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Ring structure of splines on triangulations

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INTRODUCTION

For a triangulated region Δ in $\mathbb{R}^2$, we consider the space $C^r(\Delta)$ of piecewise polynomial functions that are continuously differentiable of order r (C^r functions) on Δ. These functions are called splines, and they have many practical applications including the finite element method for solving differential equations. They are also very useful for modeling surfaces of arbitrary topology and are a widely recognized tool in isogeometric analysis, and free-form representation in Computer Aided Geometric Design.

Besides the interest that the space of splines has for applications, for every $r \geq 0$, the set $C^r(\Delta)$ forms a ring under pointwise multiplication. It was proved that, as a ring, $C^0(\Delta)$ is a quotient of the Stanley–Reisner ring A_Δ of Δ [2]. Since $C^{r+1}(\Delta) \subset C^r(\Delta)$, this result implies that there is a descending chain of subrings contained in A_Δ. We use a local characterization to study the ring structure of those elements of the Stanley–Reisner ring which correspond to splines of higher-order smoothness [6], we present some results, and a conjecture for the ring structure of splines on generic triangulations.

1. STANLEY-REISNER RING ASSOCIATED TO A SPLINE SPACE

Throughout this notes Δ will denote a simplicial complex supported on a simply connected domain $|\Delta| \subset \mathbb{R}^2$, and by $C_k^r(\Delta)$ we denote the space of C^r splines (for some integer $r \geq 0$) defined on Δ of degree less than or equal to k. The dual graph G_Δ associated to Δ is the graph with vertices corresponding to the 2-cells in Δ, and edges corresponding to adjacent pairs of 2-cells. Thus, in our setting, G_Δ will be connected and its number of cycles will depend on the number of interior vertices of Δ.

For an edge τ, let $L_\tau \in \mathbb{R}[x_1, x_2]$ be the linear polynomial vanishing on τ, and $\ell_\tau \in R := \mathbb{R}[x_1, x_2, x_3]$ its homogenization. In [2], it was proved that $C^0(\Delta)$ is isomorphic to $A_\Delta / \sum_{i=1}^n Y_i - 1$ as $\mathbb{R}$-algebras, where $A_\Delta := \mathbb{R}[Y_1, \ldots, Y_n]/I_\Delta$. The number of variables corresponds to the number of vertices $v_1, \ldots, v_n$ (on the boundary and in the interior) of Δ, and I_Δ is the ideal of nonfaces of Δ, which by definition, is the ideal generated by the square-free monomials corresponding to vertex sets which are not faces of Δ, namely

$$I_\Delta = \langle Y_{i_1} \cdots Y_{i_j} : \{v_{i_1}, \ldots, v_{i_j}\} \notin \Delta \rangle.$$

The key idea for this isomorphism is to view the variables of A_Δ as Courant functions centered at the corresponding vertex i.e., $Y_i(v_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta, $i, j = 1, \ldots, n$.

Let us embed Δ in the plane $\{x_3 = 1\} \subset \mathbb{R}^3$, and form the cone $\hat{\Delta}$ over Δ with vertex at the origin, and consider the splines $C^r(\hat{\Delta})$ defined on $\hat{\Delta}$. The Stanley–Reisner ring $A_{\hat{\Delta}}$ has

one variable more than A_Δ, but since that additional variable corresponds to the vertex of the cone it does not appear in any of the generators of I_Δ, hence

$$\tag{1} C^0(\hat{\Delta}) \cong A_\Delta.$$

Viewing the variables of the (affine) Stanley–Reisner ring as Courant functions gives a geometric picture of $C^0(\Delta)$; the *homogenization* allows to use tools from algebraic geometry for graded rings for studying $C^r(\Delta)$ [3–5]. Since there is a natural embedding

$$\tag{2} C^{r+1}(\hat{\Delta}) \hookrightarrow C^r(\hat{\Delta})$$

for every $r \geq 0$, the isomorphism (1) implies that there is a descending chain of subalgebras contained in A_Δ, each corresponding to a subalgebra of splines of increasing orders of smoothness [6].

2. Local characterization and C^r-splines

Let σ_1 and σ_2, as in Fig. 1, be two triangles which meet along an edge $\tau = \sigma_1 \cap \sigma_2$. Then, if f, g are polynomials supported on (the homogenizations) $\hat{\sigma}_1$ and $\hat{\sigma}_2$ respectively, f and g meet C^r smoothly if and only if $\ell_\tau^{r+1} | (f - g)$ 1 [1]. This condition translates into a condition on the polynomials in A_Δ as follows, [6].

For each vertex v_i in Δ let us denote by (v_{i1}, v_{i2}) its coordinates. For a triangle $\sigma = \{v_i, v_j, v_k\}$, let $x_i^\sigma, x_j^\sigma, x_k^\sigma$ be the linear functions that give the barycentric coordinates of a point in $\mathbb{R}^2$ in terms of the vertices of σ. Let $X_i^\sigma, X_j^\sigma, X_k^\sigma$ be the homogenization of x_i^σ, x_j^σ and x_k^σ with respect to x_3, respectively. Define $A_\sigma := \mathbb{R}[X_i^\sigma, X_j^\sigma, X_k^\sigma]$, and let

$$\tag{3} B_\sigma : \mathbb{R}[x_1, x_2, x_3] \to A_\sigma$$

be the automorphism defined by

$$x_1 \to v_{i1} X_i^\sigma + v_{j1} X_j^\sigma + v_{k1} X_k^\sigma$$
$$x_2 \to v_{i2} X_i^\sigma + v_{j2} X_j^\sigma + v_{k2} X_k^\sigma$$
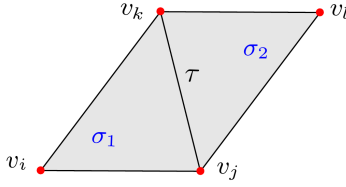$$x_3 \to X_i^\sigma + X_j^\sigma + X_k^\sigma.$$

If σ_1 and σ_2 are two triangles as in Fig. 1, the change of coordinates from A_{σ_2} to A_{σ_1} is given by the map

$$\tag{4} B_{\sigma_2 \sigma_1} : A_{\sigma_2} \to A_{\sigma_1}$$

defined by

$$X_l^{\sigma_2} \to \frac{v_{ijk}}{v_{ljk}} X_i^{\sigma_1} \qquad \text{where } v_{ijk} \text{ denotes the determinant}$$

$$X_j^{\sigma_2} \to \frac{v_{lik}}{v_{ljk}} X_i^{\sigma_1} + X_j^{\sigma_1}$$

$$X_k^{\sigma_2} \to \frac{v_{lji}}{v_{ljk}} X_i^{\sigma_1} + X_k^{\sigma_1}, \qquad v_{ijk} = \begin{vmatrix} v_{i1} & v_{j1} & v_{k1} \\ v_{i2} & v_{j2} & v_{k2} \\ 1 & 1 & 1 \end{vmatrix}.$$

Proposition 2.1 ([6]). *A polynomial $F \in A_\Delta$ corresponds to an element of $C^r(\Delta)$ if and only if $(X_i^{\sigma_1})^{r+1}$ divides $F|_{\sigma_1} - B_{\sigma_2 \sigma_1}(F|_{\sigma_2})$, with the notation as in Fig. 1, for each interior edge $\tau = \sigma_1 \cap \sigma_2$ of Δ.*

FIGURE 1. Triangles σ_1, σ_2, with $\sigma_1 \cap \sigma_2 = \tau$.

The previous condition applied to each interior edge of a given triangulation Δ yields a characterization of the elements in A_Δ which correspond to splines $C^r(\Delta)$ [6], and leads to the following result.

Proposition 2.2. *Let Δ be a 2-dimensional simplicial complex consisting of least two triangles $\sigma_1 = \{v_0, v_1, v_2\}$ and $\sigma_2 = \{v_0, v_2, v_3\}$, such that its graph G_Δ is a tree (i.e., a connected graph with no cycles). If Δ has $m + 1$ vertices then*

$$C^r(\Delta) \cong \mathbb{R}[H_0, H_1, H_2, Y_3^{r+1}, \ldots, Y_m^{r+1}]/I_\Delta,$$

where

$$H_0 = Y_0 + Y_1 + \cdots + Y_m$$
$$H_1 = v_{01}Y_0 + v_{11}Y_1 + \cdots + v_{m1}Y_m$$
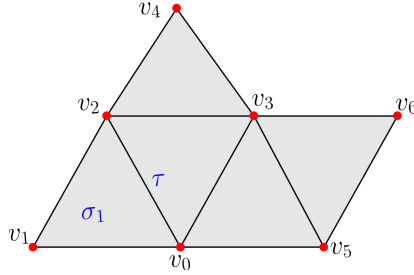$$H_2 = v_{02}Y_0 + v_{12}Y_1 + \cdots + v_{m2}Y_m.$$

FIGURE 2. Example, Proposition 2.2.

Sketch of the proof. Let us assume Δ has only two triangles $\sigma_1 = \{v_0, v_1, v_2\}$ and $\sigma_2 = \{v_0, v_2, v_3\}$. Then, the Stanley–Reisner ring associated to Δ is $A_\Delta = \mathbb{R}[Y_0, Y_1, Y_2, Y_3]/\langle Y_1 Y_3 \rangle$. Let $F \in A_\Delta$ be the linear polynomial

$$F = a_0 Y_0 + a_1 Y_1 + a_2 Y_2 + a_3 Y_3,$$

with $a_i \in \mathbb{R}$. From Proposition 2.1, if F corresponds to an element in $C^r(\Delta)$, the difference $F|_{\sigma_1} - B_{\sigma_2 \sigma_1}(F|_{\sigma_2})$ must be divisible by $(X_1^{\sigma_1})^{r+1}$, with $B_{\sigma_2 \sigma_1}$ as defined in (3). Since

$$F|_{\sigma_1} - B_{\sigma_2 \sigma_1}(F|_{\sigma_2}) = (a_0 - a_0)X_0^{\sigma_1} + (a_2 - a_2)X_2^{\sigma_1} + \left(a_1 - a_0 \frac{v_{123}}{v_{023}} - a_2 \frac{v_{013}}{v_{023}} - a_3 \frac{-v_{012}}{v_{023}}\right) X_1^{\sigma_1}$$

then (a_0, a_1, a_2, a_3) must be in the kernel of the matrix

$$(5) \qquad\qquad \begin{pmatrix} -v_{123} & v_{023} & -v_{013} & v_{012} \end{pmatrix}.$$

This kernel is spanned by the rows of the matrix

$$\begin{pmatrix} v_{01} & v_{11} & v_{21} & v_{31} \\ v_{02} & v_{12} & v_{22} & v_{32} \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and these rows define H_0, H_1, H_2. These polynomials correspond exactly to the trivial splines ($f|_\sigma = f$ for all $\sigma \in \Delta$) on Δ, which generate the ring of polynomials R (contained in $C^r(\Delta)$). The first degree we need to consider to get not trivial splines is $r + 1$. Following an analogous construction as before, it is easy to check that Y_3^{r+1}, by Proposition 2.1, corresponds to a spline in $C^r(\hat\Delta)$ which is nontrivial. Hence, all the polynomials in the ring $\mathbb{R}[H_0, H_1, H_2, Y_3^{r+1}]/I_\Delta$ correspond to elements in $C^r(\hat\Delta)$. The other inclusion follows from the dimension formula for $C_k^r(\Delta)$ [4],

$$\dim C_k^r(\Delta) = \binom{k+2}{2} + \binom{k+1-r}{2}.$$

For a simplicial complex Δ with $m \geq 3$ triangles, the proposition follows by applying the previous procedure recursively adding one new triangle at the time. $\square$

Conjecture 2.3. *For a generic central configuration Δ, where G_Δ is a cycle,*

$$C^r(\hat\Delta) \cong \mathbb{R}[H_0, H_1, H_2, S_2, \dots, S_m]/I_\Delta,$$

where $S_1, \dots, S_m$ are the polynomials in A_Δ that correspond to the generators of the module of syzygies of the ideal $\langle \ell_1^{r+1}, \dots, \ell_m^{r+1} \rangle$ in R generated by the linear forms ℓ_i corresponding to the interior edges of Δ.

Following these ideas, a similar construction leads to a conjecture for $C^r(\Delta)$ of a generic triangulation Δ, where G_Δ a connected graph with a finite number of cycles [7].

There are still many open problems concerning spline spaces, and knowing about their algebraic structure might bring some light and useful results for computation. As a consequence of the relation of $C^0(\Delta)$ with A_Δ, the dimensions as vector spaces over $\mathbb{R}$ of the subspaces $C_k^0(\Delta)$ were derived [2]. Similarly, the Proposition 2.2 and Conjecture 2.3 may lead to find the dimension of $C_k^r(\Delta)$ for splines of higher order of smoothness.

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