε and s; cf. the three books [54, 55, 56].

- The *curse of dimensionality* holds if there exist positive numbers C, τ, and ε_0 such that
$$N(\varepsilon, s) \geq C(1 + \tau)^s \quad \text{for all } \varepsilon \leq \varepsilon_0 \text{ and infinitely many } s.$$

- *Polynomial tractability* holds if there exist non-negative numbers C, τ_1, τ_2 such that
$$N(\varepsilon, s) \leq C s^{\tau_1} \varepsilon^{-\tau_2} \quad \text{for all } s \in \mathbb{N}, \ \varepsilon \in (0, 1).$$

- *Strong polynomial tractability* holds if there exist non-negative numbers C and τ such that
$$N(\varepsilon, s) \leq C \varepsilon^{-\tau} \quad \text{for all } s \in \mathbb{N}, \ \varepsilon \in (0, 1).$$

  The *exponent τ^* of strong polynomial tractability* is defined as the infimum of τ for which strong polynomial tractability holds.

It follows from a work of Ian together with Henryk Woźniakowski [77] that one has the curse of dimensionality for integration in the unweighted Korobov space. The following result is [77, Theorem 1]:

Theorem 9 (Sloan and Woźniakowski) *Consider integration in the Korobov space* $\mathcal{H}_{s,\alpha}$. *If* $N < 2^s$, *then* $e(N,s) = 1$.

In other words, the number of function evaluations required to achieve a worst case error less than one is exponential in the dimension. To prove this result, the authors constructed for given α and for given integration nodes $\mathcal{P}$ and coefficients $\boldsymbol{a}$ a so-called bump function h which belongs to the unit ball of the Korobov space $\mathcal{H}_{s,\alpha}$ and which satisfies $\int_{I^s} h(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1$ and $Q_{\mathcal{P},\boldsymbol{a}}(h) = 0$. From this it follows that

$$e(\mathcal{H}_{s,\alpha}, \mathcal{P}, \boldsymbol{a}) \geq \left| \int_{I^s} h(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_{\mathcal{P},\boldsymbol{a}}(h) \right| = \left| \int_{I^s} h(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \right| = 1.$$

To obtain positive results for tractability one has to change to the weighted setting. The concept of weighted function spaces was first introduced by Ian and Henryk Woźniakowski in the seminal paper [78] titled "When are quasi Monte Carlo algorithms efficient for high-dimensional problems?" (in fact, this question is programmatic for large parts of Ian's work during the last 20 years, see also his talk "On the unreasonable effectiveness of QMC" [66] given at the 9th *International Conference on Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing* in Warsaw in 2010, or the essay [65]). For the basic idea underlying the weighted setting, which is nowadays standard in this theory, we quote from [79, p. 699]:

> "The motivation lies in the idea [...] that it may be useful to order the coordinates $x_1, x_2, \ldots, x_d$ in such a way that x_1 is the most important coordinate, x_2 the next, and so on; and to quantify this by associating non-increasing weights $\gamma_1, \gamma_2, \ldots, \gamma_d$ to the successive coordinate directions."

The problem of tractability in weighted Korobov spaces was tackled by Ian and Henryk Woźniakowski in [79]. The main results of this paper are summarized in the following theorem:

Theorem 10 (Sloan and Woźniakowski) *Numerical integration in* $\mathcal{H}_{s,\alpha,\gamma}$ *is*

1. *strongly polynomially tractable, if and only if* $\sum_{j=1}^{\infty} \gamma_j < \infty$. *If* λ_0 *is the infimum over all* $\lambda \leq 1$ *such that* $\sum_{j=1}^{\infty} \gamma_j^{\lambda} < \infty$ *holds, then the* ε-*exponent* τ^* *of strong polynomial tractability lies in the interval* $[2\alpha^{-1}, \max(2\alpha^{-1}, 2\lambda_0)]$. *In particular, if* $\lambda_0 \leq \alpha^{-1}$, *then the* ε-*exponent* τ^* *of strong polynomial tractability is* $2\alpha^{-1}$.

2. *polynomially tractable, if and only if* $\limsup_{s \to \infty} \sum_{j=1}^{s} \gamma_j / \log s < \infty$ *holds.*

For $s > 1$ *and* N *prime, there exist lattice rules that achieve the corresponding upper bounds on the worst-case error.*

The above result guarantees the existence of lattice rules that achieve the upper bounds on the worst-case error. In fact, these lattice rules can be efficiently constructed with the CBC approach (see Section 4) as shown by Kuo [32] in 2003 and even by the fast CBC approach according to Nuyens and Cools [57]. Extensions of these results to composite N were given in [4, 34] and results for Korobov spaces with general weights were obtained in [11]. So the problem of numerical integration in weighted Korobov spaces is very well understood. However, these results for Korobov spaces are not often usable in practice

because integrands are typically not fully periodic. Since the early 1990s Ian works on the question how lattice rules can also be applied to numerical integration of not necessarily periodic functions. Early works in this direction are [60] and [50]. Nowadays, there are various strategies which can be used in order to apply lattice rules also to not necessarily periodic functions.

One such strategy are methods for periodization, i.e., methods for transforming a sufficiently smooth non-periodic integrand into a periodic integrand without changing its value of the integral. Such periodization techniques can be found, for example, in [7, Section 5.10] or in [47, Section 5.1]. A disadvantage of periodization is that the norm of the transformed integrand can be exponentially large in s. Hence this method is generally only feasible for moderate s.

Another possible strategy are randomly shifted lattice rules. For given quasi-Monte Carlo points $\mathcal{P} = \{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_N\}$ in I^s and $\boldsymbol{\Delta} \in I^s$ let

$$\mathcal{P} + \boldsymbol{\Delta} = \{\{\boldsymbol{t}_j + \boldsymbol{\Delta}\} \ : \ j = 1, 2, \ldots, N\}$$

denote the shifted point set. Then the quasi-Monte Carlo rule based on $\mathcal{P} + \boldsymbol{\Delta}$ is called a *shifted quasi-Monte Carlo rule*. In this sense a *shifted lattice rule* is of the form

$$Q_{N,s}(f, \boldsymbol{\Delta}) = \frac{1}{N} \sum_{j=1}^{N} f\left(\left\{\frac{j-1}{N}\boldsymbol{g} + \boldsymbol{\Delta}\right\}\right).$$

In the context of randomly shifted lattice rules one studies the so-called *shift-averaged worst-case error*. Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be a normed space of functions defined on the s-dimensional unit cube I^s. Then the worst-case error of a quasi-Monte Carlo rule $Q_{N,s}$ which is based on a point set $\mathcal{P}$ is

$$e(\mathcal{H}, \mathcal{P}) = \sup_f \left|\int_{I^s} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_{N,s}(f)\right|,$$

where the supremum is extended over all $f \in \mathcal{H}$ with $\|f\|_{\mathcal{H}} \leq 1$. The shift-averaged worst-case error of a shifted lattice rule is defined by

$$e^{\mathrm{sh}}(\mathcal{H}, \mathcal{P}(\boldsymbol{g}, N)) = \left(\int_{I^s} e^2(\mathcal{H}, \mathcal{P}(\boldsymbol{g}, N) + \boldsymbol{\Delta})\mathrm{d}\boldsymbol{\Delta}\right)^{1/2}.$$

This can be used to bound the root-mean-square error over uniformly distributed random shifts $\boldsymbol{\Delta}$ from I^s via the estimate

$$\sqrt{\mathbb{E}\left|\int_{I^s} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - Q_{N,s}(f, \boldsymbol{\Delta})\right|^2} \leq e^{\mathrm{sh}}(\mathcal{H}, \mathcal{P}(\boldsymbol{g}, N))\|f\|_{\mathcal{H}}.$$

Randomly shifted lattice rules are applied very successfully in weighted Sobolev spaces which shall be briefly introduced now. We just speak about the Hilbert space case with smoothness one and distinguish between the anchored space $\mathcal{H}_{s,\boldsymbol{\gamma}}^{\pitchfork}$ and the unanchored (or ANOVA) space $\mathcal{H}_{s,\boldsymbol{\gamma}}^A$. In both cases the weighted Sobolev spaces are s-fold tensor products of one-dimensional reproducing kernel Hilbert spaces $\mathcal{H}_{s,\boldsymbol{\gamma}}^\star = \mathcal{H}_{1,\gamma_1}^\star \otimes \ldots \otimes \mathcal{H}_{s,\gamma_s}^\star$

where $\star \in \{⋔, A\}$. The one-dimensional building blocks are reproducing kernel Hilbert spaces with reproducing kernel of the form

$$K_{1,\gamma}^{\star}(x, y) = 1 + \gamma \eta_{\star}(x, y)$$

where in the anchored case

$$\eta_{⋔}(x, y) = \begin{cases} \min(x, y) - c & \text{if } x, y > c, \\ c - \max(x, y) & \text{if } x, y < c, \\ 0 & \text{otherwise,} \end{cases}$$

with anchor $c \in [0, 1]$, and in the unanchored case

$$\eta_A(x, y) = \tfrac{1}{2}B_2(|x - y|) + (x - \tfrac{1}{2})(y - \tfrac{1}{2}),$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial. More detailed information on these spaces can be found, for example, in [7, Sections 4.2 and 4.3].

In [70] Ian, Frances Kuo, and Stephen Joe constructed randomly shifted lattice rules in the weighted anchored Sobolev space with the CBC algorithm. Furthermore, in [71] the three also introduced a CBC algorithm to construct a deterministic shift besides the generating vector. With these constructions one can achieve tractability for the integration problem depending on the decay of the weights. However, in these works only a convergence rate of order $O(N^{-1/2})$ was proved. Later Kuo [32] showed that even the optimal convergence rate of order of magnitude $O(N^{-1})$ can be achieved. Further results in this direction are due to Kuo and Joe [34] and due to Dick [4].

The following result (see [7, Section 5] for a proof) summarizes this development. Let

$$\widetilde{Z}_N := \{g \in \{1, 2, \ldots, N - 1\} \ : \ \gcd(g, N) = 1\}.$$

Theorem 11 (optimal CBC error bound) *Let* $\star \in \{⋔, A\}$. *The generating vector* $\boldsymbol{g} \in \widetilde{Z}_N^s$ *constructed by the CBC algorithm, minimizing the squared shift-averaged worst-case error* $e^{\text{sh}}(\mathcal{H}_{s,\gamma}^{\star}, \mathcal{P}(\boldsymbol{g}, N))$ *for the corresponding weighted Sobolev space in each step, satisfies*

$$[e^{\text{sh}}(\mathcal{H}_{s,\gamma}^{\star}, \mathcal{P}(\boldsymbol{g}, N))]^2 \leq \left(\frac{1}{\varphi(N)} \left(-1 + \prod_{j=1}^{s} \left(1 + \gamma_j^{\lambda} \left(\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}} + \beta_{\star}^{\lambda} \right) \right) \right) \right)^{1/\lambda}$$

for all $\lambda \in (1/2, 1]$, *where* φ *denotes the Euler totient function,* $\beta_A = 0$, *and* $\beta_{⋔} = c^2 - c + 1/3$ *in the anchored case with anchor* c.

The error estimate in the above theorem guarantees a convergence rate of order of magnitude $O(N^{-1+\varepsilon})$ for $\varepsilon > 0$. Furthermore, under the assumption of sufficiently fast decreasing weights, tractability for the shift-averaged worst-case error is obtained. For example, $\sum_{j=1}^{\infty} \gamma_j < \infty$ implies strong polynomial tractability. If further λ_0 is the infimum over all $\lambda \leq 1$ such that $\sum_{j=1}^{\infty} \gamma_j^{\lambda} < \infty$ holds, then the ε-exponent of strong polynomial tractability is at most $2\lambda_0$.

In practice it may happen that unbounded integrands arise resulting from the use of the cumulative inverse normal transformation in order to map the integral from the

unbounded domain $\mathbb{R}^s$ to the unit cube I^s. In [39, 85] Ian and his co-workers studied randomly shifted lattice rules for such types of integrands. Thereby they can achieve a convergence rate close to the optimal order $O(N^{-1})$.

Very briefly we report about a third strategy how to apply lattice rules to non-periodic integrands, namely tent transformed lattice rules. The tent transformation $\phi : [0, 1] \to [0, 1]$ is given by $\phi(x) = 1 - |2x - 1|$. For vectors $\boldsymbol{x}$ the tent transformed point $\phi(\boldsymbol{x})$ is understood componentwise. A tent transformed lattice rule is of the form

$$Q_{N,s}^{\phi}(f) = \frac{1}{N} \sum_{j=1}^{N} f\left(\phi\left(\left\{ \frac{j-1}{N} \boldsymbol{g} \right\} \right) \right).$$

In [8] such rules are used for the integration of cosine series which are not necessarily periodic. The authors obtained convergence rates for the worst-case errors that are optimal in the order of magnitude in N. Furthermore, under certain conditions on the weights one can also achieve tractability. We remark that these results are also valid for the unanchored Sobolev space $\mathcal{H}_{s,\gamma}^A$. Via embeddings one can transfer these results also to the anchored Sobolev space $\mathcal{H}_{s,\gamma}^{\pitchfork}$. The first who used the tent transform in the context of lattice rules was Hickernell [20].

Finally, we would like to mention that there is also the concept of polynomial lattice rules which was introduced by Niederreiter, see, for example [47]. Polynomial lattice rules are special instances of digital nets. Their overall structure is very similar to lattice rules. The main difference is that lattice rules are based on number theoretic concepts whereas polynomial lattice point sets are based on algebraic methods (polynomial arithmetic over a finite field). Also polynomial lattice rules work well for the integration of not necessarily periodic functions. A contribution of Ian to this topic is the paper [6]. This paper stood at the beginning of an intensive and fruitful collaboration of the third author with members of Ian's group in Sydney with several visits in both directions. The third author wishes to thank Ian and his colleagues at the University of New South Wales for their warm hospitality during these visits in Sydney.

6 Applications of Lattice Rules

Ian Sloan has always been an influential proponent of making theoretical results applicable to real-world problems, and telling the story of Ian and lattice rules would be incomplete without dealing with his rich work on applications to various topics. We shall try to give an overview of his numerous results on applications of lattice rules, ranging from function approximation to financial mathematics and, recently, partial differential equations.

One of the areas that are most frequently highlighted as fields of application of quasi-Monte Carlo methods is financial mathematics. Ian Sloan has not restricted his research on lattice rules to theoretical results, but has always been very interested in actually applying his methods. One of his co-authors on this subject is Xiaoqun Wang, with whom he published, e.g., the papers [81, 82, 83]. The first two of these papers, [81, 82], address an issue that is often not explicitly dealt with in theoretical studies on weighted spaces, namely: how should one, for a given problem from practice, choose the coordinate weights in order to obtain a low error of numerical integration?

The paper [81] first shows that it might be a very bad choice to "blindly" use lattice rules that have been optimized with respect to classical (i.e., non-weighted) error measures. Indeed, applying such rules to problems from financial mathematics can lead to an error behavior that is worse than applying Monte Carlo methods. However, Ian and Wang outline how to choose the weights for a given finance problem, such as pricing arithmetic average Asian options. Numerical evidence is given that, for pricing such derivatives, lattice rules with a suitable choice of weights (which have the form $\gamma_j = a\tau^j$ for some (positive real) parameters a and τ) may outperform crude Monte Carlo and also another quasi-Monte Carlo algorithm based on Sobol' sequences. The weights are chosen according to a matching strategy based on relative sensitivity indices, which are in turn related to the ANOVA decomposition, of the function at hand. Once the weights are fixed, the generating vectors of lattice rules are chosen according to one of the usual algorithms that optimize the integration rules with respect to a certain error criterion (as for example a CBC or acceptance-rejection algorithm). The choice of coordinate weights is also addressed in the paper [82], where not only product weights, but also general weights are allowed. A further focus of [82] is *dimension reduction* methods, such as principal component analysis (PCA) or the Brownian bridge. Indeed, it is shown in [82] for a model problem that dimension reduction techniques combined with a suitable choice of coordinate weights can lead to bounds on the integration error that are independent of the dimension. Ian and Wang refine their dimension reduction strategies even further in [83], where again theoretical and numerical results on option pricing are presented. We also refer to [15] for a survey on quasi-Monte Carlo methods applied to finance problems.

The idea of applying lattice rules to problems such as function approximation can already be found, e.g., in [30] and [24]. In the field of information-based complexity, dealing with high-dimensional problems in general and approximation problems in particular, several authors proposed using similar methods when studying function approximation in certain function spaces. This was for example done in the important paper [53] with Erich Novak and Henryk Woźniakowski, where (among other related questions) the approximation of functions in weighted Korobov spaces by means of lattice rules is studied. The underlying idea of the approximation algorithms analyzed in that paper is to approximate the largest Fourier coefficients of an element of a Korobov space by lattice rules. To be more precise, suppose that a one-periodic s-variate function f which is an element of a (weighted) Korobov space $\mathcal{H}_{s,\alpha,\gamma}$ and thus has an absolutely convergent Fourier series expansion,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h} \in \mathbb{Z}^s} \widehat{f}(\boldsymbol{h}) \exp(2\pi \mathtt{i} \boldsymbol{h} \cdot \boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in [0,1)^s,$$

is given. As before, $\widehat{f}(\boldsymbol{h})$ denotes the $\boldsymbol{h}^{\text{th}}$ Fourier coefficient of f. The strategy for function approximation using (rank-1) lattice rules is then to first choose a finite set $\mathcal{A} \subseteq \mathbb{Z}^s$ corresponding to the "largest" Fourier coefficients, and then to approximate the Fourier coefficients $\widehat{f}(\boldsymbol{h})$ of f for which $\boldsymbol{h} \in \mathcal{A}$. Hence, f is approximated by

$$\underbrace{\sum_{\boldsymbol{h} \in \mathcal{A}} \overbrace{\left(\frac{1}{N} \sum_{j=1}^{N} f\left(\left\{\frac{j-1}{N}\boldsymbol{g}\right\}\right) \exp\left(-2\pi \mathtt{i} \boldsymbol{h} \cdot \left\{\frac{j-1}{N}\boldsymbol{g}\right\}\right) \right)}^{\approx \widehat{f}(\boldsymbol{h})} \exp(2\pi \mathtt{i} \boldsymbol{h} \cdot \boldsymbol{x})}_{\approx f(\boldsymbol{x})}$$

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for $\boldsymbol{x} \in I^s$, where $\boldsymbol{g}$ is a suitably chosen generating vector of a lattice rule. By noting that this approximation algorithm is a *linear* algorithm, it is then possible to analyze the approximation problem using the machinery from information-based complexity. Care has to be taken of the difficulty in simultaneously choosing the set $\mathcal{A}$ and its size, as well as the lattice rule and its number of points.

This idea was further pursued in the paper [40] with Kuo and Woźniakowski. In particular, the paper [40] contains a component-by-component construction of generating vectors of lattice rules, by which one can obtain tractability of approximation in the worst-case setting, under conditions on the coordinate weights that are similar to those for high-dimensional integration. To be more precise, one of the main results in [40] is the following.

Theorem 12 (Kuo, Sloan, and Woźniakowski) *Consider function approximation based on function evaluations for weighted Korobov spaces $\mathcal{H}_{s,\alpha,\gamma}$ in the worst-case setting. Then the following statements hold true.*

(a) *If the sequence of weights $(\gamma_j)_{j\geq1}$ in the weighted Korobov space satisfies the condition*
$$\sum_{j=1}^{\infty} \gamma_j < \infty,$$

the function approximation problem is strongly polynomially tractable. A generating vector of a lattice rule yielding this result can be found by a component-by-component algorithm.

(b) *If the sequence of weights $(\gamma_j)_{j\geq1}$ in the weighted Korobov space satisfies the condition*
$$\limsup_{s\to\infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log(s+1)} < \infty,$$

the function approximation problem is polynomially tractable. A generating vector of a lattice rule yielding this result can be found by a component-by-component algorithm.

While the paper [53] considers the worst-case, randomized, and the quantum settings, [40] focuses on the worst-case setting only. A further paper of Ian Sloan with Kuo and Woźniakowski deals with the average-case setting, see [41]. Overall, it can clearly be said that Ian's research in these papers has triggered numerous further results on lattice rules employed for function approximation in modern problem settings.

One of the most recent additions to the fields of application of quasi-Monte Carlo techniques is that of partial differential equations with random coefficients. Ian's paper [17], together with Graham, Kuo, Nuyens, and Scheichl, is one of the first papers addressing quasi-Monte Carlo methods used in combination with finite elements. In this context, randomly shifted lattice rules are (among other quasi-Monte Carlo methods) used for computing the expectations of nonlinear functionals of solutions of certain elliptic PDEs with random coefficients. The observations in [17] are motivated by an example from physics, namely modeling fluid flow through porous media. After the paper [17], several others on this and related subjects followed, many of them co-authored by Ian Sloan (see, e.g., [16, 37, 38]). It is beyond the scope of this article to outline the details of the

PDE problem these observations are based on, but we refer to the paper [36] of Ian with Frances Kuo and Christoph Schwab for a summary highlighting some particular issues from the quasi-Monte Carlo point of view. Without going into great detail, it should be mentioned in this context that these papers motivate some of the latest developments in the research on quasi-Monte Carlo methods in general and lattice rules in particular. Among them are increased efforts in deriving results that hold not only for the integration of Hilbert space elements, but more general function classes, integration problems with an unbounded number of variables, and the introduction of new classes of weights. Indeed, the paper [37] introduced new kinds of coordinate weights, so-called *product and order dependent (POD)* weights. These are of the form

$$\gamma_{\mathfrak{u}} := \Gamma_{|\mathfrak{u}|} \prod_{j \in \mathfrak{u}} \gamma_j,$$

for $\mathfrak{u} \subseteq \{1, \ldots, s\}$, for two sequences $(\Gamma_j)_{j \geq 1}$ and $(\gamma_j)_{j \geq 1}$. Note that in the definition of $\gamma_{\mathfrak{u}}$, the factor $\Gamma_{|\mathfrak{u}|}$ only depends on the cardinality of $\mathfrak{u}$, while the other factor has the usual product form. The introduction of this kind of weights in [37] stems from the problem of choosing the weights for a particular integrand in such a way that the overall integration error (or a suitable upper bound on it) is small. The structure of POD weights is such that it allows for a (fast) CBC construction. For further details we refer to the aforementioned papers.

Ian also worked on quasi-Monte Carlo methods applied to integrals appearing in certain problems in statistics. Indeed, the evaluation of the likelihood in generalized response models, as for example occurring in time series regression analysis, leads to multivariate integrals over unbounded regions. This problem in combination with quasi-Monte Carlo methods was first addressed in the paper with Kuo, Dunsmuir, Wand, and Womersley, [33], where, due to certain properties of the integrands under consideration, the integration problem over an unbounded region cannot simply be solved by transforming the integral into one over the unit cube. Indeed, the authors of [33] use recentering and rescaling methods before the transformation. The theory in [33] was extended in [62], by using POD weights as they were introduced in [37]. The idea of how to choose POD weights for integration is nicely summarized in [62, p. 633] as follows:

> "*The integration error is bounded by the product of the worst-case error (depending only on the QMC point set) and the norm of the function (depending only on the integrand). We choose weights that minimise a certain upper bound on this product.*"

Let us, finally, mention yet another field to which Ian applied the theory of lattice points, namely integral equations. This idea was already dealt with in, for example, [24], [80], and [86]. The paper [5], however, considers the solution of certain Fredholm integral equations of the second kind in a weighted Korobov setting. Indeed, let us consider the Fredholm integral equation

$$f(\boldsymbol{x}) = g(\boldsymbol{x}) + \int_{I^s} \kappa(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y},$$

where κ satisfies a convolution assumption, i.e., $\kappa(\boldsymbol{x}, \boldsymbol{y}) = k(\boldsymbol{x} - \boldsymbol{y})$. Furthermore k and g are assumed to be elements of a weighted Korobov space, so they are one-periodic and

continuous. The solution f is then approximated by a lattice-Nyström method based on lattice points. Let $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N$ be the integration nodes of a lattice rule. Then f is approximated by

$$f_N(\boldsymbol{x}) = g(\boldsymbol{x}) + \frac{1}{N} \sum_{n=1}^{N} \kappa(\boldsymbol{x}, \boldsymbol{y}_n) f_N(\boldsymbol{y}_n).$$

The function values $f_N(\boldsymbol{y}_1), \ldots, f_N(\boldsymbol{y}_N)$ are obtained by solving the linear system $f_N(\boldsymbol{y}_k) = g(\boldsymbol{y}_k) + \frac{1}{N} \sum_{n=1}^{N} \kappa(\boldsymbol{y}_k, \boldsymbol{y}_n) f_N(\boldsymbol{y}_n)$, $1 \le k \le N$. Under suitable assumptions on k and N, existence and stability of a unique solution of the integral equation can be shown. The group structure of lattice points ensures that the linear system from above can be solved using the fast Fourier transform (cf. [86]). The paper [5] then studies the worst-case error of approximating f by f_N, and a CBC construction of lattice rules yielding a small error. Furthermore, conditions for obtaining tractability are discussed. The conditions on the weights in the Korobov space to obtain (strong) polynomial tractability are similar to those for numerical integration.

On a personal note, the paper [5] was partly written when the first author of the current paper visited Australia and Ian Sloan's group at the University of New South Wales for the first time—a trip which resulted in fruitful scientific collaborations and wonderful friendships with Ian and his colleagues.

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