

# **Number Systems and Tilings over Laurent Series**

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# NUMBER SYSTEMS AND TILINGS OVER LAURENT SERIES

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ABSTRACT. Let  $\mathbb{F}$  be a field and  $\mathbb{F}[x, y]$  the ring of polynomials in two variables over  $\mathbb{F}$ . Let  $f \in \mathbb{F}[x, y]$  and consider the residue class ring  $R := \mathbb{F}[x, y]/f\mathbb{F}[x, y]$ . Our first aim is to study digit representations in  $R$ , i.e., we ask for which  $f$  each element  $r \in R$  admits a digit representation of the form  $d_0 + d_1x + \dots + d_\ell x^\ell$  with digits  $d_i \in \mathbb{F}[y]$  satisfying  $\deg_y d_i < \deg_y f$ . These digit systems are motivated by the well-known notion of canonical number system. In a next step we enlarge the ring in order to allow for representations including negative powers of the “base”  $x$ . More precisely we define and characterize digit representations for the ring  $\mathbb{F}((x^{-1}, y^{-1}))/f\mathbb{F}((x^{-1}, y^{-1}))$  and give easy to handle criteria for finiteness and periodicity. Finally, we attach fundamental domains to our number systems. The fundamental domain of a number system is the set of all numbers having only negative powers of  $x$  in their “ $x$ -ary” representation. Interestingly, the fundamental domains of our number systems turn out to be unions of boxes. If we choose  $\mathbb{F} = \mathbb{F}_q$  to be a finite field, these unions become finite.

## 1. INTRODUCTION

Since the end of the 19th century various generalizations of the usual radix representation of the ordinary integers to other algebraic structures have been introduced and extensively investigated. A prominent class of number system are *canonical number systems* in residue class rings of polynomials over  $\mathbb{Z}$ . A special instance of a canonical number systems has first been investigated by KNUTH [14]. Later on, they have been studied thoroughly by GILBERT, GROSSMAN, KÁTAI, KOVÁCS and SZABÓ (cf. for instance [6, 7, 11, 12, 13]). The present definition of canonical number system goes back to PETHŐ [15] and reads as follows. Let  $P \in \mathbb{Z}[x]$  and consider the residue class ring  $\mathbb{Z}[x]/P\mathbb{Z}[x]$ . Then  $(x, \{0, 1, \dots, |P(0)| - 1\})$  is called a *canonical number system* if each  $z \in \mathbb{Z}[x]/P\mathbb{Z}[x]$  can be represented by an element

$$\sum_{j=0}^{\ell} d_j x^j \in \mathbb{Z}[x] \quad \text{with “digits” } 0 \leq d_j < |P(0)|.$$

More recently, canonical number systems gained considerable interest and were studied extensively. We refer the reader for instance to AKIYAMA et al. [1] where they are embedded in a more general framework.

The present paper is motivated by the definition of canonical number systems. Indeed, we replace  $\mathbb{Z}$  in the definition of canonical number systems by  $\mathbb{F}[y]$ , where  $\mathbb{F}$  is an arbitrary field. For finite fields this concept has been introduced and studied by the third and fourth authors [18] (similar generalizations have been investigated in recent years; see e.g. [4, 8, 16]). Indeed, let  $\mathbb{F} = \mathbb{F}_q$  be a finite field. In [18] number systems in the residue class ring  $R := \mathbb{F}[x, y]/f\mathbb{F}[x, y]$ , where  $f \in \mathbb{F}[x, y]$  have been investigated. In particular, all polynomials  $f$  with the property that each  $r \in R$  admits a finite representation

$$r \equiv d_0 + d_1x + \dots + d_\ell x^\ell \pmod{f}.$$

with “digits”  $d_i \in \mathbb{F}[y]$  satisfying  $\deg_y d_i < \deg_y f$  (see Section 2 for a formal definition). Moreover, eventually periodic representations have been investigated in this paper. In the first part of the

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present paper we are able to extend their result to arbitrary fields  $\mathbb{F}$ . Even in this more general case the characterization problem of all such number systems turns out to be completely solvable.

In order to motivate the second aim of our paper we again go back to canonical number systems. Already KNUTH, who studied the special instance  $P(x) = x^2 + 2x + 2$  observed that canonical number systems have interesting geometrical properties. Indeed, considering representations involving negative powers of the base and defining so-called “fundamental domains” yields connections to fractals and the theory of tilings (see for instance [2, 10, 17]).

In the present paper we would like to carry over representations with respect to negative powers as well as the definition of fundamental domains to our new notion of number system. The fundamental domain turns out to be a union of “boxes”. If  $\mathbb{F} = \mathbb{F}_q$  is a finite field, this union becomes finite which makes the fundamental domain easy to describe.

The paper is organized as follows. In Section 2 we give the formal definition of digit systems in residue class rings  $R$  of the polynomial ring  $\mathbb{F}[x, y]$  over a field  $\mathbb{F}$ . Moreover, we give a simple proof of a theorem by SCHEICHER and THUSWALDNER [18] which characterized all polynomials that admit finite representations. Moreover, we enlarge the space of interest to the residue class ring  $\mathbb{F}((x^{-1}, y^{-1}))/f\mathbb{F}((x^{-1}, y^{-1}))$ . We are able to prove that each element of this ring admits a unique “digital” representation. Section 3 shows that our digit systems are symmetric in the variables  $x$  and  $y$  and gives a way to switch between the “ $x$ -” and “ $y$ -digit representation”. The switching algorithm gives insights in the arithmetic structure of the digit systems. In Section 4 we investigate which elements allow periodic digit representations. It turns out that this question can be answered with help of the total ring of fractions of  $R$ . In this section we have to confine ourselves to the case where  $\mathbb{F}$  is a finite field. Finally, in Section 5 we construct “fundamental domains” for our digit systems. Interestingly, it turns out that these sets are bounded, closed and open and can be written as unions of cylinders. If  $\mathbb{F}$  is a finite field these unions are finite which makes the set totally bounded and therefore compact. The paper ends with an example.

## 2. DIGIT SYSTEMS

Let  $\mathbb{F}$  be a field,  $G_x := \mathbb{F}[x][y]$  and  $f \in G_x$ . The main objects of the present paper are digit representations for the elements of the residue class ring  $R_x := G_x/fG_x$ . Set

$$\mathcal{N}_x = \{d \in \mathbb{F}[y] : \deg_y d < \deg_y f\}.$$

We say that  $g \in G_x$  is an  $x$ -digit representation of  $r \in R_x \setminus \{0\}$  iff the following two conditions hold:

- $g$  maps to  $r$  under the canonical projection map
- $g = \sum_{i=0}^{\ell} d_i x^i$  for some  $\ell \in \mathbb{Z}$  and  $d_i \in \mathcal{N}_x$  with  $d_{\ell} \neq 0$

The polynomials  $d_i$  are called the  $x$ -digits of  $r$ . If each element of  $R_x \setminus \{0\}$  has a unique  $x$ -digit representation we say that  $(x, \mathcal{N}_x)$  is a *digit system* in  $R_x$  with *base*  $x$  and *set of digits*  $\mathcal{N}_x$ .

**Remark 2.1.** *Note that if  $\mathbb{F}$  is a finite field,  $R_x$  is an  $(\mathcal{N}_x, x)$ -ring in the sense of ALLOUCHE et al. [3]. If we choose  $\mathbb{F}$  to be a finite field the associated digit systems coincide with the notion of digit systems studied in [18].*

It is easy to characterize  $x$ -digit systems.

**Theorem 2.2** (Representations I). *Let  $f \in G_x$ . The pair  $(x, \mathcal{N}_x)$  is a digit system in  $R_x$  iff  $f$  is monic in  $y$ .*

*Proof.* Assume first that  $f$  is monic in  $y$ . Let  $r \in R_x$  and pick any representative  $g' \in G_x$ . Since  $f$  is monic in  $y$  we can find  $g \in G_x$  s.t.  $g = af + g'$  with  $a \in R_x$  and  $\deg_y g < \deg_y f$  using division with remainder. Writing  $g$  as a polynomial in  $x$  with coefficients in  $y$  we see that it is an  $x$ -digit representation of  $r$ .

The kernel of the quotient map is obviously generated by  $f$ . Assume  $g'' \in G_x$  is another  $x$ -digit representation. Then  $0 \neq g'' - g$  maps to zero in  $R_x$  and also has degree in  $y$  smaller than  $\deg_y f$ . Therefore it cannot be divisible by  $f$ , contradiction.

Assume now that  $(x, \mathcal{N}_x)$  is a digit system and  $\sum_{i=0}^{\ell} d_i x^i$  the  $x$ -digit representation of  $y^{\deg_y f}$ . Then  $y^{\deg_y f} - \sum_{i=0}^{\ell} d_i x^i$  is monic in  $y$  and divisible by  $f$ , therefore  $f$  is monic in  $y$  as well.  $\square$

**Remark 2.3.** Obviously, expanding an element  $r \in R_x$  in base  $x$  with  $y$ -digits is the same as reducing the exponents of  $y$ .

**Remark 2.4.** Note that if  $\mathbb{F}$  is a finite field, we obtain a generalization of [18, Theorem 2.5].

If we further define

$$G_y := \mathbb{F}[y][x], G := \mathbb{F}[x, y]$$

then all three rings  $G$ ,  $G_x$  and  $G_y$  are trivially  $\mathbb{F}$ -isomorphic by sending  $x \mapsto x$  and  $y \mapsto y$ . Accordingly we get isomorphisms of the residue class rings

$$R := G/fG, \quad R_x, \quad R_y := G_y/fG_y.$$

**Remark 2.5.** Assume that we are given a polynomial  $f \in G$  s.t. both  $R_x$  admits an  $x$ -digit system and  $R_y$  admits a  $y$ -digit system. In view of Theorem 2.2 this is equivalent to the fact that  $f$  is monic w.r.t. both  $x$  and  $y$ . If we think of the  $x$ -digit (resp.  $y$ -digit) representation as being the canonical representation for elements of  $R_x$  (resp.  $R_y$ ) then we may consider the isomorphism  $R_x \rightarrow R_y$  as switching between  $x$ - and  $y$ -digit representations. For this reason, later on we will confine ourselves to these choices of  $f$ .

So far we have carried over the notion of canonical number systems to polynomial rings over a field. However, we also aim at an analogy for representations including negative powers of the base. With these representations we finally wish to define ‘‘fundamental domains’’ for our number systems. Thus for our further constructions we make use of the complete fields  $\mathbb{F}((x^{-1}))$  and  $\mathbb{F}((y^{-1}))$ . First define the rings

$$H := \mathbb{F}((x^{-1}, y^{-1})), \quad H_x := \mathbb{F}((x^{-1}))[y], \quad H_y := \mathbb{F}((y^{-1}))[x]$$

and their quotients modulo  $f$

$$S := H/fH, \quad S_x := H_x/fH_x, \quad S_y := H_y/fH_y.$$

Note that  $S_x$  is a vector space over the complete field  $\mathbb{F}((x^{-1}))$  of dimension  $\deg_y f$ .

In what follows we will rarely mention symmetric statements obtained by interchanging the roles of  $x$  and  $y$ .

By  $\pi_G$ ,  $\pi_{G_x}$ ,  $\pi_H$  and  $\pi_{H_x}$  we denote the respective quotient maps of the rings defined above. The elements of  $G_x$ ,  $H_x$  and  $H$  are naturally represented as formal sums  $h = \sum_{(i,j) \in \mathbb{Z}^2} h_{i,j} x^i y^j$  with certain support restrictions. The support of  $h$ , denoted by  $\text{supp}(h)$ , is the set of lattice points  $(i, j) \in \mathbb{Z}^2$  s.t.  $h_{i,j} \neq 0$ . For a binary relation  $*$  on  $\mathbb{Z}$  we use the shorthand notation  $\mathbb{Z}_{*a} := \{i \in \mathbb{Z} \mid i * a\}$ . We have the following assertions.

- $h \in H$  iff there are  $a, b \in \mathbb{Z}$  s.t.  $\text{supp}(h) \subseteq \mathbb{Z}_{<a} \times \mathbb{Z}_{<b}$
- $h \in H_x$  iff there are  $a, b \in \mathbb{Z}$  s.t.  $\text{supp}(h) \subseteq \mathbb{Z}_{\leq a} \times (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{<b})$
- $h \in G_x$  iff  $\text{supp}(h) \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$  is finite

Setting

$$m := \deg_x f \quad \text{and} \quad n := \deg_y f$$

we will assume from now on that  $f$  is of the following shape

$$f = \sum_{0 \leq i \leq m} b_i x^i \quad \text{with} \quad b_i \in \mathbb{F}[y], b_m \neq 0$$

and we define two standard areas in the exponent lattice

$$A_x := \mathbb{Z} \times (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{<n}) \subset \mathbb{Z}^2$$

and

$$A_y := (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{<m}) \times \mathbb{Z} \subset \mathbb{Z}^2.$$

We are now able to extend the notion of digit system to the larger ring  $S_x$ .

**Definition 2.6** (Digit Representation). *We say that  $h \in H_x$  is an  $x$ -digit representation of  $s \in S_x$  iff the following two conditions hold:*

- $\pi_{H_x}(h) = s$
- $h = \sum_{i=-\infty}^{\ell} d_i x^i$  for some  $\ell \in \mathbb{Z}$  and  $d_i \in \mathcal{N}_x$

The second condition is equivalent to  $\deg_y h < n$  or  $\text{supp}(h) \subset A_x$ . For short we write

$$s = (d_\ell \dots d_0.d_{-1}d_{-2}\dots)_x.$$

If there exist  $p$  and  $q$  such that  $d_i = d_{i-p}$  for all  $i < -q$  we say that  $s$  admits an eventually periodic  $x$ -digit representation. This will be written as

$$s = (d_\ell \dots d_0.d_{-1}\dots d_{-q}\overline{d_{-q-1}\dots d_{-q-p}})_x$$

Moreover, we say that  $s$  admits a purely periodic  $x$ -digit representation if

$$s = (\overline{d_{-1}\dots d_{-p}})_x.$$

If each element of  $S_x \setminus \{0\}$  has a unique  $x$ -digit representation we say that  $(x, \mathcal{N}_x)$  is a digit system in  $S_x$  with base  $x$  and set of digits  $\mathcal{N}_x$ .

By copying the proof of Theorem 2.2, we immediately get the following. We may however drop the constraint of being monic since we are now working in a polynomial ring over a field.

**Theorem 2.7** (Representations II). *Let  $f \in G$ . The pair  $(x, \mathcal{N}_x)$  is a digit system in  $S_x$ .*

**Remark 2.8.** *Note that by our definition the pair  $(x, \mathcal{N}_x)$  forms a digit system in  $S_x$  for arbitrary  $f \in G$ . However, it forms a digit system in  $R_x$  only if  $f$  is monic in  $y$ . Indeed, if  $f$  is not monic in  $y$  then the elements of  $R_x$  (regarded as a subset of  $S_x$ ) generally only admit  $x$ -digit representations in  $H_x$  but not in  $G_x$ .*

### 3. THE TRANSFORMATION BETWEEN $x$ - AND $y$ -DIGIT REPRESENTATIONS

In what follows we want to study the relation between  $x$ - and  $y$ -digit representations. Since we built our theory starting from  $(x, \mathcal{N}_x)$  digit systems in  $R_x$  in view of Remarks 2.5 and 2.8 we assume that  $f$  is monic in  $x$  and  $y$ . Under this assumption we will prove that the isomorphism

$$(3.1) \quad \varphi_{xy} : R_x \rightarrow R_y; \quad x \mapsto x, \quad y \mapsto y$$

can be extended to an isomorphism between  $S_x$  and  $S_y$ . Recall that by Theorem 2.7 each element of  $S_x$  admits a unique  $x$ -digit representation. In the present section we want to describe an explicit transformation procedure that turns an  $x$ -digit representation of an element of  $S_x$  into a  $y$ -digit representation of an element of  $S_y$  (see Theorem 3.8).

In what follows we will decompose a formal series  $h = \sum_{(i,j)} h_{i,j} x^i y^j$  into its  $y$ -fractional part  $\{h\}_y := \sum_{j < 0} (\sum_i h_{i,j} x^i) y^j$  and its  $y$ -integer part  $[h]_y := \sum_{j \geq 0} (\sum_i h_{i,j} x^i) y^j$ .

The idea for establishing the transformation procedure is to show that elements of  $S$  – the ring not favoring one of its variables – are representable uniquely w.r.t. both standard bases, i.e., have a unique  $x$ - and a unique  $y$ -digit representation. This transformation process is done by applying so-called *atomic steps* which will be treated in the two following lemmas.

First we define the atomic step of the first kind. It cuts off the negative powers of  $y$  for a single coefficient  $h_k$  in a representation  $\sum_i h_i(y) x^i$ .

**Lemma 3.1** (Atomic Step of The First Kind). *Let  $h \in H$  be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \quad h_i \in \mathbb{F}((y^{-1})).$$

*Let  $k \leq \ell$  be an integer and set  $l_k := \{h_k\}_y$ . Then we say that*

$$h' = \sum_{i=-\infty}^{\ell} h'_i x^i, \quad h'_i \in \mathbb{F}((y^{-1}))$$

*emerges from  $h$  by an atomic step of the first kind at index  $k$  (notation:  $h' = A_k(h)$ ) if*

$$h' = h - \frac{l_k}{b_m} x^{k-m} f.$$

In this case we have  $h'_k \in \mathbb{F}[y]$  and the estimates

$$\begin{aligned} \deg_y h'_k &\leq \deg_y h_k, \\ \deg_y h'_{k-m+i} &\leq \max(\deg_y l_k + n - 1, \deg_y h_{k-m+i}) \quad (0 < i < m), \\ \deg_y h'_{k-m} &\leq \max(\deg_y l_k + n, \deg_y h_{k-m}). \end{aligned}$$

Moreover  $\pi_H(h) = \pi_H(h')$  and  $h'_i = h_i$  for  $i > k$  or  $i < k - m$ .

*Proof.* This follows immediately from the equality

$$h' = h - \frac{l_k}{b_m} \sum_{i=0}^m b_i x^{k-m+i}$$

because  $\deg_y b_i < n$  for  $0 < i < m$ ,  $\deg_y b_0 = n$  and  $\deg_y b_m = 0$ .  $\square$

The following lemma shows how a combination of several atomic steps of the first kind affects a given representation.

**Lemma 3.2** (Succession of Atomic Steps of the First Kind). *Let  $h \in H$  be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \quad h_i \in \mathbb{F}((y^{-1})),$$

and  $k_0 := \min(\ell, \ell - \lceil \frac{-\deg_y h}{n-1} \rceil)$ . Then for  $k \leq k_0$  the element

$$A_k \circ \dots \circ A_\ell(h) = \sum_{i=-\infty}^{k_0} h'_i x^i, \quad h'_i \in \mathbb{F}((y^{-1})),$$

satisfies  $h'_i \in \mathbb{F}[y]$  for all  $k \leq i \leq k_0$ . In particular its  $x$ -degree is bounded by  $k_0$ .

*Proof.* Assume that  $\deg_y h \geq 0$ . Then  $k_0 = \ell$  and the lemma is easily seen by  $\ell - k + 1$  successive applications of Lemma 3.1. Now assume  $\deg_y h < 0$ . For  $k_0 < j \leq \ell$  set

$$h^{(j)} := A_j \circ \dots \circ A_\ell(h) = \sum_{i=-\infty}^{\ell} h_i^{(j)} x^i, \quad h_i^{(j)} \in \mathbb{F}((y^{-1})).$$

We claim that

$$(3.2) \quad \deg_y h_i^{(j)} \leq \begin{cases} -\infty, & i \geq j, \\ \deg_y h + (\ell - j + 1)(n - 1), & j - m < i < j, \\ \deg_y h + (\ell - j + 1)(n - 1) + 1, & i = j - m, \\ \deg_y h, & \text{otherwise.} \end{cases}$$

This assertion is proved by induction. Since  $h^{(j)} = A_\ell(h)$  for  $j = \ell$ , the induction start is an immediate consequence of Lemma 3.1. Also, the induction step follows directly from this lemma and works as long as  $\deg_y h_j^{(j+1)} < 0$ , which holds because  $j > k_0$ .

With the choice  $j = k_0 + 1$  Equation (3.2) first implies that

$$\deg_x h^{(k_0+1)} \leq k_0.$$

An application of  $A_j$  cannot increase the  $x$ -degree of the argument. So by another  $k_0 - k + 1$  successive applications of Lemma 3.1 we have  $\deg_x h^{(k)} \leq k_0$  and  $h_i^{(k)} \in \mathbb{F}[y]$  for  $i \leq k_0$ .  $\square$

The atomic step of the second kind contained in the next lemma cuts off a single coefficient  $h_k$  in a representation  $\sum_i h_i(y)x^i$  in a way that its degree becomes less than  $n$ .

**Lemma 3.3** (Atomic Step of The Second Kind). *Let  $h \in H$  be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \quad h_i \in \mathbb{F}((y^{-1})).$$

Let  $k \leq \ell$  be an integer and set  $l_k := y^n \lfloor y^{-n} h_k \rfloor_y$ . Then we say that

$$h' = \sum_{i=-\infty}^{\max(\ell, k+m)} h'_i x^i, \quad h'_i \in \mathbb{F}((y^{-1}))$$

emerges from  $h$  by an atomic step of the second kind at index  $k$  (notation:  $h' = B_k(h)$ ) if

$$h' = h - \frac{l_k}{b_0} x^k f.$$

In this case we have the estimates

$$\begin{aligned} \deg_y h'_k &< n, \\ \deg_y h'_{k+i} &\leq \max(\deg_y h_{k+i}, \deg_y h_k - 1) \quad (0 < i < m), \\ \deg_y h'_{k+m} &\leq \max(\deg_y h_{k+m}, \deg_y h_k - n). \end{aligned}$$

Moreover  $\pi_H(h) = \pi_H(h')$  and  $h'_i = h_i$  for  $i < k$  or  $i > k + m$ .

*Proof.* This follows immediately from the equality

$$h' = h - \frac{l_k}{b_0} \sum_{i=0}^m b_i x^{k+i}$$

because  $\deg_y b_i < n$  for  $0 < i < m$ ,  $\deg_y b_0 = n$  and  $\deg_y b_m = 0$ .  $\square$

Again we need to dwell upon the effect of a combination of atomic steps of the second kind to a given representation. This is done in the following lemma.

**Lemma 3.4** (Succession of Atomic Steps of the Second Kind). *Let  $h \in H$  be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \quad h_i \in \mathbb{F}((y^{-1})),$$

$k \leq \ell$  and  $\ell_0 := \max(\ell, \ell + (m-1)(\deg_y h - n))$ . Then for  $\ell' \geq \ell_0$  the element

$$B_{\ell'} \circ \cdots \circ B_k(h) = \sum_{i=-\infty}^{\ell_0+m} h'_i x^i$$

satisfies  $\deg_y h'_i < n$  for all  $i \geq k$ . In particular its  $x$ -degree is bounded by  $\ell_0 + m$ .

*Proof.* Assume that  $\deg_y h < n$ . Then  $\ell_0 = \ell$  and by the definition of  $B_*$  in Lemma 3.1 the operator  $B_{\ell'} \circ \cdots \circ B_k$  doesn't change  $h$ . Now assume  $\deg_y h \geq n$ . For  $0 \leq j \leq \deg_y h - n$  set

$$h^{(j)} := B_{\ell+(m-1)j} \circ \cdots \circ B_k(h) := \sum_{i=-\infty}^{\ell+(m-1)j+m} h_i^{(j)} x^i.$$

We claim that

$$(3.3) \quad \deg_y(h_i^{(j)}) < \begin{cases} n & \text{for } k \leq i \leq \ell + (m-1)j, \\ \deg_y h - j & \ell + (m-1)j < i < \ell + (m-1)j + m, \\ \deg_y h - j - (n-1) & i = \ell + (m-1)j + m. \end{cases}$$

This is shown by induction on  $j$ . For  $j = 0$  this is easily seen by  $\ell - k + 1$  successive applications of Lemma 3.3. Each induction step follows by another  $m - 1$  applications. After  $\deg_y h - n$  steps we arrive at the element

$$h^{(\deg_y h - n)} = B_{\ell_0} \circ \cdots \circ B_k(h)$$

which, in view of (3.3), has the desired properties. Again the operator  $B_{\ell'} \circ \cdots \circ B_{\ell_0+1}$  doesn't change  $h^{(\deg_y h - n)}$ .  $\square$

**Corollary 3.5** (A Cauchy Property). *Let  $h \in H$  be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \quad h_i \in \mathbb{F}((y^{-1})),$$

$k \leq \ell$ ,  $\ell'_1 := \max(k+m, k+m+(m-1)(\deg_y h - n))$  and  $\ell'_2 := \max(\ell'_1, \ell, \ell+(m-1)(\deg_y h - n))$ . Then

$$h' := B_{\ell'_2} \circ \dots \circ B_k(h) = \sum_{i=-\infty}^{\ell'_2+m} h'_i x^i \quad \text{and} \quad h'' := B_{\ell'_2} \circ \dots \circ B_{k+1}(h) = \sum_{i=-\infty}^{\ell'_2+m} h''_i x^i$$

satisfy  $\max(\deg_y h'_i, \deg_y h''_i) < n$  for  $i > k$  and  $h'_i = h''_i$  for  $i > \ell'_1 + m$ .

*Proof.* The claim about the maximum of the  $y$ -degrees and the bounds on the  $x$ -degrees of  $h'$  and  $h''$  follow from direct applications of Lemma 3.4 with  $\ell' = \ell'_2$ , so it remains to show the last claim or equivalently  $\deg_x(h'' - h') \leq \ell'_1 + m$ . Note that by the definition of  $l_k$  in Lemma 3.3 it is clear that the operators  $B_*$  are  $\mathbb{F}$ -linear. Hence setting  $g := B_k(h)$  we may write

$$h'' - h' = B_{\ell'_2} \circ \dots \circ B_{k+1}(h - g).$$

By the definition of  $B_k$  we have

$$h - g = \sum_{i=k}^{k+m} g_i x^i$$

for certain  $g_i$  with  $\deg_y g_i \leq \deg_y h$ . Another application of Lemma 3.4 with  $\ell' = \ell'_1$  shows that

$$B_{\ell'_1} \circ \dots \circ B_{k+1}(h - g)$$

has  $x$ -degree bounded by  $\ell'_1 + m$  and  $y$ -degree less than  $n$ . Finally

$$B_{\ell'_2} \circ \dots \circ B_{k+1}(h - g) = B_{\ell'_1} \circ \dots \circ B_{k+1}(h - g)$$

because applying  $B_{\ell'_2} \circ \dots \circ B_{\ell'_1+1}$  to an element of  $y$ -degree smaller than  $n$  is the identity and the result follows.  $\square$

We can define a norm  $|h| := e^{\deg h}$  for  $h \in H$  (where  $\deg$  denotes the total degree). Then  $H$  becomes a topological ring, i.e., addition and multiplication are continuous w.r.t. the induced metric. Multiplication is even an open mapping since the degree of a product is equal to the sum of the degrees of the factors. Note that  $H$  is not complete, for example, the sequence  $(\sum_{0 \leq i \leq j} x^i y^{-2i})_j$  is Cauchy but has no limit in  $H$ . Nevertheless it is easy to see, that if  $(f_j)_j$  is a sequence in  $H$  which is Cauchy and s.t.  $\text{supp}(f_j) \subseteq \mathbb{Z}_{\leq a} \times \mathbb{Z}_{\leq b}$  for certain fixed  $a, b \in \mathbb{Z}$  and  $j$  sufficiently high, then  $f := \lim_{j \rightarrow \infty} f_j$  exists.

The following two lemmas form the basis of the switching process between  $x$ - and  $y$ -digit representations.

**Lemma 3.6** (Uniformization Lemma I). *Let  $h \in H$ . Then there is also  $h' \in H$  and  $\ell \in \mathbb{Z}$  with  $\pi_H(h') = \pi_H(h)$  and  $\text{supp}(h') \subseteq \mathbb{Z}_{\leq \ell} \times \mathbb{Z}_{< n}$ .*

*Proof.* Let  $\ell'$  be defined as in Lemma 3.4 and set

$$h^{(k)} = B_{\ell'} \circ \dots \circ B_k(h)$$

for  $k$  sufficiently small. Then the sequence  $(h^{(k)})_{-k}$  is Cauchy by Corollary 3.5 and  $\text{supp}(h^{(k)}) \subseteq \mathbb{Z}_{\leq \ell'+m} \times \mathbb{Z}_{\leq \deg_y(h)}$ . Hence  $h' := \lim_{k \rightarrow -\infty} h^{(k)}$  exists. Moreover it is easy to see that in fact  $\text{supp}(h') \subseteq \mathbb{Z}_{\leq \ell'+m} \times \mathbb{Z}_{< n}$ . Setting  $\ell := \ell' + m$  it remains to show that  $\pi_H(h') = \pi_H(h)$ , in other words  $h' - h \in fH$ .

By construction we have  $h^{(k)} - h = e^{(k)} f$  for certain  $e^{(k)} \in H$ . Since  $(h^{(k)} - h)_{-k}$  is a Cauchy sequence and multiplication by  $f$  is an open mapping, we conclude that also  $(e^{(k)})_{-k}$  is Cauchy and the support of its elements is suitably bounded. Hence

$$h' - h = \lim_{k \rightarrow -\infty} (h^{(k)} - h) = \lim_{k \rightarrow -\infty} e^{(k)} f = \left( \lim_{k \rightarrow -\infty} e^{(k)} \right) f. \quad \square$$

**Lemma 3.7** (Uniformization Lemma II). *Let  $h \in H$  and assume  $\text{supp}(h) \subseteq \mathbb{Z}_{\leq \ell} \times \mathbb{Z}_{< n}$  for some  $\ell \in \mathbb{Z}$ . Then there is also  $h' \in H$  with  $\pi_H(h') = \pi_H(h)$  and  $\text{supp}(h') \subseteq \mathbb{Z}_{\leq \ell} \times (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{< n}) \subset A_x$ .*

*Proof.* Set

$$h^{(k)} = A_k \circ \cdots \circ A_\ell(h) = \sum_{i=-\infty}^{\ell} h_i^{(k)} x^i, \quad h_i^{(k)} \in \mathbb{F}((y^{-1})),$$

for  $k$  sufficiently small. Then  $h_i^{(k)} = h_i^{(k')}$  for  $i \geq \max(k, k')$  from the definition of the operators  $A_*$  in Lemma 3.1. It follows immediately that  $(h^{(k)})_{-k}$  is Cauchy. Also  $\text{supp}(h^{(k)}) \subseteq \mathbb{Z}_{\leq \ell} \times \mathbb{Z}_{< n}$  and hence  $h' := \lim_{k \rightarrow -\infty} h^{(k)}$  exists. Moreover it is easy to see that in fact  $\text{supp}(h') \subseteq \mathbb{Z}_{\leq \ell} \times (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{< n})$ . The fact that  $\pi_H(h') = \pi_H(h)$  is shown exactly as in the proof of Lemma 3.6.  $\square$

The following theorem forms the main result of the present section.

**Theorem 3.8** (Representations III). *For each  $s \in S$  there is a unique  $h \in H$  with  $\text{supp}(h) \subset A_x$  and  $\pi_H(h) = s$ . (In other words each  $s \in S$  admits a unique  $x$ -digit representation.)*

*Proof.* Choose  $h'' \in H$  arbitrary s.t.  $\pi_H(h'') = s$ . Apply Lemma 3.6 to  $h''$  in order to produce an element  $h' \in H$  with  $\text{supp}(h') \subseteq \mathbb{Z}_{\leq \ell} \times \mathbb{Z}_{< n}$ . In a second step apply Lemma 3.7 to  $h'$  to get  $h \in H$  with  $\text{supp}(h) \subset A_x$  and note that still  $\pi_H(h) = s$ . It remains to show that such  $h$  is unique.

If we had two distinct representations, their difference would be an element  $g \in H \setminus \{0\}$  with  $\text{supp}(g) \subset A_x$ . It will be sufficient to prove that  $\pi_H(g) \neq 0$ . Assume on the contrary that  $g = af$  for some  $a \in H$ . Let  $\Pi(g)$  be the Minkowski sum of the convex hull of  $\text{supp}(g)$  and the negative quadrant. Further let  $(i_1, j_1) \in \mathbb{Z}^2$  be the vertex on the horizontal face of  $\Pi(g)$  and  $(i_2, j_2) \in \mathbb{Z}^2$  be the vertex on the vertical face of  $\Pi(g)$  and set  $d(g) := j_1 - j_2 \in \mathbb{Z}_{\geq 0}$ . Define  $\Pi(a)$ ,  $\Pi(f)$ ,  $d(a)$  and  $d(f)$  analogously. Then  $\Pi(g) = \Pi(a) + \Pi(f)$  and  $d(g) = d(a) + d(f)$ . But  $d(f) = n$  because  $f$  is monic in  $x$  and  $d(g) \leq n - 1$  because  $\text{supp}(g) \subset A_x$ , contradiction.  $\square$

**Remark 3.9.** *Note that the argument of the proof makes essential use of the fact that  $f$  is monic in  $x$ .*

In what we did so far we also constructed an explicit isomorphism between the sets  $S_x$ ,  $S$  and  $S_y$ . This is emphasized in the following corollary.

**Corollary 3.10** (The Isomorphism). *We have  $S \cong S_x \cong S_y$ . In fact, the isomorphism  $\varphi_{xy}$ , see (3.1), extends to  $S_x \rightarrow S_y$ .*

*Proof.* We show  $S_x \cong S$ . The natural inclusion  $H_x \hookrightarrow H$  induces a homomorphism  $\psi : S_x \rightarrow S$ . Let  $s \in S_x \setminus \{0\}$ . Then  $s$  has a unique representative  $h \in H_x \setminus \{0\}$  with  $\text{supp}(h) \subset A_x$  by Theorem 2.7. But then  $\psi(s)$  is represented by the same  $h$  regarded as element of  $H$ , and by Theorem 3.8 we find  $\psi(s) \neq 0$ , hence  $\psi$  is injective.

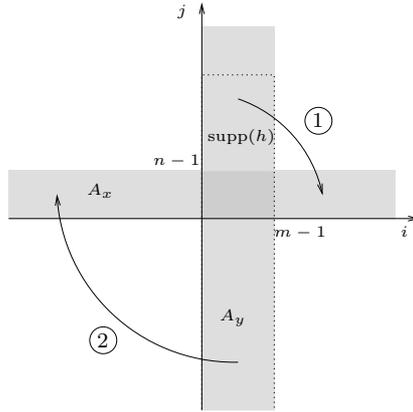
On the other hand, let  $s \in S$  and  $h \in H$  be a representative with  $\text{supp}(h) \subset A_x$ . Then obviously  $h$  is in the image of  $H_x$ , hence  $\psi$  is also surjective. By the same reasoning we have  $S \cong S_y$  and by composition we get an isomorphism  $\psi_{xy} : S_x \rightarrow S_y$ .

Now let  $r \in R_x$ , i.e.,  $r = \pi_{G_x}(h)$  for some  $h \in G_x$  with  $\text{supp}(h) \subset A_x$ . Also there is  $h' \in G_y$  with  $\text{supp}(h') \subset A_y$  and  $\varphi_{xy}(r) = \pi_{G_y}(h')$  by Theorem 2.2. Now consider  $h$  and  $h'$  as elements of  $H$ , then  $h - h' \in fH$  and hence  $\pi_H(h) = \pi_H(h')$ . Now the construction of  $\psi_{xy}$  and the uniqueness statement in Theorem 3.8 imply that  $\psi_{xy}|_{R_x} = \varphi_{xy}$ .  $\square$

We write again  $\varphi_{xy} : S_x \rightarrow S_y$  for the extended isomorphism. For the representation Theorem 3.8 we had to show that we can transform an arbitrary support to fit into the region  $A_x$ . For depicting the isomorphism, say  $\varphi_{yx}$ , we have to transform a support in  $A_x$  into a support in  $A_y$ . This is illustrated in Figure 1.

#### 4. THE TOTAL RING OF FRACTIONS AND PERIODIC DIGIT EXPANSIONS

Our next objective is to investigate under what condition the  $x$ -digit representation of an element of  $S_x$  is periodic. It turns out that periodic representations are related to the *total ring*


 FIGURE 1. The Isomorphism  $\varphi_{yx} : S_y \rightarrow S_x$ 

How to think about that isomorphism? If we represent  $s \in S_y$  by an element  $h$  supported on  $A_y$ , we can depict the isomorphism as follows: One application of Lemma 3.6 moves the upper part of  $h$  into  $A_x$ . A subsequent application of Lemma 3.7 moves the lower part into  $A_x$ . The only part which is affected by the transformation of both lemmas is the overlapping region  $A_y \cap A_x$ , which has been shaded in dark grey.

of fractions  $\mathcal{Q}(R)$  of  $R$ . Recall that  $\mathcal{Q}(R) := \mathcal{S}^{-1}R$  where  $\mathcal{S} \subset R$  denotes the multiplicative set of non-zero divisors. Let  $F := \mathcal{Q}(R)$  and set

$$F_x := \mathbb{F}(x)[y]/f\mathbb{F}(x)[y] \quad \text{and} \quad F_y := \mathbb{F}(y)[x]/f\mathbb{F}(y)[x].$$

**Lemma 4.1** (Total Rings of Fractions). *We have  $F \cong F_x \cong F_y$ . In fact, the isomorphism  $\varphi_{xy}$ , see (3.1), extends uniquely to  $F_x \rightarrow F_y$ .*

*Proof.* Being monic in  $y$ , the polynomial  $f$  is in particular primitive as a polynomial with coefficients in  $\mathbb{F}[x]$ . Hence the set  $\mathcal{T} := \pi_G(\mathbb{F}[x] \setminus \{0\})$  consists of non-zero divisors and we have embeddings  $R \hookrightarrow \mathcal{T}^{-1}R \cong F_x \hookrightarrow F$ . This implies that  $\mathcal{Q}(F_x) = F$ .

Now let  $h \in F_x$  be a non-zero divisor. In other words it is represented by a polynomial  $h' \in \mathbb{F}(x)[y]$  s.t.  $h'$  and  $f$  are coprime in the principal ideal domain  $\mathbb{F}(x)[y]$ . Then the image of  $h'$  is invertible in  $F_x$ , i.e.,  $h^{-1} \in F_x$ . Hence  $F_x$  is its own total ring of fractions and the last embedding above is already an isomorphism.

By composing  $F_x \rightarrow F$  and the inverse of  $F_y \rightarrow F$  we find an isomorphism  $\psi_{xy} : F_x \rightarrow F_y$  clearly extending  $\varphi_{xy}$ . Let  $a/b \in F_x$ , then  $b(a/b) = a$  and for any such isomorphism we have  $\varphi_{xy}(b)\psi_{xy}(a/b) = \varphi_{xy}(a)$ . Since  $\varphi_{xy}(b)$  is a non-zero divisor this already determines  $\psi_{xy}(a/b)$  uniquely.  $\square$

Note that it is easy to compute the isomorphic images effectively, using the Extended Euclidean Algorithm. We write again  $\varphi_{xy} : F_x \rightarrow F_y$  for the extended isomorphism. This is justified; Indeed, we have the following commuting diagram

$$\begin{array}{ccccc}
 & & \varphi_{xy} & & \\
 S_x & \xrightarrow{\quad} & S & \xleftarrow{\quad} & S_y \\
 \uparrow & & \uparrow & & \uparrow \\
 F_x & \xrightarrow{\quad} & F & \xleftarrow{\quad} & F_y \\
 \uparrow & & \uparrow & & \uparrow \\
 R_x & \xrightarrow{\quad} & R & \xleftarrow{\quad} & R_y \\
 & & \varphi_{xy} & & 
 \end{array}$$

where all the maps (except the  $\varphi_{xy}$ ) are induced by the respective maps of representatives. We have to argue, why the middle row fits into this diagram. The reason is that the vertical arrows are inclusions, the top and bottom horizontal arrows are all isomorphisms. The objects in the

middle are the total rings of fractions that obviously have to be mapped isomorphically onto each other.

In the remainder of this section we assume that  $\mathbb{F} = \mathbb{F}_q$  is a finite field. In view of the above diagram we will from now on consider digit expansions of elements in the rings  $R$ ,  $F$  and  $S$ .

**Lemma 4.2** (Rational Laurent Series). *Let  $z \in \mathbb{F}_q((x^{-1}))$ . Then  $z$  is eventually periodic iff  $z \in \mathbb{F}_q(x)$ .*

*So let  $z \in \mathbb{F}_q(x)$  be arbitrary. Write  $z = a + x^{-\kappa}(b + c/d)$  with  $\kappa \in \mathbb{Z}_{\geq 0}$ ,  $a, b, c, d \in \mathbb{F}_q[x]$ ,  $\deg_x(c) < \deg_x(d)$ ,  $\deg_x(b) < \kappa$ ,  $\gcd(c, d) = 1$  and  $x \nmid d$ . Let  $\mu$  be minimal s.t.  $x^\mu \equiv 1 \pmod{d}$ . Then we have an expansion*

$$z = (a_\rho \dots a_0 \cdot b_{\kappa-1} \dots \overline{b_0 p_{\mu-1} \dots p_0})$$

*with  $\rho = \deg_x(a)$  and certain  $a_i, b_i, p_i \in \mathbb{F}_q$ . Further  $\mu$  is the minimal length of the period.*

**Remark 4.3.** *Representing elements  $z \in \mathbb{F}_q(x)$  as above is always possible. Choosing a  $\mu$  is only possible because  $\mathbb{F}_q$  is finite and, hence,  $x$  maps to an element of the finite group of units modulo  $d$ .*

*Proof.* First assume  $z' = c/d$  with the same conditions. Then  $x^\mu = 1 + ed$  and  $x^\mu c/d = c/d + p$  with  $p := ec$ . The first summand is purely non-integral, whereas the second summand is integral. Moreover  $\deg_x(p) = \deg_x(x^\mu c/d) < \mu$ . So we have an expansion  $p = (p_{\mu-1} \dots p_0)$ . By repeating the argument we see that  $z'$  is purely periodic,  $z' = (\overline{p_{\mu-1} \dots p_0})$ .

On the other hand assume that  $z'$  is purely periodic of this form. Then an easy calculation shows

$$(4.1) \quad z' = \frac{\sum_{i=0}^{\mu-1} p_i x^i}{x^\mu - 1},$$

so, reducing this fraction, we get a representation  $z' = c/d$  as above.

Now any element  $z \in \mathbb{F}_q(x)$  can be written as in the claim and any eventually periodic  $z \in \mathbb{F}_q((x^{-1}))$  can be written as  $z = a + x^{-\kappa}(b + z')$  with  $a, b \in \mathbb{F}_q[x]$  and  $z' \in \mathbb{F}_q((x^{-1}))$  purely periodic.

It remains to prove minimality of  $\mu$ . Assume that  $\mu' \leq \mu$  is the length of the minimal period. Then Equation (4.1) with  $\mu$  replaced by  $\mu'$  implies that  $d \mid x^{\mu'} - 1$ , in other words  $x^{\mu'} \equiv 1 \pmod{d}$  and hence also  $\mu \leq \mu'$ .  $\square$

**Corollary 4.4** (Periodicity). *Let  $\mathbb{F} = \mathbb{F}_q$  be a finite field and let  $s \in S$ . The following assertions are equivalent:*

- $s \in F$
- $s$  has an eventually periodic  $x$ -digit representation
- $s$  has an eventually periodic  $y$ -digit representation

*Proof.* By symmetry, it suffices to show equivalence of the first two statements. By Lemma 4.1,  $s \in F$  iff  $s$  can be represented by an element  $h = \sum_{i=0}^{n-1} h_i y^i \in \mathbb{F}_q(x)[y] \subset H_x$ . By Lemma 4.2 this is the case iff all the coefficients have eventually periodic expansions in  $\mathbb{F}_q((x^{-1}))$ , hence, iff  $h$  is eventually periodic when written as a sum of  $x$ -digits.  $\square$

The difficulty of making statements about the periodic expansions now depends heavily on the representation of an element  $s \in F$ . The easy case is when  $s$  is represented by an element of  $\mathbb{F}_q(x)[y]$  and we want to study the  $x$ -digit representation.

**Theorem 4.5** (Shape of the Period). *Let  $\mathbb{F} = \mathbb{F}_q$  be a finite field and let  $s \in F$  be represented by*

$$h = \sum_{i=0}^{n-1} h_i y^i$$

*with  $h_i = a_i + x^{-\kappa_i}(b_i + c_i/d_i) \in \mathbb{F}_q(x)$  s.t.  $\kappa_i \in \mathbb{Z}_{\geq 0}$ ,  $a_i, b_i, c_i, d_i \in \mathbb{F}_q[x]$ ,  $\deg_x(c_i) < \deg_x(d_i)$ ,  $\deg_x(b_i) < \kappa_i$ ,  $\gcd(c_i, d_i) = 1$  and  $x \nmid d_i$ . Let  $\mu_i$  be minimal such that  $x^{\mu_i} \equiv 1 \pmod{d_i}$ . Then we have an expansion*

$$s = (a_\rho \dots a_0 \cdot b_{\kappa-1} \dots \overline{b_0 p_{\mu-1} \dots p_0})$$

for certain  $x$ -digits  $a_i, b_i, p_i$ . Here  $\rho = \max_i(\deg_x(a_i))$ ,  $\kappa = \max_i(\kappa_i)$  and  $\mu = \text{lcm}_i(\mu_i)$  is the minimal length of the period.

*Proof.* The  $x$ -digit representation is inferred directly from the set of Laurent series expansions of the  $h_i$ . So this corollary is an immediate consequence of Lemma 4.2.  $\square$

The more difficult case is when  $s$  is represented by an element of  $\mathbb{F}_q(y)[x]$  in the same situation. In this case, we first use the Extended Euclidean Algorithm to convert the representation and then apply Theorem 4.5.

**Example 4.6.** Let  $f := x^3 + 3xy + x + 2y + y^2 \in \mathbb{F}_5[x, y]$  and consider the element

$$s := \frac{y^5 + xy + 2y + 4x}{y^3 + 2y^2x + y^2 + y} \in \mathcal{Q}(\mathbb{F}_5[x, y]/f\mathbb{F}_5[x, y]).$$

We want to compute, say, its  $y$ -digit expansion. Using the Extended Euclidean Algorithm, we first compute a nicer representation by

$$\begin{aligned} h &= h_0 + h_1x + h_2x^2 \\ &= \left( \frac{y^8 + 4y^7 + 2y^6 + 2y^5 + 4y^4 + 4y^3 + 2y^2 + 4y + 2}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1} \right) + \\ &\quad \left( \frac{2y^4 + 2y^3 + 3y^8 + 3y^7 + 3y^6 + y^5 + 4y + 4}{y^7 + 2y^6 + y^5 + 3y^4 + 3y^2 + y} \right) x + \\ &\quad \left( \frac{4y^6 + 3y^3 + 3y^2 + 2}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1} \right) x^2. \end{aligned}$$

Decomposing coefficients as in Theorem 4.5 yields:

$$\begin{aligned} h_0 &= (y^2 + 2y + 2) + y^0 \left( (0) + \frac{3y^5 + y^4 + y}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1} \right), \\ h_1 &= (3y + 2) + y^{-1} \left( (1) + \frac{3y^5 + y^2 + 4y + 3}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1} \right), \\ h_2 &= (4) + y^0 \left( (0) + \frac{2y^5 + y^4 + y^3 + 3y^2 + 3y + 3}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1} \right) \end{aligned}$$

One computes that  $y$  has order 208 modulo  $y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1$ , indeed

$$y^{208} - 1 \equiv (4 + 3y + \dots + 3y^{201} + y^{202})(y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1) \pmod{5}.$$

In this example  $\rho = \max_i(\deg_x(a_i)) = 2$ ,  $\kappa = \max_i(\kappa_i) = 1$  and  $\mu = \text{lcm}_i(\mu_i) = 208$ . And indeed, if we develop the coefficients separately and combine them to  $y$ -digits, we get the expansion

$$s = ((1)(2 + 3x)(2x + 2 + 4x^2) \cdot \underbrace{(x + 3 + 2x^2)(2x^2 + 3x)(4x + 2) \dots (2x^2)(2x + 3 + 2x^2)}_{208 \text{ digits}}).$$

## 5. THE FUNDAMENTAL DOMAINS ASSOCIATED TO A DIGIT SYSTEM

Now we want to give a more detailed study of the isomorphism  $\varphi_{xy} : S_x \rightarrow S_y$ . Instead of considering the map  $\varphi_{xy}$ , we compare the  $x$ - and  $y$ -digit representations of elements of  $s$ .

**Definition 5.1** (Height of Elements). Let  $s \in S \setminus \{0\}$  and  $h \in H$  be the  $x$ -digit representation of  $s$ . Then we define  $\text{hgt}_x(s) := \deg_x(h)$ .

In other words, if  $s$  is represented by  $h = \sum_{j=-\infty}^{\ell} h_j x^j$  where the  $h_j$  are  $x$ -digits and  $h_\ell \neq 0$ , then  $\text{hgt}_x(s) = \ell$ . By Theorem 3.8, we can convert from one representation into the other, so we may ask how to bound one height in terms of the other.

**Lemma 5.2** (Mutual Bounds on Height). Let  $s \in S \setminus \{0\}$ . Then we have the following implications:

- (1)  $\text{hgt}_y(s) < 0 \quad \Rightarrow \quad \text{hgt}_x(s) \leq (m - 1) - \left\lceil \frac{-\text{hgt}_y(s)}{n-1} \right\rceil$
- (2)  $0 \leq \text{hgt}_y(s) \leq n - 1 \quad \Rightarrow \quad \text{hgt}_x(s) \leq m - 1$
- (3)  $n - 1 < \text{hgt}_y(s) \quad \Rightarrow \quad \text{hgt}_x(s) \leq (\text{hgt}_y(s) - n + 2)(m - 1) + 1$

*Proof.* Let

$$h = \sum_{i=0}^{m-1} h_i x^i, \quad h_i \in \mathbb{F}((y^{-1}))$$

be the  $y$ -digit representation of an element  $s \in S$ .

- (1) Applying Lemma 3.2 with  $\ell = m - 1$  and  $\deg_y h = \text{hgt}_y s$  yields (with  $h^{(k)}$  defined as in this lemma)

$$\deg_x h^{(k)} \leq m - 1 - \left\lfloor \frac{-\text{hgt}_y s}{n - 1} \right\rfloor.$$

for  $k$  sufficiently small. Setting  $h' = \lim_{k \rightarrow \infty} h^{(k)}$  this implies that

$$\deg_x h' \leq m - 1 - \left\lfloor \frac{-\text{hgt}_y s}{n - 1} \right\rfloor.$$

However, since  $h'$  is the  $x$ -digit representation of  $s$  this implies that  $\deg_x(h') = \text{hgt}_x(s)$  and we are done.

- (2) This follows directly from Lemma 3.7.  
 (3) If  $\deg_y h$  satisfies the bounds in (3) we need to apply Lemma 3.6 first. Since  $\deg_x(h) \leq m - 1$  by Lemma 3.4 (setting  $\ell = m - 1$  and  $\deg_y h = \text{hgt}_y s$ ) this yields  $h'$  with  $\pi_H(h) = \pi_H(h')$  and

$$\deg_x(h') \leq (m - 1)(\text{hgt}_y s - n + 2) + 1.$$

By Lemma 3.7 we obtain an  $x$ -representation  $h''$  of  $s$  satisfying the same bounds on  $\deg_x$  as  $h'$ . This yields (3).  $\square$

The following example illustrates the fact that the bounds are actually *sharp*.

**Example 5.3.** Let  $f := x^3 - x^2y^3 + y^4 \in \mathbb{F}_5[x, y]$ ,  $a := x^2y^{-10}$ ,  $b := x^2y^3$  and  $c := x^2y^{10}$ . Here  $m = 3$ ,  $n = 4$ ,  $\text{hgt}_y(a) = -10$ ,  $\text{hgt}_y(b) = 3$  and  $\text{hgt}_y(c) = 10$ . In these cases one checks:

$$a \equiv 4y^2x^{-7} + 3yx^{-5} + 4y^3x^{-4} + 2x^{-3} + y^2x^{-2} \pmod{f}$$

$$b \equiv y^3x^2 \pmod{f}$$

$$c \equiv y^2x^8 + 2yx^{10} + y^3x^{11} + 3x^{12} + 4y^2x^{13} + 4yx^{15} + y^3x^{16} + 4x^{17} \pmod{f}$$

Hence  $\text{hgt}_x(a) = -2$ ,  $\text{hgt}_x(b) = 2$  and  $\text{hgt}_x(c) = 17$ . These are exactly the bounds of the above lemma.

We may consider  $S_x$  (and also  $S$ ) an  $\mathbb{F}((x^{-1}))$ -vector space or  $S_y$  (and also  $S$ ) an  $\mathbb{F}((y^{-1}))$ -vector space. In both cases we are dealing with normed topological vector spaces; we just set  $|s|_x := q^{\text{hgt}_x s}$  and  $|s|_y := q^{\text{hgt}_y s}$  (with  $q$  being the size of the finite field or  $e$  in case of an infinite ground field). The two vector spaces are obviously complete w.r.t. to the respective metric.

**Definition 5.4** (Fundamental Domains). *We define*

$$\mathcal{F}_x := \{s \in S \mid \text{hgt}_x(s) < 0\} = \{s \in S \mid |s|_x < 1\}$$

and call it the fundamental domain w.r.t.  $x$ .

In other words  $\mathcal{F}_x \subset S$  is the set of those elements that have a purely fractional expansion in  $x$ -digits. We are interested in the structure of  $\mathcal{F}_x$  when written in terms of  $y$ -digits. A similar question is how the fundamental domains  $\mathcal{F}_x$  and  $\mathcal{F}_y$  are related or how the two different norms on  $S$  compare. This yields information about the arithmetic properties of the isomorphism  $\phi_{xy}$ . From the above lemma, we get immediately the following corollary.

**Corollary 5.5** (Continuity of  $\varphi_{xy}$ ). *Let  $s \in S$ . We have the following:*

- (1) *The isomorphism  $\varphi_{xy}$  is a continuous map.*
- (2) *If  $\text{hgt}_y(s) < -(n - 1)(m - 1)$  then  $s \in \mathcal{F}_x$ .*
- (3) *If  $s \in \mathcal{F}_x$  then  $\text{hgt}_y(s) \leq n - 1$ .*

*Proof.* All straight-forward consequences of the above lemma:

- (1) We may restrict our attention to neighborhoods of 0 because  $\varphi_{xy}$  can be viewed as a homomorphism between topological groups. Lemma 5.2 now becomes a very explicit statement of the  $\epsilon\delta$  definition of continuity at 0.
- (2) In particular we have  $\text{hgt}_y(s) < 0$ , so we are in the first case of Lemma 5.2. We get the following chain of inequalities:

$$\begin{aligned}
 & -\text{hgt}_y(s) > (n-1)(m-1) \\
 \Rightarrow & (n-1)\lceil \frac{-\text{hgt}_y(s)}{n-1} \rceil > (n-1)(m-1) \\
 \Rightarrow & \lceil \frac{-\text{hgt}_y(s)}{n-1} \rceil > m-1 \\
 \Rightarrow & (m-1) - \lceil \frac{-\text{hgt}_y(s)}{n-1} \rceil < 0
 \end{aligned}$$

Hence by Lemma 5.2 we have  $\text{hgt}_x(s) < 0$  and thus  $s \in \mathcal{F}_x$ .

- (3) This follows directly from the symmetric statement of the last case of Lemma 5.2.  $\square$

**Corollary 5.6** (Mutual Composition of Fundamental Domains). *Let  $\rho := (n-1)(m-1)$  and*

$$V := \{s \in \mathcal{F}_x \mid s = \pi_H(h) \text{ for some } h \in H \text{ with } \text{supp}(h) \subseteq \{0, \dots, m-1\} \times \{-\rho, \dots, n-1\}\}.$$

*Then  $V$  is an  $\mathbb{F}$ -vector space and  $\mathcal{F}_x = \bigcup_{s \in V} (s + y^{-\rho}\mathcal{F}_y)$ . In particular  $\mathcal{F}_x$  is a clopen, bounded subset of the topological  $\mathbb{F}((y^{-1}))$ -vector space  $S$ . If  $\mathbb{F} = \mathbb{F}_q$  is a finite field then  $V$  is finite and  $\mathcal{F}_x$  is even compact.*

*Proof.* By Number (3) of Corollary 5.5 we can write any element  $s \in \mathcal{F}_x$  as  $s = \pi_H(h')$  for some  $h' \in H$  with  $\text{supp}(h') \subset A_y$  and  $\text{deg}_y h' < n$ . Write now  $h' = h + h''$  with  $\text{supp}(h) \subseteq \{0, \dots, m-1\} \times \{-\rho, \dots, n-1\}$  and  $\text{deg}_y h'' < -\rho$ . Then by Number (2) of the same Corollary we always have  $\pi_H(h'') \in \mathcal{F}_x$ . Therefore  $\pi_H(h') \in \mathcal{F}_x$  if and only if  $\pi_H(h) \in \mathcal{F}_x$ . But elements of the form  $\pi_H(h'')$  are exactly the elements in the set  $y^{-\rho}\mathcal{F}_y$ , showing the first part of the claim.

Set

$$W = \{s \in S \mid s = \pi_H(h) \text{ for some } h \in H \text{ with } \text{supp}(h) \subseteq \{0, \dots, m-1\} \times \{-\rho, \dots, \infty\}\}.$$

Now  $S$  is the disjoint union of the open sets of the form  $s + y^{-\rho}\mathcal{F}_y$  where  $s \in W$ .  $V$  is a subset of  $W$ , hence  $\mathcal{F}_x$  and  $S \setminus \mathcal{F}_x$  can both be written as infinite unions of open sets. Therefore  $\mathcal{F}_x$  is open, closed and bounded (again by Number (3) of Corollary 5.5).

If  $\mathbb{F} = \mathbb{F}_q$  is finite, it is easily seen that  $\mathcal{F}_x$  is *totally* bounded and therefore compact.  $\square$

**Remark 5.7.** *The interest of this and the following corollaries lies in the fact that we view  $\mathcal{F}_x$  (which corresponds to the open unit ball of  $S$  as an  $\mathbb{F}((x^{-1}))$ -vector space) as a subset of  $S$  considered as an  $\mathbb{F}((y^{-1}))$ -vector space. The statement about the compactness becomes wrong in case of an infinite ground field  $\mathbb{F}$ . Then  $\mathcal{F}_x$  is not even compact in  $S$  as an  $\mathbb{F}((x^{-1}))$ -vector space. For example let  $a_i$  for  $i \in \mathbb{N}$  be an infinite sequence of pairwise distinct  $x$ -digits. Then  $b_i := (\cdot a_i)$  is a sequence in  $\mathcal{F}_x$  without accumulation point.*

We will now sketch an algorithm for computing  $\mathcal{F}_x$  in terms of  $\mathcal{F}_y$ . Indeed, using the Corollary, we only have to determine  $V$ .

First, for each  $(i, j) \in \{0, \dots, m-1\} \times \{-\rho, \dots, n-1\}$  we compute an element  $h_{i,j} \in H$  with  $\text{supp}(h_{i,j}) \subseteq \{0, \dots, m-1\} \times \{0, \dots, n-1\}$  as follows: We can compute the  $x$ -digit representation of  $\pi_H(x^i y^j)$ , in other words, using the Extended Euclidean Algorithm, we compute an element  $g_{i,j} \in \mathbb{F}(x)[y]$  with  $\text{deg}_y g_{i,j} < n$  s.t.  $\pi_H(x^i y^j) = \pi_H(g_{i,j})$ . Set now  $h_{i,j} := \lceil g_{i,j} \rceil_x$ .

Now if  $h := \sum_{(i,j)} a_{i,j} x^i y^j \in H$  with indices  $(i, j)$  running over the set  $\{0, \dots, m-1\} \times \{-\rho, \dots, n-1\}$  is a generic element then  $\pi_H(h) \in V$  if and only if  $\sum_{(i,j)} a_{i,j} h_{i,j} = 0$ . Extracting the coefficients of the monomials  $x^i y^j$  for  $(i, j) \in \{0, \dots, m-1\} \times \{0, \dots, n-1\}$  gives a set of linear conditions that determine  $V$ .

One can view a fundamental domain as a solution of an iterated function system in case  $\mathbb{F} = \mathbb{F}_q$  is a finite field.

**Corollary 5.8** (Fundamental Domains as Self-Affine Sets). *Let  $\mathbb{F} = \mathbb{F}_q$  be a finite field. Then the fundamental domain  $\mathcal{F}_x$  is the unique non-empty compact subset of  $S$  (seen as  $\mathbb{F}((y^{-1}))$ -vector space) satisfying the set equation*

$$\mathcal{F}_x = \bigcup_{d \in \mathcal{N}_x} x^{-1}(\mathcal{F}_x + d).$$

*Proof.* This is an easy consequence of the general theory of self-affine sets (see for instance HUTCHINSON [9]).  $\square$

**Remark 5.9.** *Note that if  $\mathbb{F}$  is infinite,  $\mathcal{F}_x$  cannot be seen as a solution of an infinite iterated function system in  $S$  (in the sense of FERNAU [5], for instance), because the contraction ratios of the mappings  $\chi_d(z) := x^{-1}(z + d)$ , where  $d \in \mathcal{N}_x$ , do not converge to zero. We could regard it only as a non-compact solution of*

$$T = \bigcup_{d \in \mathcal{N}_x} x^{-1}(T + d).$$

*However, this does not help very much, as this set equation has more than one non-empty non-compact solution.*

**Corollary 5.10** (Tilings Induced by Fundamental Domains). *The collection  $\{\mathcal{F}_x + r \mid r \in R\}$  forms a (non-trivial) tiling of the space  $S$  (seen as  $\mathbb{F}((y^{-1}))$ -vector space) in the sense that*

$$\bigcup_{r \in R} (\mathcal{F}_x + r) = S$$

and

$$(\mathcal{F}_x + r_1) \cap (\mathcal{F}_x + r_2) = \emptyset$$

for all  $r_1, r_2 \in R$  distinct.

*Proof.* Any  $s \in S$  has a unique  $x$ -digit representation  $h$  by Theorem 3.8. Setting  $s_0 := \pi_H(\{h\}_x)$  and  $s_1 := \pi_H([h]_x)$  we find that it decomposes uniquely as  $s = s_0 + s_1$  with  $s_0 \in \mathcal{F}_x$  and  $s_1 \in R \subset S$ , hence  $S = \mathcal{F}_x \oplus R$  is a direct sum of  $\mathbb{F}$ -vector spaces. The claim of the Corollary is just another formulation of the same fact.  $\square$

Let now again  $\mathbb{F} = \mathbb{F}_q$  be a finite field and  $\nu : \mathbb{F}_q \rightarrow \{0, \dots, q-1\}$  an enumeration of the finite field. We define the following map to the real numbers:

$$\mu : \mathbb{F}_q((y^{-1})) \rightarrow \mathbb{R} : \sum_{i=-\infty}^l c_i y^i \mapsto \sum_{i=-\infty}^l \nu(c_i) q^i$$

Note that the right sum converges since it can be bounded by a geometric sum.

Assume now  $n = 2$ . Then we may consider elements  $s \in S$  as vectors of the plane  $\mathbb{F}_q((y^{-1})) \times \mathbb{F}_q((y^{-1}))$  and visualize them via the map  $\mu \times \mu$  in  $\mathbb{R}^2$ . If we use our algorithm, sketched above, to express  $\mathcal{F}_x$  in terms of  $\mathcal{F}_y$  we see immediately that  $\mathcal{F}_x$  corresponds to a finite union of closed boxes of side length  $q^{-\rho}$  where  $\rho$  is defined as in Corollary 5.6. Since elements of  $R$  have integral coordinates in  $\mathbb{R}^2$  under the map  $\mu \times \mu$  it is also clear that the overall volume of the image of the fundamental domain  $\mathcal{F}_x$  is 1. This gives an intuitive picture of the above tiling.

**Example 5.11.** *We illustrate this with an example. Let  $f := x^3 + 3xy + x + 2y + y^2 \in \mathbb{F}_5[x, y]$ , hence  $q = 5$ . We want to express, say,  $\mathcal{F}_y$  in terms of  $\mathcal{F}_x$ . Therefore we compute the vector space of Corollary 5.6 where  $\rho = (3-1)(2-1) = 2$  and get*

$$V = \langle 1 + 2x^{-1}y, x^{-2}y, x^{-1}, x^{-2} \rangle_{\mathbb{F}_5} \subset \mathbb{F}_5((x^{-1}, y^{-1}))/f\mathbb{F}_5((x^{-1}, y^{-1}))$$

which reads in vector notation as

$$V = \langle (1, 2x^{-1}), (0, x^{-2}), (x^{-1}, 0), (x^{-2}, 0) \rangle_{\mathbb{F}_5}.$$

Since  $\mathbb{F}_5$  is finite also  $V$  is finite and one computes exactly 625 different  $\mathbb{F}_5$ -linear combinations of the basis elements in  $V$ . Applying the map  $\mu \times \mu$  to each of these elements one gets the lower left vertex of a box in  $\mathbb{R}^2$  of side length  $5^{-\rho} = 0.04$ . The union of these boxes visualizes the fundamental domain and has volume 1, see Figure 2.

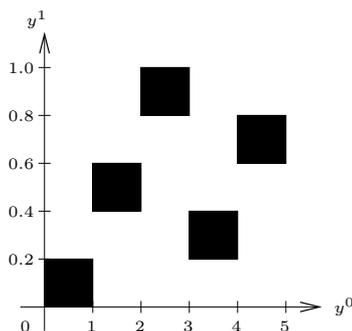


FIGURE 2. Fundamental Domain

This shows the fundamental domain  $\mathcal{F}_y$  in  $S$  (when viewed as a vector space over  $\mathbb{F}_5((x^{-1}))$ ) induced by the polynomial  $x^3 + 3xy + x + 2y + y^2 \in \mathbb{F}_5[x, y]$ . For illustration purposes elements of  $\mathbb{F}_5((x^{-1}))$  were sent to  $\mathbb{R}$  by mapping  $x$  to 5 and the field elements to their canonic representatives in  $\{0, \dots, 4\}$ .

## REFERENCES

1. S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *Generalized radix representations and dynamical systems. I*, Acta Math. Hungar. **108** (2005), no. 3, 207–238.
2. S. Akiyama and J. M. Thuswaldner, *The topological structure of fractal tilings generated by quadratic number systems*, Comput. Math. Appl. **49** (2005), no. 9-10, 1439–1485.
3. J.-P. Allouche, E. Cateland, H.-O. Peitgen, W. J. Gilbert, J. O. Shallit, and G. Skordev, *Automatic maps in exotic numeration systems*, Theory Comput. Syst. (Math. Systems Theory) **30** (1997), 285–331.
4. G. Barat, V. Berthé, P. Liardet, and J. Thuswaldner, *Dynamical directions in numeration*, Annal. Inst. Fourier **56** (2006), 1987–2092.
5. H. Fernau, *Infinite iterated function systems*, Math. Nachr. **170** (1994), 79–91.
6. W. J. Gilbert, *Radix representations of quadratic fields*, J. Math. Anal. Appl. **83** (1981), 264–274.
7. E. H. Grossman, *Number bases in quadratic fields.*, Stud. Sci. Math. Hung. **20** (1985), 55–58.
8. M. Hbaib and M. Mkaour, *Sur le bêta-développement de 1 dans le corps des séries formelles*, Int. J. Number Theory **2** (2006), no. 3, 365–378.
9. J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
10. I. Kátai and I. Környei, *On number systems in algebraic number fields*, Publ. Math. Debrecen **41** (1992), no. 3-4, 289–294.
11. I. Kátai and B. Kovács, *Kanonische Zahlensysteme in der Theorie der Quadratischen Zahlen*, Acta Sci. Math. (Szeged) **42** (1980), 99–107.
12. ———, *Canonical number systems in imaginary quadratic fields*, Acta Math. Hungar. **37** (1981), 159–164.
13. I. Kátai and J. Szabó, *Canonical number systems for complex integers*, Acta Sci. Math. (Szeged) **37** (1975), 255–260.
14. D. E. Knuth, *The art of computer programming, vol 2: Seminumerical algorithms*, 3rd ed., Addison Wesley, London, 1998.
15. A. Pethő, *On a polynomial transformation and its application to the construction of a public key cryptosystem*, Computational number theory (Debrecen, 1989), de Gruyter, Berlin, 1991, pp. 31–43.
16. K. Scheicher,  *$\beta$ -expansions in algebraic function fields over finite fields*, Finite Fields Appl. **13** (2007), 394–410.
17. K. Scheicher and J. M. Thuswaldner, *Canonical number systems, counting automata and fractals*, Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 1, 163–182.
18. ———, *Digit systems in polynomial rings over finite fields*, Finite Fields Appl. **9** (2003), no. 3, 322–333.

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