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Stable recovery of the time-dependent source term from one measurement for the wave equation.

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Abstract: In this work, we discuss the inverse problem which consists of the determination of an unknown time-dependent force function from one time-dependent measurement collected in any space point for the one-dimensional wave equation. This problem is motivated by the question of estimating the time-dependent body force needed to exert on a given string to reach a desired shape at the final time.

We prove the unique solvability using, as data, a linear combination of the displacement and the flux measured at one arbitrary fixed point of the string. We also derive a conditional Hölder stability estimate of this inverse problem. The numerical solution of the problem is investigated by means of the Ritz-Galerkin technique along with applying the satisfier function to obtain the cost-effective and stable results. Some numerical examples are provided to show the performance of the proposed scheme.

Keywords: Hyperbolic equation; Inverse wave problem; Ritz-Galerkin method; Landweber iteration.

AMS subject classification: 65N12; 65N15; 65N20; 65N21

1 Introduction

Inverse problems appear in many branches of science and technology [8, 13, 14] as in the scattering, vibration of materials and in general wave propagation in different types of media such as the surface and interior of the earth in geophysics, human tissues in medical imaging and other various industrial applications [1, 4, 8, 19]. The multidisciplinary features of these problems has drawn the attention of many researchers coming from different backgrounds to develop mathematical techniques for solving them. A large number of publications related to this rapidly growing field has appeared, see for instance [7, 13, 14] to cite only a few of the monographs treating the related mathematical aspects.

Of particular interest to us is the inverse problems for the wave equation, [2, 3, 5, 11, 13, 25]. The propagation of waves in a string of length $L > 0$ acted upon a time-dependent force $f(t)$ is modeled by the following hyperbolic equation

$$A_{tt}(x,t) = c^2 A_{xx}(x,t) + f(t), \quad in \quad \Omega_T = \{(x,t)|\ 0 \le x \le L,\ 0 \le t \le T\}, \tag{1.1}$$

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supplemented with the boundary conditions, corresponding to the flux tension of the string at the end points $x \in \{0, L\}$, namely

$$A_x(0,t) = b_1(t), \quad A_x(L,t) = b_2(t), \quad 0 < t < T, \tag{1.2}$$

and at the time terminals

$$A_t(x,0) = 0, \quad A(x,T) = h_1(x), \quad 0 < x < L. \tag{1.3}$$

The mathematical model (1.1)-(1.3) describes the vibration of a uniform string of length L subject to body and boundary forces. Precisely, at its two extremities $x = 0$ and $x = L$, we impose respectively two boundary forces (or fluxes) b_1 and b_2, as in (1.2), while we use only time-dependent body force $f(t)$, as in (1.1), i.e. uniform in space. Finally, the conditions in (1.3) mean, respectively, that initially the string was at rest and at the final time T it reaches a shape described by $h_1(x)$.

The inverse problem we wish to study is motivated by the following question. Assume that the extremities of our string are subject to fixed forces b_1 and b_2, can we estimate the (uniform in space) body force $f(t)$ needed to exert on the whole string so that it reaches, at the final time T, the given shape $h_1(x)$? Note that initially the string is at rest, i.e. $A_t(x,0) = 0, x \in (0,L)$, but we do not assume to know its shape, i.e. $A(x,0), x \in (0,L)$. To answer to this question, we assume that we have at hand the extra information $h(t)$

$$c_1 A(x^*,t) + c_2 A_x(x^*,t) = h(t), \quad 0 < t < T, \tag{1.4}$$

describing the behavior of the string at a given but arbitrary point $x^* \in [0,L]$. Here c_1, c_2, T, L are considered as arbitrary real numbers provided that c_1 is nonzero. Observe that after estimating $f(t)$, we can estimate the general displacement $A(x,t)$, especially the initial shape of the string $A(x,0) = A_0(x)$, by solving the problem (1.1)-(1.3).

Without lose of generality, we take the speed of sound c to be the unity, i.e. $c = 1$. For the same problem, although with the given function $A_0(x)$ and the unknown forcing function depending only upon the space variable i.e. $f(x)$, studies based on the least squares method and the combination of the boundary elements method (BEM) with the separation of variables, have been proposed in [5, 11]. In addition, the questions of existence and uniqueness for the solution of the inverse source problems in the wave equation have been discussed in [13, 24]. In those investigations, the authors have focused on the problems which involve exclusively either the backward wave problems with known right-hand side or the inverse wave problems with unknown source problems but with the given initial condition.

In this work, we have considered the three natural questions of uniqueness, stability and reconstruction. Indeed, first, we have justified the unique solvability of this inverse problem under the natural condition $\frac{T}{L} \notin \mathcal{C} := \{\frac{2m+1}{2n} |\ m,n \in \mathbb{N}\}$. Let us notice here that this result can be derived for more general Sturm Liouville equations, see Remark 2.4. Observe also that the set $\mathcal{C}$ is not that restrictive since $\forall m,n \in \mathbb{N}$, $\frac{T}{L} := \frac{2m}{2n+1} \notin \mathcal{C}$, for instance. Second, we have estimated the modulus of continuity of the inverse problem. Precisely, we have derived a conditional Hölder stability estimate of $f(t)$ (and then of $A_0(x)$) in terms of h, assuming the other parameters b_1, b_2 and h_1 to be fixed. This conditional Hölder stability estimates implies that our inverse problem is at most moderately instable. But this mild instability can indeed occur as it is shown in the following example [1]. Given the values $\alpha,\ \beta \in \mathbb{R},\ \epsilon \in (0,2)$ we consider

[1] From the relation (4.16), we observe that our inverse problem has a similar degree of instability as the numerical differentiation.

the boundary conditions as:

$$b_{1n}(t) = b_{2n}(t) = \frac{\alpha}{n^{\epsilon}},\ h_{1n}(x) = \frac{1 + \alpha x + \beta T^2 + \cos\left(nT\right)}{n^{\epsilon}},\ h_n(t) = c_1 \frac{1 + \alpha x^* + \beta t^2 + \cos(nt)}{n^{\epsilon}} + \frac{c_2 \alpha}{n^{\epsilon}}. \tag{1.5}$$

Then, the solution of the problem (1.1)-(1.4) is given by: [2]

$$\begin{cases} A_n(x,t) = \frac{1+\alpha x+\beta t^2+\cos(nt)}{n^{\epsilon}}, \\ f_n(t) = \frac{2\beta}{n^{\epsilon}} - n^{2-\epsilon}\cos(nt). \end{cases} \tag{1.6}$$

It is clear that f_n is unbounded while the whole boundary conditions tend to zero as $n \longrightarrow \infty$ which shows the instability in retrieving the source function $f(t)$. Therefore dealing with this problem requires employing appropriate procedures to produce stable solutions [17, 20]. The third part of our study is devoted to the numerical solution of the problem using the Ritz-Galerkin method. Most contributions in the numerical methods for solving the inverse wave problems have been devoted to iterative techniques and classic finite difference method [2, 3, 6, 10, 23, 25]. Prior to our work, the Ritz-Galerkin method, known as a domain Galerkin technique, dealt with several linear and nonlinear partial differential equations, see for instance [9, 20, 21, 27]. Briefly stated, we solved the problem for not only the standard initial and boundary conditions [9, 21], but also the nonlocal boundary conditions [20] through an auxiliary function called "satisfier function". In conclusion, the large sets of collocation points are not needed for applying the supplemented boundary conditions which naturally leads to a system of algebraic equations of smaller size and hence reduces the computation time. Although the authors in [20, 21] reported satisfactory results with relatively low-cost computational efforts, their solution could suffer from propagation of errors because improperly posed problems are always involved with noisy input data. Since the satisfier function incorporates all initial and boundary conditions including the possibly erroneous ones, the technique has some shortcomings. [3] Regarding the nonlinear problems [9], apart from the aforementioned problems, the difficulty in dealing with nonlinear system of equations was not addressed. On the other hand, the issue of convergence analysis was barely touched in the previous literature. In our model, we are dealing with a linear inverse problem where both initial condition and right-hand side are unknown. By extending the application of Ritz-Galerkin method to the solution of such inverse problems, we aim to diminish the discussed difficulties. From the numerical point of view, we discuss the advantage of our technique in taking into account the cost-effective features. Furthermore, an upper bound for the error function corresponding to the approximate solution is derived. This estimate shows that although the error increases as time increases, the increase of error is not significant for small values of T.

The rest of the paper is organized as follows. The uniqueness of the solution for the inverse problem problem is presented in Section 2 and in Section 3, we derive the corresponding Hölder stability estimate. We describe the Ritz-Galerkin technique to solve the problem and state the continuous dependence of the solution on the data in Section 4. Finally, Section 5 contains some numerical examples to test the performance of the suggested procedures.

[2] To get an example to our actual inverse problem, i.e. with homogeneous boundary and initial conditions, we split $A_n(x,t)$ into two parts $A_n(x,t) = A_{n,1}(x,t) + A_{n,2}(x,t)$ where $A_{n,1}$ is solution of (1.1)-(1.3) with homogeneous source term and $A_{n,2}$ is solution of (1.1)-(1.3) with homogeneous boundary and initial conditions and the source term f_n. The well posedness of (1.1)-(1.3) implies that $A_{n,1}$ is bounded since its boundary and initial conditions are bounded. Hence, also $c_1 A_{n,1}(x^*,t) + c_2 (A_{n,1})_x(x^*,t)$ is bounded. As h_n is bounded, we deduce that $c_1 A_{n,2}(x^*,t) + c_2 (A_{n,2})_x(x^*,t)$ is bounded. But still f_n is unbounded.

[3] For more clarification see [21] and Remark 4.1 of the present work.

2 Uniqueness of the solution of the inverse problem

We start with the following lemma.

Lemma 2.1. *The unique solution to the problem*

$$\theta_{tt}(x,t) = \theta_{xx}(x,t), \quad in \quad \Omega_T, \tag{2.1}$$

$$\theta(0,t) = \theta(L,t) = 0, \quad t \in (0,T), \tag{2.2}$$

$$\theta(x,T) = \theta_t(x,0) = 0, \quad x \in (0,L), \tag{2.3}$$

is the trivial function $\theta(x,t) = 0$ *in* $C^{2,2}([0,L]\times[0,T])$ *provided that* $\frac{T}{L} \notin \mathcal{C} = \{\frac{2m+1}{2n} |\ m,n \in \mathbb{Z}\}$.

Proof. Any solution of Equation (4.7) can be represented as

$$\theta(x,t) = \sum_{k=1}^{\infty} \theta_k(t) X_k(x), \tag{2.4}$$

where the $X_k(x)$'s are the eigenfunctions of the auxiliary self-adjoint spectral problem

$$\begin{cases} X''(x) + \lambda^2 X(x) = 0, & 0 \le x \le L, \\ X(0) = X(L) = 0, \end{cases} \tag{2.5}$$

that is, $X_k(x) = \sin(\frac{k\pi x}{L})$, $\lambda_k = \frac{k\pi}{L}$. By inserting the equation (2.4) in (4.7) we find that the functions $\theta_k(t)$, $k = 1,2,...$ satisfy the following system of equations:

$$\theta_k''(t) + (\frac{k\pi}{L})^2 \theta_k(t), \quad k = 1,2,... \tag{2.6}$$

The general solution for (2.6) can be considered as

$$\theta_k(t) = \alpha_k \cos(\frac{k\pi t}{L}) + \beta_k \sin(\frac{k\pi t}{L}), \quad k = 1,2,... \tag{2.7}$$

Since we require (2.4) to satisfy the boundary conditions (2.3) then the Fourier coefficients α_k, β_k of the expansion (2.7) are equal to zero if $\frac{T}{L} \notin \mathcal{C} = \{\frac{2m+1}{2n} |\ m,n \in \mathbb{N}\}$ and this implies that the solution (2.4) becomes $\theta(x,t) = 0$.■

Suppose that two pairs of functions

$$\big(A_1(x,t), f_1(t)\big) \text{ and } \big(A_2(x,t), f_2(t)\big),$$

are solutions for system (1.1)-(1.4). By setting $\Delta A(x,t) = A_2(x,t) - A_1(x,t)$, $\Delta f(t) = f_2(t) - f_1(t)$ we obtain:

$$\Delta A_{tt}(x,t) = \Delta A_{xx}(x,t) + \Delta f(t), \quad in \quad \Omega_T = \{(x,t) |\ 0 \le x \le L,\ 0 \le t \le T\}, \tag{2.8}$$

$$\Delta A_x(0,t) = \Delta A_x(L,t) = 0, \quad 0 < t < T, \tag{2.9}$$

$$\Delta A_t(x,0) = 0, \quad 0 < x < L. \tag{2.10}$$

Assume that $\Delta A(x,t)$ is the solution to system (2.8)-(2.10) that fulfills the initial condition $\Delta A(x,0) = A_2(x,0) - A_1(x,0) = \Delta A_0(x)$. Now, we introduce the auxiliary transformations, following [26],

$$G(t) = \int_0^t \int_0^s \Delta f(\tau) d\tau ds, \quad V(x,t) = \Delta A(x,t) - G(t), \tag{2.11}$$

and use the equations (2.8)-(2.10) to get

$$V_{tt}(x,t) = V_{xx}(x,t), \quad in \quad \Omega_T = \{(x,t)|\ 0 \le x \le L,\ 0 \le t \le T\}, \tag{2.12}$$

$$V_x(0,t) = V_x(L,t) = 0, \quad 0 < t < T, \tag{2.13}$$

$$V_t(x,0) = 0,\ V(x,0) = \Delta A_0(x) \quad 0 < x < L. \tag{2.14}$$

Since the solution to the system (2.12)-(2.14) is unique [18], therefore we can write

$$\Delta A(x,t) = V(x,t) + \int_0^t \int_0^s \Delta f(\tau) d\tau ds. \tag{2.15}$$

Lemma 2.2. *Concerning the system (2.12)-(2.15) we have the equivalence*

$$\Delta A(x,T) = 0,\ 0 < x < L \Longleftrightarrow \Delta A_0(x) + G(T) = 0, \quad 0 \le x \le L. \tag{2.16}$$

Proof. $\Longrightarrow$) From $\Delta A(x,T) = 0$, (2.15) implies $V(x,T) = -G(T)$ which shows that $V(x,T)$ is a constant function in x. Setting $\theta(x,t) = V_x(x,t)$ and using equations (2.12)-(2.14) we end up with the system (4.7)-(2.3) that, according to Lemma 2.1, admits a unique solution $\theta(x,t) = 0$. Thus, $V(x,t)$ is a constant function in x. On the other hand, using equation (2.12), we have $V_{tt}(x,t) = 0$ and hence $V_t(x,t) = K$ is a constant function where, from $V_t(x,0) = 0$, we get $V_t(x,t) = 0$. Therefore $V(x,t)$ is a constant function in t too, thus $V(x,t) = \Delta A_0(x) =$ constant. Using (2.11) we obtain

$$\Delta A(x,t) = \Delta A_0(x) + G(t) \tag{2.17}$$

and obviously $\Delta A(x,T) = 0$ shows that $\Delta A_0(x) + G(T) = 0$.

($\Longleftarrow$ If $\Delta A_0(x) + G(T) = 0$ one deduces that the unique solution to the system (2.12)-(2.14) is the constant function $V(x,t) = \Delta A_0(x)$ in Ω_T. Applying (2.15) in (2.17) and using (2.16) we obtain $\Delta A(x,T) = 0$. ■

The main result of this section is stated in the following theorem.

Theorem 2.3. *Assume that $\frac{T}{L} \notin \mathcal{C} = \{\frac{2m+1}{2n}|\ m,n \in \mathbb{N}\}$, then the inverse problem (1.1)-(1.4) possesses a unique solution.*

Proof. Since $A_1(x,t),\ A_2(x,t)$ are supposed to satisfy all the initial and boundary conditions (1.2)-(1.4), then we have

$$\Delta A(x,T) = 0, \quad c_1 \Delta A(x^*,t) + c_2 \Delta A_x(x^*,t) = 0. \tag{2.18}$$

Combining $\Delta A(x,T) = 0$ with Lemma 2.2 and using (2.17), we deduce that

$$0 = c_1 \Delta A(x^*,t) + c_2 \Delta A_x(x^*,t) = c_1\big(\Delta A_0(x^*) + G(t)\big) + c_2 \frac{\partial\big(\overbrace{V(x,t)}^{=constant} + G(t)\big)}{\partial x}|_{x=x^*}. \tag{2.19}$$

Setting $t = 0$ in (2.19) we find $\Delta A_0(x^*) = 0$ and then $G(t) = 0$, that is,

$$0 = \Delta f(t) = f_2(t) - f_1(t) \Longrightarrow f_2(t) = f_1(t).$$

Finally, using (2.19) we get $0 = \Delta A_0(x) = A_2(x,0) - A_1(x,0) \Longrightarrow A_2(x,0) = A_1(x,0)$. ■

Remark 2.4. *In the previous uniqueness results, we have assumed that the speed c is constant (normalized to be the unity). Actually, the same result holds for, known, variable and smooth speeds $c := c(x)$. Using the same arguments as before, we can show that if $\lambda_k T \neq (\frac{\pi}{2} + n\pi)$, $n = 1, 2, ...$, then the corresponding inverse problem has a unique solution. Here λ_k's are the eigenvalues of the reduced problem $c(x)X''(x) + \lambda X(x) = 0$, in $(0, L)$ with the boundary conditions $X(0) = X(L) = 0$. Moreover, similar uniqueness result can be shown for the more general Sturm Liouville equation $\rho(x)(X_t)_t - (c(x)X_x)_x = f(t)$ with, known, variable and smooth coefficients ρ and c.*

3 Conditional stability estimate of the inverse problem

In this section, we derive a conditional Hölder stability estimate for our inverse problem. We use the usual notations for the norms of the needed Sobolev spaces: $\|f^\diamond\|_2^2 := \int_0^T |f^\diamond(t)|^2 dt$, $\|f^\diamond\|_{W^{1,\infty}(0,T)} := \|f^\diamond\|_{L^\infty(0,T)} + \|\frac{df^\diamond}{dt}\|_{L^\infty(0,T)}$ and $\|A^\diamond\|_2^2 := \int_0^T \int_0^L |A^\diamond(x,t)|^2 dxdt$.

Theorem 3.1. *Let $A^\diamond(x,t)$ be the unique solution to the system*

$$A^\diamond_{tt}(x,t) - A^\diamond_{xx}(x,t) = f^\diamond(t), \quad in \quad \Omega_T = \{(x,t)|\ 0 \le x \le L,\ 0 \le t \le T\}, \tag{3.1}$$

$$A^\diamond_x(0,t) = 0, \quad A^\diamond_x(L,t) = 0, \quad 0 < t < T, \tag{3.2}$$

$$A^\diamond_t(x,0) = 0, \quad A^\diamond(x,T) = 0, \quad 0 < x < L. \tag{3.3}$$

Then, there exist two positive constants σ_1 and σ_2 such that [4]

$$\|f^\diamond\|_2^2 \le \sigma_1 \mu^{\frac{1}{3}} \ \text{and}\ \|A^\diamond\|_2^2 \le \sigma_2 \mu^{\frac{1}{3}} \tag{3.4}$$

for any $f^\diamond \in W^{1,\infty}\big(0,T\big)$ satisfying

$$\|f^\diamond\|_{W^{1,\infty}(0,T)} \le U \tag{3.5}$$

where

$$\|c_1 A^\diamond(x^*,\cdot) + c_2 A^\diamond_x(x^*,\cdot)\|_{L^\infty(0,T)} \le \mu. \tag{3.6}$$

The constants σ_1 and σ_2 are estimated as:

$$\sigma_1 \le C\, U^{\frac{5}{3}} c_1^{\frac{-1}{3}} T^{\frac{1}{3}} \max\{1, T^{\frac{5}{3}}\} \ \text{and}\ \sigma_2 = L(\frac{4T^2}{\pi^2})^2 \sigma_1 \tag{3.7}$$

where C is the universal constant given in (3.25).

Proof. We apply the standard Fourier method and introduce the solution to the system (3.1)-(3.3) as:

$$A^\diamond(x,t) = \sum_{k=0}^{\infty} \alpha_k(x) \cos\big(\frac{(2k+1)\pi t}{2T}\big). \tag{3.8}$$

Considering the governing equation (3.1) we get:

$$A^\diamond_{tt}(x,t) - A^\diamond_{xx}(x,t) = \sum_{k=0}^{\infty} \big\{ -\big(\frac{(2k+1)\pi}{2T}\big)^2 \alpha_k(x) - \alpha_k''(x) \big\} \cos\big(\frac{(2k+1)\pi t}{2T}\big) = f^\diamond(t). \tag{3.9}$$

[4] The norms used in the estimates (3.5) and (3.6) can be changed with the ones of $H^1(0,T)$ and $L^2(0,T)$ respectively. With these norms the constants σ_1 and σ_2 will involve $T^{-\frac{1}{3}}$ as a multiplicative term which grows as T is small.

$f^{\diamond}(t)$ can be represented by the series $f^{\diamond}(t) = \sum_{k=0}^{\infty} f_k \cos\big(\frac{(2k+1)\pi t}{2T}\big)$ that substituting it in equation (3.9) we have:

$$\sum_{k=0}^{\infty} \big\{ -\big(\frac{(2k+1)\pi}{2T}\big)^2 \alpha_k(x) - \alpha_k^{''}(x)\big\} \cos\big(\frac{(2k+1)\pi t}{2T}\big) = \sum_{k=0}^{\infty} f_k \cos\big(\frac{(2k+1)\pi t}{2T}\big). \tag{3.10}$$

Hence, the unknown functions $\alpha_k(x)$ should satisfy the following system of second order differential equations:

$$\big(\frac{(2k+1)\pi}{2T}\big)^2 \alpha_k(x) + \alpha_k^{''}(x) = -f_k, \quad \alpha_k^{'}(0) = \alpha_k^{'}(L) = 0, \tag{3.11}$$

where $f_k = \frac{2}{T}\int_0^T f^{\diamond}(t)\cos\big(\frac{(2k+1)\pi t}{2T}\big)$ and the boundary conditions $\alpha_k^{'}(0) = \alpha_k^{'}(L) = 0$ are induced from $A_x^{\diamond}(0,t) = A_x^{\diamond}(L,t) = 0$. The general solution to the system of equations (3.11) is

$$\alpha_k(x) = \Gamma_1 \cos\big(\frac{(2k+1)\pi x}{2T}\big) + \Gamma_2 \sin\big(\frac{(2k+1)\pi x}{2T}\big) + \xi_k. \tag{3.12}$$

Now by applying the homogeneous boundary conditions we find that:

$$\alpha_k^{'}(0) = 0 \Longrightarrow \Gamma_2 = 0,\ \alpha_k^{'}(L) = 0 \Longrightarrow \Gamma_1 \times \big(\frac{(2k+1)\pi}{2T}\big) \sin\big(\frac{(2k+1)\pi L}{2T}\big) = 0. \tag{3.13}$$

Clearly, if $\sin\big(\frac{(2k+1)\pi L}{2T}\big) = 0$, then

$$\frac{(2k+1)\pi L}{2T} = k^{'}\pi, k^{'} \in \mathbb{Z} \Longrightarrow \frac{T}{L} = \frac{2k+1}{2k^{'}} \in \mathcal{C}.$$

This contradicts our assumption $\frac{T}{L} \notin \mathcal{C}$. Thus, from the last equality in (3.13) we have $\Gamma_1 = 0$. Therefore, using (3.11) we get $\xi_k = \frac{-f_k}{\big(\frac{(2k+1)\pi}{2T}\big)^2}$. Accordingly, the solution can be written as:

$$A^{\diamond}(x,t) = \sum_{k=0}^{\infty} \frac{-4T^2 f_k}{(2k+1)^2\pi^2} \cos\big(\frac{(2k+1)\pi t}{2T}\big), \quad f^{\diamond}(t) = \sum_{k=0}^{\infty} f_k \cos\big(\frac{(2k+1)\pi t}{2T}\big). \tag{3.14}$$

Using the relations

$$\forall\ m,n \in \mathbb{N}, < \cos\big(\frac{(2n+1)\pi t}{2T}\big), \cos\big(\frac{(2m+1)\pi t}{2T}\big) >= \begin{cases} 0, & m \neq n. \\ \frac{T}{2} & m = n, \end{cases} \tag{3.15}$$

we deduce that

$$\|f^{\diamond}\|_2^2 = \int_0^T |f^{\diamond}(t)|^2 dt = \int_0^T \big\{\sum_{k=0}^{\infty} f_k \cos\big(\frac{(2k+1)\pi t}{2T}\big)\big\}^2 dt = \frac{T}{2}\sum_{n=0}^{\infty} |f_n|^2. \tag{3.16}$$

Moreover, integration by parts implies that

$$f_k \frac{T}{2} = \int_0^T f^{\diamond}(t)\cos\big(\frac{(2k+1)\pi t}{2T}\big)dt = \frac{f^{\diamond}(t)}{\big(\frac{(2k+1)\pi}{2T}\big)} \sin\big(\frac{(2k+1)\pi t}{2T}\big)\Big]_{t=0}^{t=T}$$

$$-\frac{1}{\big(\frac{(2k+1)\pi}{2T}\big)} \int_0^T \frac{df^{\diamond}(t)}{dt} \sin\big(\frac{(2k+1)\pi t}{2T}\big)dt. \tag{3.17}$$

Since $f^\diamond \in W^{1,\infty}(0,T)$, hence it is continuous and by the mean value theorem there exists $t^* \in (0,T)$ such that $\frac{1}{T}\int_0^T f^\diamond(t)dt = f^\diamond(t^*)$. We write $f^\diamond(T) - f^\diamond(t^*) = \int_{t^*}^T \frac{df^\diamond}{dt}(t)dt$, then we get $|f^\diamond(T)| \leq \frac{1}{T}\int_{t^*}^T |f^\diamond(t)|dt + \int_{t^*}^T |\frac{df^\diamond}{dt}(t)|dt$ and hence $|f^\diamond(T)| \leq \max\{1,T\}\, U$. The equation (3.17) becomes

$$|f_k \frac{T}{2}| \leq \frac{1}{\big(\frac{(2k+1)\pi}{2T}\big)} \max\{1,T\}\, U \Longrightarrow |f_k| \leq \frac{4\max\{1,T\}\, U}{\pi(2k+1)}. \tag{3.18}$$

Also, taking $A^\diamond(x,t) = \sum_{k=0}^\infty \frac{-4T^2 f_k}{(2k+1)^2\pi^2}\cos\big(\frac{(2k+1)\pi t}{2T}\big)$ in view of equation (3.15) we arrive at:

$$\int_0^T \big\{c_1 A^\diamond(x^*,t) + c_2 A_x^\diamond(x^*,t)\big\}\cos\big(\frac{(2k+1)\pi t}{2T}\big)dt = \frac{-4c_1T^2 f_k}{(2k+1)^2\pi^2}\frac{T}{2}, \tag{3.19}$$

and using (3.6) we obtain $\frac{4|c_1|T^2 f_k}{(2k+1)^2\pi^2}\frac{T}{2} \leq \mu\, T$ and hence

$$|f_k| \leq \frac{\mu\pi^2(2k+1)^2}{2|c_1|T^2}. \tag{3.20}$$

We split (3.16) as

$$\|f^\diamond\|_2^2 = \frac{T}{2}\big(\sum_{k=0}^{\Lambda}|f_k|^2 + \sum_{k=\Lambda+1}^{\infty}|f_k|^2\big), \tag{3.21}$$

and use the upper bounds presented in equations (3.20) and (3.18) for $|f_k|$, then we get

$$\|f^\diamond\|_2^2 \leq \frac{\pi^4\mu^2}{8|c_1|^2T^3}\sum_{n=0}^{\Lambda}(2n+1)^4 + \frac{8U^2\max\{1,T^2\}\, T}{\pi^2}\big(\frac{\pi 2}{8} - \sum_{n=0}^{\Lambda}(2n+1)^{-2}\big) \tag{3.22}$$

$$\leq \Phi(\Lambda) := \frac{\pi^4\mu^2}{8|c_1|^2T^3}\widetilde{\alpha_1}\Lambda^5 + \frac{8U^2\max\{1,T^2\}T}{\pi^2}\frac{\widetilde{\alpha_2}}{\Lambda}, \quad \widetilde{\alpha_1},\ \widetilde{\alpha_2} \in \mathbb{R}^+. \tag{3.23}$$

One can fairly take $\widetilde{\alpha_1} = 90,\ \widetilde{\alpha_2} = \frac{1}{2}$, then the lower bound of $\Phi(\Lambda)$ holds for

$$\Lambda_* = \big(\frac{8}{5\pi^6}\big)^{\frac{1}{6}}\mu^{\frac{-1}{3}}T^{\frac{2}{3}}(\max\{1,T^2\})^{\frac{1}{6}}(\widetilde{\alpha_2})^{\frac{1}{6}}(\widetilde{\alpha_1})^{\frac{-1}{6}}c_1^{\frac{1}{3}}U^{\frac{1}{3}}.$$

Thus

$$\|f^\diamond\|_2^2 \leq \sigma_1\mu^{\frac{1}{3}},\ \text{where } \sigma_1 \leq C\, U^{\frac{5}{3}}c_1^{\frac{-1}{3}}T^{\frac{1}{3}}(\max\{1,T^2\})^{\frac{5}{6}} \tag{3.24}$$

and

$$C := \big(\sqrt[6]{5\times 8^5} + \frac{1}{\sqrt[6]{8\times 5^5}}\big)\frac{1}{\pi}(\widetilde{\alpha_1})^{\frac{1}{6}}(\widetilde{\alpha_2})^{\frac{5}{6}}. \tag{3.25}$$

Furthermore, we know that

$$\|A^\diamond\|_2^2 = \int_0^T\int_0^L \big\{\sum_{k=0}^{\infty}\frac{-4T^2 f_k}{(2k+1)^2\pi^2}\cos\big(\frac{(2k+1)\pi t}{2T}\big)\big\}^2 dxdt \tag{3.26}$$

$$= \sum_{k=0}^{\infty} L\big(\frac{-4T^2}{(2k+1)^2\pi^2}\big)^2\frac{T}{2}|f_k|^2 \leq L\big(\frac{4T^2}{\pi^2}\big)^2\|f^\diamond\|_2^2 \leq L\big(\frac{4T^2}{\pi^2}\big)^2\sigma_1\mu^{\frac{1}{3}}, \tag{3.27}$$

which, by setting $\sigma_2 = L\big(\frac{4T^2}{\pi^2}\big)^2\sigma_1$, becomes

$$\|A^\diamond\|_2^2 \leq \sigma_2\mu^{\frac{1}{3}}. \tag{3.28}$$

In addition

$$\|A_0(x)\|_2^2 = \|A(x,0)\|_2^2 = L\big(\frac{32T^3}{\pi^4}\big)\sum_{k=0}^{\infty}\frac{T}{2}\frac{f_k^2}{(2k+1)^4} \leq L\big(\frac{32T^3}{\pi^4}\big)\sigma_1\mu^{\frac{1}{3}}.\blacksquare$$

In other words, by considering $(A(x,t), f(t))$ and $(\widetilde{A(x,t)}, \widetilde{f(t)})$ as the exact and approximate solutions to the problem (1.1)-(1.4) in case the boundary data (1.4) is contaminated with errors such that $|error| \leq \mu$, then:

$$\exists \;\; \sigma_1,\; \sigma_2 \in \mathbb{R}^+ \;\; s.t \;\; \|f(t) - \widetilde{f(t)}\|_2^2 \leq \sigma_1\mu^{\frac{1}{3}}, \;\; \|A(x,t) - \widetilde{A(x,t)}\|_2^2 \leq \sigma_2\mu^{\frac{1}{3}}.$$

4 Solution procedure

To solve the inverse problem given by (1.1)-(1.4), we first make use of the transformation

$$W(x,t) = A(x,t) - \int_0^t\int_0^s f(\tau)d\tau ds, \quad Z(x,t) = W_x(x,t), \tag{4.1}$$

and obtain the following equations

$$Z_{tt}(x,t) = Z_{xx}(x,t), \quad in \quad \Omega_T \tag{4.2}$$

$$Z(0,t) = b_1(t), \quad Z(L,t) = b_2(t), \quad 0 < t < T, \tag{4.3}$$

$$Z(x,T) = h_1^{'}(x), \quad Z_t(x,0) = 0, \quad 0 < x < L. \tag{4.4}$$

We divide the procedure into several steps.

Step 1 : This stage is devoted to solve the problem (4.2)-(4.4) by applying the Ritz-Galerkin method. First, by introducing the auxiliary functions [16, 20, 21, 27]

$$B(x,t) = b_1(t) + \frac{x}{L}\big(b_2(t) - b_1(t)\big), \quad I(x,t) = \frac{t^2}{T^2}h_1^{'}(x), \tag{4.5}$$

the satisfier function to the system (4.3)-(4.4) is

$$SF(x,t) = B(x,t) + I(x,t) - \{I(0,t) + \frac{x}{L}\big(I(L,t) - I(0,t)\big)\}, \tag{4.6}$$

provided that the following compatibility conditions hold

$$b_1^{'}(0) = b_2^{'}(0) = 0,\; h_1^{'}(0) = b_1(T),\; h_1^{'}(L) = b_2(T),\; h(T) = c_1h_1(x^*) + c_2h_1^{'}(x^*). \tag{4.7}$$

Hence, the Ritz type approximation to the problem (4.2)-(4.4) is presented by

$$\overline{Z(x,t)} = \sum_{n_j=0}^{k_j-1}\sum_{j=0}^{\mathbf{O}}\sum_{n_i=0}^{k_i-1}\sum_{i=0}^{\mathbf{O}} z_{ijn_in_j}t^2(t-T)x(x-L)\psi_{i,n_i}(x)\psi_{j,n_j}(t) + SF(x,t). \tag{4.8}$$

The bases functions $\psi_{i,n_i}(x),\; \psi_{j,n_j}(t)$ can be selected as the Bernstein Multi-scaling functions described by, see [9, 20],

$$\psi_{i,n}(x) = \begin{cases} B_{i,\mathbf{O}}(\frac{k}{F}x - n), & \frac{n\;F}{k} \leq x < \frac{(n+1)F}{k}, \\ 0, & otherwise, \end{cases} \tag{4.9}$$

where $n = \overline{0, k-1}$, $[0, F) \subset \mathbb{R}$, $\mathbf{O}$ is the order of Bernstein polynomial $B_{i,\mathbf{O}}(x)$ defined by:

$$B_{i,\mathbf{O}}(x) = \frac{\mathbf{O}!}{i!(\mathbf{O}-i)!} \frac{(x)^i (b-x)^{\mathbf{O}-i}}{(b)^{\mathbf{O}}}, \quad 0 \leq i \leq \mathbf{O}, \tag{4.10}$$

and b is the maximum range of the interval $[0, b]$ over which the polynomials are defined to form a complete basis [12, 20]. Using the residual function $Res\big(Z(x,t)\big) = \overline{Z(x,t)}_{tt} - \overline{Z(x,t)}_{xx}$ and solving the final system of equations $Az = y$, resulted from the Galerkin equations

$$\int_0^T \int_0^L Res\big(Z(x,t)\big)\psi_{i,n_i}(x)\psi_{j,n_j}(t)dxdt = 0, \quad i,j = \overline{0,\mathbf{O}},\ n_{s\in\{i,j\}} = \overline{0, k_s - 1}, \tag{4.11}$$

we derive the approximation (4.8).

Step 2 : Here we approximate $A(x,0) = A_0(x)$. From (4.1), it is obvious that $A_0(x) = W(x,0)$ and $W_x(x,0) = Z(x,0)$. Thus

$$A(x,0) = W(x,0) = \int_0^x Z(s,0)ds + \gamma. \tag{4.12}$$

Employing the extra condition (1.4) we get

$$h(0) = c_1 A(x^*,0) + c_2 A_x(x^*,0) = c_1 W(x^*,0) + c_2 W_x(x^*,0), \tag{4.13}$$

then, using (4.13) and inserting the approximation $\overline{Z(x,t)}$ in (4.12) we show that

$$\gamma = \frac{h(0) - c_2\overline{Z(x^*,0)} - c_1 \int_0^{x^*} \overline{Z(x,0)}dx}{c_1}. \tag{4.14}$$

Step 3 : Here we deal with the approximation of $f(t)$. Considering an arbitrary point $x^* \in (0, L)$ and taking the governing equation (1.1) into account we have:

$$A_{tt}(x^*,t) - A_{xx}(x^*,t) = A_{tt}(x^*,t) - W_{xx}(x^*,t). \tag{4.15}$$

Using the extra condition (1.4) and taking advantage of the relation $A_x(x,t) = Z(x,t)$ we deduce that

$$f(t) = \frac{h''(t) - c_2 Z_{tt}(x^*,t)}{c_1} - Z_x(x,t)|_{x=x^*}. \tag{4.16}$$

By substituting the approximation $\overline{Z(x,t)}$ from (4.8) in (4.16), we derive the approximation $\overline{f(t)}$ of $f(t)$.

Remark 4.1. *It is worth mentioning that recovering $f(t)$ from equation (4.16) is possible as long as the function $h(t)$, given as the overdetermination of the problem, is twice differentiable or at least piecewise continuously differentiable. Indeed, this assumption is almost impossible because the extra measurement of the problem involves perturbations. Even if we add the smooth function $p(t) = \lambda \sin(\frac{t}{\lambda^2}), \lambda \longrightarrow 0$ to the boundary condition $h(t)$, as the error with input data, that is, $\widetilde{h(t)} = h(t) + p(t)$ and differentiate two times, we get $\widetilde{h''(t)} = h''(t) - \frac{\sin(\frac{t}{\lambda^2})}{\lambda^3}$. This value tends to infinity if $\lambda \longrightarrow 0$ and then enters a large error in our computations. Thus, for the general case, we disregard the relation (4.16) and propose the following substitute.*

Step $3^{'}$: From the governing equation (1.1) we have

$$A(x,t) - A_0(x) = \int_0^t \int_0^{t^{'}} \big\{A_{xx}(x,s) + f(s)\big\} ds dt^{'}. \tag{4.17}$$

Using the approximations for $A_0(x),\ Z(x,t)$ obtained in equations (4.12) and (4.8) respectively and using the condition (1.4), we get

$$h(t) - \big(c_1 \overline{A(x,0)}|_{x=x^*} + c_2 \overline{A_x(x,0)}|_{x=x^*}\big) = A^{\#}(x^*,t) + B^{\#}(x^*,t), \tag{4.18}$$

where

$$A^{\#}(x,t) = c_1 \int_0^t \int_0^{t^{'}} \big\{\overline{Z_x(x,s)} + f(s)\big\} ds dt^{'}, \quad B^{\#}(x,t) = c_2 \frac{\partial}{\partial x} \int_0^t \int_0^{t^{'}} \big\{\overline{Z_x(x,s)} + f(s)\big\} ds dt^{'}. \tag{4.19}$$

Introducing the approximation

$$\overline{f(t)} = \sum_{n_r=0}^{k_r-1} \sum_{r=0}^{\mathbf{O}} f_{rn_r} \psi_{r,n_r}(t), \tag{4.20}$$

and substituting (4.20) in (4.18) together with applying the Galerkin equations

$$\int_0^T \big\{h(t) - \big(c_1 \overline{A(x,0)}|_{x=x^*} + c_2 \overline{A_x(x,0)}|_{x=x^*}\big) - B^{\#}(x^*,t) - A^{\#}(x^*,t)\big\} \psi_{r,n_r}(t) dt = 0, \tag{4.21}$$

we get a linear system of equations to solve and determine the unknown coefficients

$$f_{rn_r}, \quad r = \overline{0, \mathbf{O}}, \quad n_r = \overline{0, k_r - 1},$$

in the approximation (4.20).

Step 4 : The final step is to seek for the approximation of $A(x,t)$. From the relations (4.1) we obtain

$$W(x,t) = \int_0^x Z(x,t) + H(t) \Longrightarrow A(x,t) = \int_0^x Z(x,t) + H(t) + \int_0^t \int_0^s f(\tau) d\tau ds, \tag{4.22}$$

in $0 < x < L,\ 0 < t < T$ and reapplying the extra condition (1.4) we arrive at

$$H(t) = \frac{h(t) - \big\{c_1\big(\int_0^{x^*} Z(s,t) ds + \int_0^t \int_0^s f(\tau) d\tau ds\big) + c_2 Z(x^*,t)\big\}}{c_1}. \tag{4.23}$$

Substituting the approximations $\overline{Z(x,t)},\ \overline{f(t)}$ in $Z(x,t)$ and $f(t)$ respectively and finally inserting $H(t)$ in equation (4.22), we derive the approximation $\overline{A(x,t)}$ of $A(x,t)$. Here ends Step 4 and hence the solution procedure.

We finish this section by deriving an estimate between the true solution $A(x,t)$ and the approximated one $\overline{A(x,t)}$. On the Hilbert space $\mathcal{H} = \mathcal{L}_2(0,L)$, equipped with scalar product $< u, v >:= \int_0^L u(x)v(x)dx$, we consider the operator $(\mathcal{F}, Dom(\mathcal{F}))$ defined by

$$\mathcal{F}u := -\frac{\partial}{\partial x}\big(\frac{\partial u}{\partial x}\big) \text{ and } Dom(\mathcal{F}) = \{u \in \mathcal{H}, \text{ such that } \mathcal{F}u \in \mathcal{H} \text{ and } \frac{\partial u(0)}{\partial x} = \frac{\partial u(L)}{\partial x} = 0\}. \tag{4.24}$$

Observe that $Dom(\mathcal{F}) = \{u \in H^2(0,L), \text{ such that } \frac{\partial u(0)}{\partial x} = \frac{\partial u(L)}{\partial x} = 0\}$. Taking into account the procedure given by **Step 1-Step 4** and setting

$$e(x,t) = A(x,t) - \overline{A(x,t)}, \ \delta f(t) = f(t) - \overline{f(t)}, \tag{4.25}$$

we can write the Cauchy-problem in the form of a second-order operator differential equation for $e(t) \in \mathcal{H}$ as:

$$\frac{d^2 e}{dt^2} + \mathcal{F}e = \delta f(t), \quad 0 < t < T, \quad e(0) = e_0, \ e_t(0) = 0. \tag{4.26}$$

Then, we have the following estimate.

Proposition 4.2. *The following upper bound for the error function $e(t)$ holds*

$$\|e(t)\|_*^2 \leq \exp(t)\big(\|e_0\|_{\mathcal{F}}^2 + \int_0^t \exp(-s)\|\delta f(s)\|^2 ds\big) \quad 0 < t < T \tag{4.27}$$

where

$$\|u\|_*^2 := \|\frac{du}{dt}\|_2^2 + \|u\|_{\mathcal{F}}^2 \text{ and } \|v\|_{\mathcal{F}}^2 := < \mathcal{F}v, v > \tag{4.28}$$

for the self-adjoint and positive operator $\mathcal{F}$. In particular, if $\sup_{0<t<T}\{\delta f(t), e_0\} \leq \epsilon$ then

$$\|e(t)\|_*^2 \leq 3\epsilon^2 \exp(T). \tag{4.29}$$

Proof. First, by integration by parts we see that the operator $\mathcal{F}$ is self-adjoint and non-negative in $\mathcal{H}$. Second, we have

$$\frac{1}{2}\frac{d}{dt}\big(\|\frac{de}{dt}\|_2^2 + \|e\|_{\mathcal{F}}^2\big) = \frac{1}{2}\frac{d}{dt}\big(\int_0^L (\frac{de}{dt})^2 + (\frac{\partial e}{\partial x})^2\big)dx$$

$$= \int_0^L \frac{d^2 e}{dt^2}\frac{de}{dt} + \frac{\partial}{\partial x}(\frac{de}{dt})\frac{\partial e}{\partial x}dx = \int_0^L \frac{de}{dt}\{\underbrace{\frac{d^2 e}{dt^2} - \frac{\partial^2 e}{\partial x^2}}_{=\delta f(t)}\}dx + \underbrace{\frac{\partial e}{\partial x}\frac{de}{dt}|_{x=0}^{x=L}}_{=0}$$

$$\leq \|\frac{de}{dt}\|_2 \|\delta f(t)\|_2 \leq \frac{1}{2}\big(\|\frac{de}{dt}\|_2^2 + \|\delta f(t)\|_2^2 + \|e\|_{\mathcal{F}}^2\big). \tag{4.30}$$

The desired estimate is derived by applying the Gronwall's lemma to the inequality (4.30). For the case $\sup_{0<t<T}\{\delta f(t), e_0\} \leq \epsilon$, it is obvious that

$$\|e(t)\|_*^2 \leq \exp(t)\big(\|e_0\|_{\mathcal{F}}^2 + \int_0^t \exp(-s)\|\delta f(s)\|^2 ds\big), \qquad 0 < t < T, \tag{4.31}$$

$$\leq \exp(t)\big(\epsilon^2 + \epsilon^2 \int_0^t \exp(-s)ds\big) = \epsilon^2 \exp(t)\big(1 + 1 + \exp(-t)\big) \leq 3\epsilon^2 \exp(T). \tag{4.32}$$

■

5 Numerical experiments

We solve two benchmark test examples which are chosen for reporting the results of implementing the Ritz-Galerkin method in the presence of both exact and contaminated data with the noise level $\lambda\% = \lambda \times 10^{-4}$. The numerical implementation is carried out in MATHEMATICA 7, with hardware configuration: desktop 32-bit Intel Core 2 Duo CPU, 4 GB of RAM, 32-bit Operating System (Windows 7).

5.1 Example 1

As a first example, the algorithm given by **Step 1-Step** 3 and **Step 4** is tested for approximating the piecewise continuously differentiable functions:

$$f(t) = \begin{cases} 12t^2 + 2, & 0 \le t < \frac{1}{4}, \\ -25\cos(5t), & \frac{1}{4} \le t < \frac{1}{2}, \\ 6t, & \frac{1}{2} \le t < \frac{3}{4}, \\ -2, & \frac{3}{4} < t < 1 \end{cases} \tag{5.1}$$

and

$$A(x,t) = \begin{cases} \frac{2}{3} - x - x^2 + t^4 - \sin(x)\cos(t), & 0 \le t < \frac{1}{4}, \\ \cos(5t), & \frac{1}{4} \le t < \frac{1}{2}, \\ \exp(x+t) + t^3, & \frac{1}{2} \le t < \frac{3}{4}, \\ x^2 - t, & \frac{3}{4} < t < 1 \end{cases} \tag{5.2}$$

with the following properties

$$c_1 = T = L = 1,\ c_2 = -1,\ x^* = \frac{9}{10}.$$

Here, we consider the exact boundary data and use the formulas (4.8) and (4.16) to illustrate the approximations as depicted in Figure 1. It should be noticed that we have taken the satisfier function in (4.8) as an approximation for $Z(x,t)$, [20]. Thus, the solution is straightforward and we need to solve no system of algebraic equations.

5.2 Example 2

Consider the inverse problem (1.1)-(1.4) with the following properties

$$b_1(t) = -1 + \cos(t), \quad c_1 = T = L = 1,\ c_2 = -1,\ x^* = \frac{9}{10},$$

$$h_1(x) = 3 - x + x^2 - \sec(1) + \cos(1)\sin(x) + \tan(1), \quad b_2(t) = 1 + \cos(1)\cos(t),$$

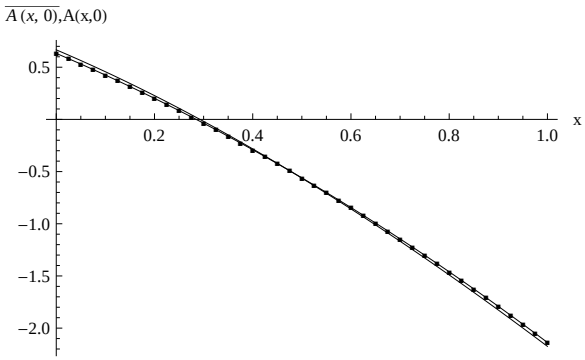
$$h(t) = \frac{111}{100} + t^3 - \big(\cos(\frac{9}{10}) - \sin(\frac{9}{10})\big)\cos(t) - \sec(t) + t\tan(t) + \lambda\sin(\frac{t}{\lambda^2}),\ \lambda = 3\%.$$

We aim to approximate the solutions for the continuous functions

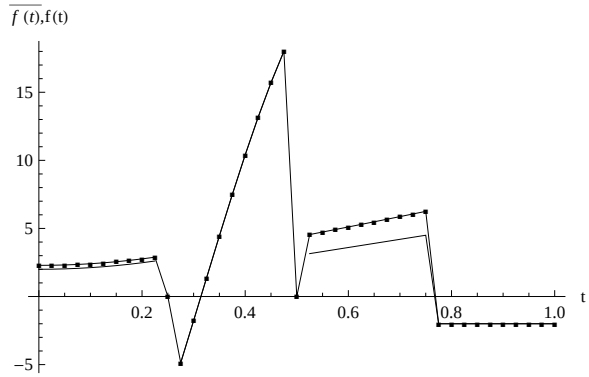
$$f(t) = -2 + 6t + 2\sec^2(t)\big(1 + \tan^2(t)\big) - \sec(t)\big(\sec^2(t) + \tan^2(t)\big),$$

$$A(x,t) = \sin(x)\cos(t) + 2 - x + x^2 - \sec(t) + t^3 + t\tan(t), \quad A_0(x) = 1 - x + x^2 + \sin(x).$$

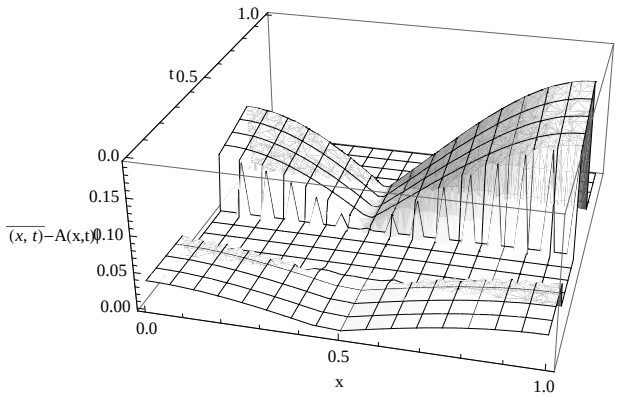
By applying the method presented in **Step 1-Step 2** and **Step** $3'$**-Step 4** along with employing the Bernstein basis functions of order 3, we obtain the results presented in Figure 2. The approximations in the presence of the contaminated input data are derived using the Landweber's iterations [14, 15] with $a = 1,\ m(\delta) = [\csc(\delta)],\ \delta = 5 \times 10^{-5}$. Following them, it is seen that the proposed method provides the approximations which have the acceptable agreement with the exact solutions.



(a) Exact (*Line*) and approximate (**. . .**) solutions for $A_0(x)$

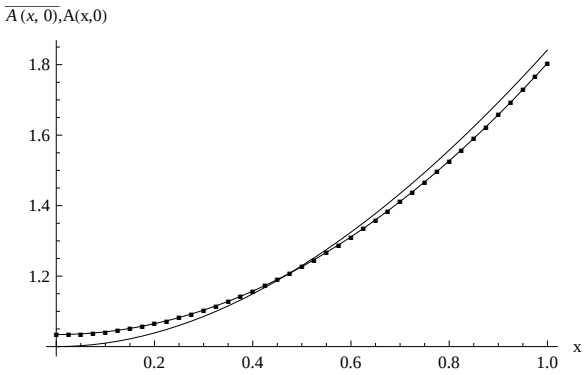


(b) Exact (*Line*) and approximate (**. . .**) solutions for $f(t)$

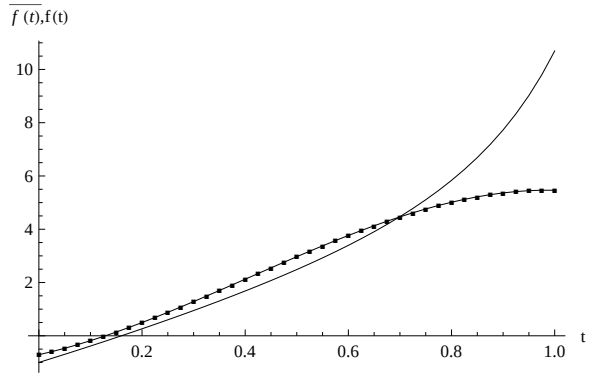


(c) Absolute error between the approximate and exact solutions for $A(x,t)$

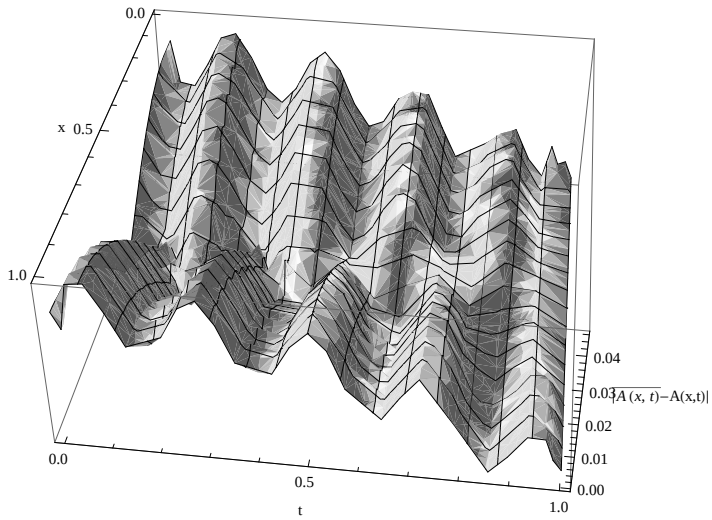
Figure 1: Graphs of the exact and approximate solutions for Example 5.1. All plots for exact data.



(a) Exact (*Line*) and approximate (**...**) solutions for $A_0(x)$



(b) Exact (*Line*) and approximate (**...**) solutions for $f(t)$



(c) Absolute error between the approximate and exact solutions for $A(x,t)$

Figure 2: Graphs of the exact and approximate solutions for Example 5.2. All plots for contaminated data with noise level $\lambda = 3\%$.

6 Conclusion

We considered the $1D$ inverse wave problem of recovering the time dependent source term (and then the initial data) from the linear combination of the displacement and the flux measured at an arbitrary space point. We first showed the unique solvability under natural conditions on the time length. Then, we derived a conditional Hölder stability estimate of the inverse problem. Finally, we applied the Ritz-Galerkin method along with the satisfier function to obtain low cost numerical results.

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