

A Practical Optimal Control Approach to the Stationary MHD System in Velocity–Current Formulation

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A PRACTICAL OPTIMAL CONTROL APPROACH TO THE STATIONARY MHD SYSTEM IN VELOCITY–CURRENT FORMULATION

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ABSTRACT. An optimal control problem for the equations governing the stationary problem of magnetohydrodynamics (MHD) is considered. Control mechanisms by external and injected currents and magnetic fields are treated. Motivated by practical applications, an optimal control problem is formulated. First order necessary and second order sufficient conditions are developed. An operator splitting scheme for the numerical solution of the MHD state equations is proposed and analyzed.

1. INTRODUCTION

Magnetohydrodynamics, or MHD, deals with the mutual interaction of electrically conducting fluids and magnetic fields. The nature of the coupling between fluid motion and the electromagnetic quantities arises from the following three phenomena:

- (1) The relative movements of a conducting fluid and a magnetic field induce an electromotive force (Faraday's law) to the effect that an electric current develops in the fluid.
- (2) This current in turn induces a magnetic field (Ampères law).
- (3) The magnetic field interacts with the current in the fluid and exerts a Lorentz force in the fluid.

It is the third feature in the nature of MHD which renders it so phenomenally attractive for exploitation especially in metallurgical processes. The Lorentz force offers a unique possibility of generating a volume force in the fluid and hence to control its motion in a contactless fashion and without any mechanical interference. Therefore it comes as no surprise that MHD technology is used routinely today by engineers, for instance in order to stir molten metals during solidification, to dampen their undesired convection-driven flow during casting, to filter out impurities, and to melt and even levitate metals.

It seems that in the majority of these industrial applications of MHD, the magnetic fields used to influence the fluid flow in a desired way are designed by engineering expertise. With the present paper, we wish to contribute to the application of the powerful methods from mathematical optimization to compute tailored magnetic fields for MHD flow control. Although this work intends primarily to lay the mathematical foundations of MHD optimal control, we believe to have chosen a problem setup of practical relevance allowing our results to be directly exploited in numerical methods and numerous applications.

Before we turn to the problem description, let us put our work into perspective. Throughout the paper, we always refer to stationary incompressible magnetohydrodynamics involving viscous fluids. Instationary problems will require an investigation in their own right and compressible MHD mostly occurs in the realm of plasma physics whereas we focus on the engineering aspects of MHD. While a remarkable amount of attention in the past decade was devoted to the analysis of optimal control of the Navier-Stokes equations, see, *e.g.*, [9–11, 14], we are aware of only two contributions so far concerning the optimal control of the MHD system, [13, 17], both of which treat problems quite different from ours. The MHD state equations alone have been investigated in a number of papers including [12, 19, 20].

We organized the material in the following way: In the remainder of this section, we briefly recall the stationary MHD state equations for the reader's convenience, and convert it to the velocity–current formulation. In Section 2, we introduce the variational form of the state equation following [20]. Our main results are given in Section 3, where we propose and analyze an optimal control problem for the MHD system. In particular, we derive and discuss its first order necessary optimality system (Theorem 3.11) and establish second order sufficient conditions (Proposition 3.12). We also present a new proof concerning the existence of solutions to the MHD state equations (Proposition 3.3). In Section 4, we propose an operator splitting scheme for the solution of the MHD state equation which makes use of existing solvers for the Navier-Stokes equations and div–curl systems. We conclude with an outlook on follow-up work in Section 5.

Essentially, the MHD system consists of the Navier-Stokes equation with Lorentz force, yielding the fluid velocity \mathbf{u} and its pressure p , plus Maxwell's equations describing the interaction of the electric field \mathbf{E} and the magnetic field \mathbf{B} .¹ In the stationary case, the complete MHD system is given by

$$\begin{aligned} \nabla \cdot \mathbf{J} &= 0 & \nabla \times \mathbf{E} &= 0 & (1.1) \\ \text{(Charge conservation)} & & \text{(Faraday's law)} & & \end{aligned}$$

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 & \nabla \times (\mu^{-1}\mathbf{B}) &= \mathbf{J} & (1.2) \\ \text{(No magnetic monopoles)} & & \text{(Ampère's Law)} & & \end{aligned}$$

$$\begin{aligned} \mathbf{J} &= \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) & (1.3) \\ \text{(Ohm's law)} & & \end{aligned}$$

together with the Navier-Stokes system with Lorentz force

$$\varrho(\mathbf{u} \cdot \nabla)\mathbf{u} - \eta\Delta\mathbf{u} + \nabla p = \mathbf{J} \times \mathbf{B} \quad (1.4)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (1.5)$$

We refer to [4, 22] for more details. Here and throughout, μ denotes the magnetic permeability of the matter occupying a certain point in space, and ϱ , η and σ denote the fluid's density, viscosity and conductivity. All of these numbers are positive. We emphasize that we consider μ a constant throughout space, hence we assume a non-magnetic fluid and no magnetic material present in its relevant vicinity.

It is an outstanding feature in magnetohydrodynamics that from the set of state variables $(\mathbf{u}, p, \mathbf{E}, \mathbf{B}, \mathbf{J})$, the electric and magnetic fields \mathbf{E} and \mathbf{B} extend to all of \mathbb{R}^3 , whereas the velocity \mathbf{u} and pressure p are confined to the bounded region

¹Strictly speaking, \mathbf{B} should be called the magnetic induction, while $\mathbf{H} = \mu^{-1}\mathbf{B}$ is the magnetic field. It is however common usage in MHD literature to call \mathbf{B} the magnetic field.

$\Omega \subset \mathbb{R}^3$ occupied by the fluid. The current density \mathbf{J} is defined within the fluid region and possibly also in external conductors.

Rather than treating the full set of variables $(\mathbf{u}, p, \mathbf{E}, \mathbf{B}, \mathbf{J})$, MHD systems are described by a properly chosen subset, which is frequently taken as the pair of primal variables (\mathbf{u}, \mathbf{B}) . This entails that either \mathbf{B} has to be considered on all of \mathbb{R}^3 , or that artificial shielding boundary conditions have to be assigned on $\partial\Omega$ so that the coupled system can be considered on the fluid region Ω alone. Physically, shielding boundary conditions represent a fluid being surrounded on all sides by a perfectly conducting vessel. Such boundary conditions exclude the control action by means of distant magnetic fields. This is an especially attractive feature of MHD control. In what appear to be the practically more relevant cases where the outside of the fluid region Ω is finitely conducting or non-conducting, hence permitting control by distant magnetic fields, the proper boundary condition for \mathbf{B} is an interface condition requiring \mathbf{B} to be continuous across $\partial\Omega$ in both its normal and tangential components, *i.e.*,

$$[\mathbf{B}]_{\partial\Omega} = 0 \quad (1.6)$$

where $[\cdot]_{\partial\Omega}$ denotes the jump of any quantity when going from the interior of Ω to its exterior. As a consequence, \mathbf{B} has to be considered on all of \mathbb{R}^3 .

These shortcomings of the (\mathbf{u}, \mathbf{B}) formulation are not present in the velocity–current formulation in the variables (\mathbf{u}, \mathbf{J}) of the state equation system (1.1)–(1.5) as introduced in [20]. In this formulation, the magnetic field \mathbf{B} is eliminated by means of a solution operator $\mathcal{B}(\mathbf{J})$ which uniquely solves the div–curl system (1.2) for divergence-free currents \mathbf{J} and respects the interface condition (1.6). Moreover, the irrotational electric field \mathbf{E} is replaced by its potential ϕ (unique only up to a constant). In our case of constant permeability μ , the operator $\mathcal{B}(\mathbf{J})$ is known as the Biot-Savart law,

$$\mathcal{B}(\mathbf{J})(\mathbf{x}) = -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \times \mathbf{J}(\mathbf{y}) \, d\mathbf{y}. \quad (1.7)$$

Some properties of \mathcal{B} are summarized in Lemma 2.4 below. Inserting $\mathbf{B} = \mathcal{B}(\mathbf{J})$ into (1.1)–(1.5), we are left with the velocity–current formulation of the stationary MHD system,

$$\begin{aligned} \varrho(\mathbf{u} \cdot \nabla)\mathbf{u} - \eta\Delta\mathbf{u} + \nabla p - \mathbf{J} \times \mathcal{B}(\mathbf{J}) &= 0 & \nabla \cdot \mathbf{u} &= 0 & (1.8) \\ \sigma^{-1}\mathbf{J} + \nabla\phi - \mathbf{u} \times \mathcal{B}(\mathbf{J}) &= 0 & \nabla \cdot \mathbf{J} &= 0 & (1.9) \end{aligned}$$

for the unknowns $(\mathbf{u}, p, \mathbf{J}, \phi)$. Here \mathbf{u} and p and the electric potential ϕ are confined to the region Ω occupied by our conducting fluid, while \mathbf{J} may additionally extend to external conductors.

In general, the total magnetic field \mathbf{B} is a superposition of the induced magnetic field $\mathcal{B}(\mathbf{J})$ and other magnetic fields, such as fields belonging to permanent magnets or fields generated by *given* electric currents, see (3.6). We note in passing that in the case of weakly conducting fluids, as for example salt water, or more generally, in the case of low magnetic Reynolds numbers $R_m = \mu\sigma ul$ (where u and l are typical velocity and length scales), the magnetic field associated with the induced current is neglected in comparison with an imposed field [4]. Hence, $\mathcal{B}(\mathbf{J})$ can be replaced by a *given* field \mathbf{B}_0 , and system (1.8)–(1.9) decouples into a Navier-Stokes system with Lorentz force term and the div–curl system

$$\nabla \cdot \mathbf{J} = 0 \quad \nabla \times \mathbf{J} = \sigma \nabla \times (\mathbf{u} \times \mathbf{B}_0).$$

We refer to [16] for control approaches concerning the instationary von Kármán flow for a weakly conducting fluid.

2. FUNCTION SPACES AND OPERATORS

In this section, we present the proper functional analytic setting for the stationary MHD problem following [20]. In order to obtain its variational formulation, we multiply (1.8)–(1.9) by smooth test functions $(\mathbf{v}, q, \mathbf{K}, \psi)$ where \mathbf{v} has zero Dirichlet boundary values. Integration by parts yields

$$\begin{aligned} & \varrho \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} + \eta \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} (\nabla \cdot \mathbf{v}) \mathcal{P}(p) - \int_{\Omega} (\mathbf{J} \times \mathcal{B}(\mathbf{J})) \cdot \mathbf{v} - \int_{\Omega} (\nabla \cdot \mathbf{u}) q \\ & + \sigma^{-1} \int_{\Omega} \mathbf{J} \cdot \mathbf{K} + \int_{\Omega} \mathbf{K} \cdot (\nabla \phi) - \int_{\Omega} (\mathbf{u} \times \mathcal{B}(\mathbf{J})) \cdot \mathbf{K} + \int_{\Omega} \mathbf{J} \cdot \nabla \psi = \int_{\partial\Omega} j \psi \quad (2.1) \end{aligned}$$

where $j = \mathbf{J} \cdot \mathbf{n}$ denotes the given boundary values in normal direction for the current \mathbf{J} . In (2.1),

$$\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i=1}^3 \int_{\Omega} \nabla u_i \cdot \nabla v_i$$

and $\mathcal{P}(p)$ denotes the projection of p on the functions with zero mean,

$$\mathcal{P}(p) = p - \frac{1}{|\Omega|} \int_{\Omega} p.$$

Based on (2.1), we introduce the bilinear forms

$$\begin{aligned} a_1(\mathbf{u}, \mathbf{v}) &= \eta \int_{\Omega} (\nabla \mathbf{u} : \nabla \mathbf{v}) & a_2(\mathbf{J}, \mathbf{K}) &= \sigma^{-1} \int_{\Omega} \mathbf{J} \cdot \mathbf{K} \\ d_1(\mathbf{u}, p) &= - \int_{\Omega} (\nabla \cdot \mathbf{u}) \mathcal{P}(p) & d_2(\mathbf{J}, \phi) &= \int_{\Omega} \mathbf{J} \cdot (\nabla \phi) \end{aligned}$$

and trilinear forms

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \varrho \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \quad c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.$$

Throughout, let Ω denote a bounded domain in \mathbb{R}^3 with Lipschitzian boundary and let $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ and in general $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the usual Sobolev spaces [1], for $1 < p < \infty$. In addition, for $l = 1, 2$, let $V^l(\mathbb{R}^3)$ stand for the completion of $H^l(\mathbb{R}^3)$ with respect to the seminorm which measures only the l -th order derivatives [19]. Furthermore, $H^{1/2}(\partial\Omega)$ is the trace space of $H^1(\Omega)$, endowed with the the infimum norm

$$\|\phi\|_{H^{1/2}(\partial\Omega)} = \inf \|\Phi\|_{H^1(\Omega)},$$

where the infimum extends over all Φ whose trace coincides with ϕ . The norm duals of $H_0^1(\Omega)$ and $H^{1/2}(\partial\Omega)$ are $H^{-1}(\Omega)$ and $H^{-1/2}(\partial\Omega)$, respectively. The norm dual of $W_0^{1,p}(\Omega)$ is $W^{-1,p'}(\Omega)$, where p' is the dual of p , *i.e.*, $p' = p/(p-1)$. Boldface notation indicates the triple cartesian product of a space with itself, *e.g.*, $\mathbf{L}^2(\Omega) = [L^2(\Omega)]^3$ and the symbol $\mathbf{L}_{\text{div}}^2(\Omega)$ denotes the subspace of divergence free (solenoidal) functions in $\mathbf{L}^2(\Omega)$. Finally, we denote by A^* the adjoint of a bounded linear operator A .

Let us now turn to the forms introduced above. Besides the obvious continuity properties, they satisfy:

Lemma 2.1 (LBB conditions).

The constraint forms d_1 and d_2 satisfy the following Ladyzhenskaya–Babuska–Brezzi

(LBB) conditions on $\mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$, and on $\mathbf{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$, respectively:

$$\inf_{p \in L^2(\Omega)/\mathbb{R}} \sup_{\mathbf{u} \in \mathbf{H}_0^1(\Omega)} \frac{d_1(\mathbf{u}, p)}{\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|p\|_{L^2(\Omega)/\mathbb{R}}} \geq \beta_1$$

$$\inf_{\phi \in H^1(\Omega)/\mathbb{R}} \sup_{\mathbf{J} \in \mathbf{L}^2(\Omega)} \frac{d_2(\mathbf{J}, \phi)}{\|\mathbf{J}\|_{\mathbf{L}^2(\Omega)} \|\phi\|_{H^1(\Omega)/\mathbb{R}}} \geq \beta_2$$

for some $\beta_1, \beta_2 > 0$.

Let us define the following spaces associated to the constraint forms d_1 and d_2 :

$$V_1 = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : d_1(\mathbf{v}, p) = 0 \text{ for all } p \in L^2(\Omega)/\mathbb{R}\}$$

$$V_1^0 = \{\Phi_1 \in \mathbf{H}^{-1}(\Omega) : \langle \Phi_1, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{v} \in V_1\}$$

$$V_2 = \{\mathbf{K} \in \mathbf{L}^2(\Omega) : d_2(\mathbf{K}, \phi) = 0 \text{ for all } \phi \in H^1(\Omega)/\mathbb{R}\}$$

$$V_2^0 = \{\Phi_2 \in \mathbf{L}^2(\Omega)' : \langle \Phi_2, \mathbf{K} \rangle = 0 \text{ for all } \mathbf{K} \in V_2\}.$$

Note that

$$V_1 = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ on } \Omega\}$$

$$V_2 = \{\mathbf{K} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{K} = 0 \text{ on } \Omega \text{ and } \mathbf{K} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Lemma 2.2 (Properties of constraint forms).

If $\Phi_1 \in V_1^0$ and $\Phi_2 \in V_2^0$, then the equations

$$d_1(\mathbf{v}, p) = \langle \Phi_1, \mathbf{v} \rangle \text{ for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$

$$d_2(\mathbf{K}, \phi) = \langle \Phi_2, \mathbf{K} \rangle \text{ for all } \mathbf{K} \in \mathbf{L}^2(\Omega)$$

are uniquely solvable for $p \in L^2(\Omega)/\mathbb{R}$ and $\phi \in H^1(\Omega)/\mathbb{R}$, and $\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c_1 \|\Phi_1\|_{\mathbf{H}^{-1}(\Omega)}$ and $\|\phi\|_{H^1(\Omega)/\mathbb{R}} \leq c_2 \|\Phi_2\|_{\mathbf{L}^2(\Omega)'}$ hold for some $c_1, c_2 > 0$.

Proof. See [7, Ch. I, Lemma 4.1]. \square

Lemma 2.3 (Passing to the limit in c).

- (1) Let $\mathbf{u}^n \rightharpoonup \mathbf{u}$ in $\mathbf{L}^2(\Omega)$, $\mathbf{v}^n \rightarrow \mathbf{v}$ in $\mathbf{L}^3(\Omega)$ and $\mathbf{w} \in \mathbf{L}^6(\Omega)$. Then $c(\mathbf{u}^n, \mathbf{v}^n, \mathbf{w}) \rightarrow c(\mathbf{u}, \mathbf{v}, \mathbf{w})$.
- (2) Let $\mathbf{u} \in \mathbf{L}^2(\Omega)$, $\mathbf{v}^n \rightarrow \mathbf{v}$ in $\mathbf{L}^3(\Omega)$ and $\mathbf{w}^n \rightharpoonup \mathbf{w}$ in $\mathbf{L}^6(\Omega)$. Then $c(\mathbf{u}, \mathbf{v}^n, \mathbf{w}^n) \rightarrow c(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

Proof. For the first claim, we use the estimate

$$|c(\mathbf{u}^n, \mathbf{v}^n, \mathbf{w}) - c(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \left| \int_{\Omega} (\mathbf{u}^n \times (\mathbf{v}^n - \mathbf{v})) \cdot \mathbf{w} \right| + \left| \int_{\Omega} ((\mathbf{u}^n - \mathbf{u}) \times \mathbf{v}) \cdot \mathbf{w} \right|.$$

We apply Hölder's inequality to the first term, using the native norms of all three factors involved. It converges to zero since $\|\mathbf{v}^n - \mathbf{v}\|_{\mathbf{L}^3(\Omega)}$ converges to zero and $\|\mathbf{u}^n\|_{\mathbf{L}^2(\Omega)}$ is bounded by assumption (1). Hölder's inequality again shows that $\int_{\Omega} (\cdot \times \mathbf{v}) \cdot \mathbf{w}$ is a continuous linear functional on $\mathbf{L}^2(\Omega)$ so that also the second term converges to zero. The second claim follows alike, using the splitting

$$|c(\mathbf{u}, \mathbf{v}^n, \mathbf{w}^n) - c(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \left| \int_{\Omega} (\mathbf{u} \times (\mathbf{v}^n - \mathbf{v})) \cdot \mathbf{w}^n \right| + \left| \int_{\Omega} (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w}^n - \mathbf{w}) \right|.$$

\square

Lemma 2.4 (Properties of \mathcal{B}).

The following properties hold:

- (1) The Biot-Savart operator (1.7) maps any given $\mathbf{J} \in \mathbf{L}^2(\mathbb{R}^3)$ to $\mathcal{B}(\mathbf{J}) \in \mathbf{V}^1(\mathbb{R}^3)$. The restriction of $\mathcal{B}(\mathbf{J})$ to Ω lies in $\mathbf{H}^1(\Omega)$.

- (2) In this sense, the Biot-Savart operator defines a continuous linear map between $\mathbf{L}^2(\mathbb{R}^3)$ and $\mathbf{H}^1(\Omega)$, i.e.,

$$\|\mathcal{B}(\mathbf{J})\|_{\mathbf{H}^1(\Omega)} \leq c_B \|\mathbf{J}\|_{\mathbf{L}^2(\Omega)}$$

for all $\mathbf{J} \in \mathbf{L}^2(\Omega)$ and some $c_B > 0$.

- (3) If $\mathbf{J} \in \mathbf{L}_{\text{div}}^2(\Omega)$, then $\mathcal{B}(\mathbf{J})$ is the unique solution of the div-curl system (1.2).
- (4) The operator \mathcal{B} is self-adjoint in $\mathbf{L}^2(\mathbb{R}^3)$.

Remark 2.5. Whenever \mathbf{J} has compact support, as will be the case in our applications, $\mathcal{B}(\mathbf{J})$ belongs to $\mathbf{H}^1(\mathbb{R}^3)$. However, it is sufficient for our purpose that the restriction of $\mathcal{B}(\mathbf{J})$ to Ω is in $\mathbf{H}^1(\Omega)$, as guaranteed by the lemma.

Proof of Lemma 2.4. Consider the Newton potential

$$(\mathcal{L}_0 \mathbf{J})(\mathbf{x}) = -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{J}(\mathbf{y}) \, d\mathbf{y},$$

which defines an isomorphism from $\mathbf{L}^2(\mathbb{R}^3)$ to $\mathbf{V}^2(\mathbb{R}^3)$ [3, 19]. One argues that $\mathcal{B}(\mathbf{J}) = \nabla \times \mathcal{L}_0(\mathbf{J})$ holds for all $\mathbf{J} \in \mathbf{L}^2(\mathbb{R}^3)$. This implies that $\mathcal{B}(\mathbf{J}) \in \mathbf{V}^1(\mathbb{R}^3)$. Since $\mathbf{V}^1(\mathbb{R}^3)$ embeds into $\mathbf{H}_{\text{loc}}^1(\mathbb{R}^3)$ [3, Ch.], claims (1) and (2) follow. Theorem 2.7 and Remark 2.8(b) in [19] imply that (3) holds. To prove self-adjointness, we multiply $\mathcal{B}(\mathbf{J})$ by a function $\mathbf{C} \in \mathbf{L}^2(\Omega)$ and integrate over \mathbb{R}^3 :

$$\begin{aligned} \int_{\mathbb{R}^3} \mathbf{C} \cdot \mathcal{B}(\mathbf{J}) &= -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \mathbf{C}(\mathbf{x}) \cdot \left(\int_{\mathbb{R}^3} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \times \mathbf{J}(\mathbf{y}) \, d\mathbf{y} \right) \, d\mathbf{x} \\ &= -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbf{C}(\mathbf{x}) \cdot \left(\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \times \mathbf{J}(\mathbf{y}) \right) \, d\mathbf{x} \, d\mathbf{y} \\ &= -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \mathbf{J}(\mathbf{y}) \cdot \left(\int_{\mathbb{R}^3} \mathbf{C}(\mathbf{x}) \times \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{x} \right) \, d\mathbf{y}. \\ &= -\frac{\mu}{4\pi} \int_{\mathbb{R}^3} \mathbf{J}(\mathbf{x}) \cdot \left(\int_{\mathbb{R}^3} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \times \mathbf{C}(\mathbf{y}) \, d\mathbf{y} \right) \, d\mathbf{x}. \end{aligned}$$

Since the left hand side by definition equals $\int_{\mathbb{R}^3} \mathbf{J} \cdot \mathcal{B}^*(\mathbf{C})$, we have found $\mathcal{B}^*(\mathbf{C}) = \mathcal{B}(\mathbf{C})$. \square

We conclude this section by a compact embedding result whose proof can be found in [7]:

Lemma 2.6 (Compact maps and embeddings).

For a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, the embeddings $H^1(\Omega) \hookrightarrow L^{6-\varepsilon}(\Omega)$ and $L^2(\Omega) \hookrightarrow W^{-1,6-\varepsilon}(\Omega)$ are compact for all $\varepsilon > 0$. In addition, the pointwise product map $(u, v) \mapsto uv$ is continuous from $H^1(\Omega) \times H^1(\Omega)$ to $W^{1,3/2}(\Omega)$ and the latter embeds compactly into $L^{3-\varepsilon}(\Omega)$ for all $\varepsilon > 0$.

3. THE OPTIMAL CONTROL PROBLEM

In the course of this section we propose and analyze an optimal control problem for the stationary MHD system. Our goal is to devise a framework of practical relevance which will allow our results to be exploited in industrial applications. Considering the applications mentioned in [4], it appears that the problem setup depicted in Figure 3.1 satisfies our requirements. The geometry of an actual problem may be quite different from the one on display. To begin with, we assume that the electrically conducting fluid, *e.g.*, a liquid metal, is contained in some vessel Ω . On part of the boundary $\partial\Omega$ of the vessel, an external conductor Ω_{inj} is attached.

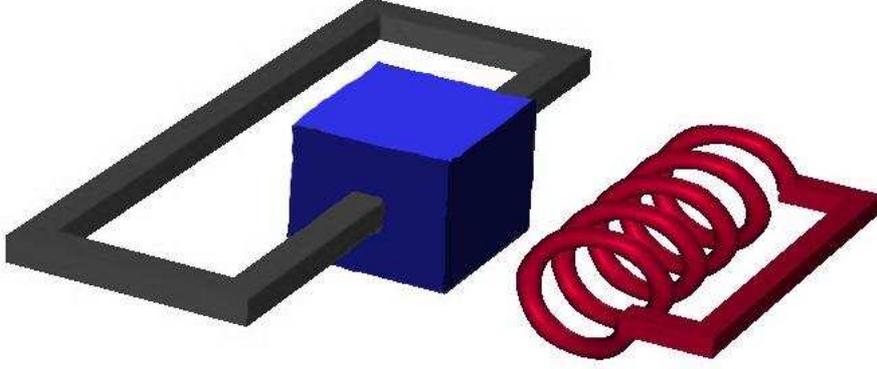


FIGURE 3.1. General Setup: Fluid region Ω (blue cube), external conductor Ω_{inj} attached to the fluid region (grey), and external conductor Ω_{ext} separate from the fluid region (red).

The current distribution in Ω_{inj} is assumed to be known, but its magnitude can be adjusted. The purpose of such a propulsion device is to drive the fluid in Ω in a desired way, both through the action of the magnetic field induced by the current in Ω_{inj} and through the current which is "injected" into the fluid region Ω through the electrodes attached to its surface. The same assembly can in principle be found in electromagnetic filtration devices. In some cases, it may be undesirable to attach the external conductor to the surface of the fluid vessel Ω . Thus we have included a second conductor Ω_{ext} separate from the fluid region in which again the current distribution is given but the magnitude of which can be controlled. This external conductor has an impact on the fluid motion in Ω solely through its induced magnetic field. An assembly where a number of such coils is distributed around the fluid vessel can be found, *e.g.*, in electromagnetic stirring devices [4,21], albeit their magnetic field is usually amplified by yokes in the coil centers which are currently not included in our model in view of the assumption that the permeability μ is constant. In addition, we envision another external magnetic field \mathbf{B}_{ext} which is subject to optimization. Of course, in practice such a field cannot be shaped at will. We consider it as originating from a permanent magnet whose field is known again except for its magnitude which serves as an optimization parameter. With the necessary modifications, also its location and geometric dimensions may be subject to optimization. Finally, the fluid can be driven by injection and suction or a moving boundary. However, in this paper we focus on the control by currents and magnetic fields alone. For analytical and numerical aspects of the Dirichlet boundary control for the Navier-Stokes system we refer to [5, 11, 15].

Let us recall from (1.8)–(1.9) the stationary MHD system in velocity–current formulation:

$$\varrho(\mathbf{u} \cdot \nabla)\mathbf{u} - \eta\Delta\mathbf{u} + \nabla p = \mathbf{J} \times \mathbf{B} \quad \nabla \cdot \mathbf{u} = 0 \quad (3.1)$$

$$\sigma^{-1}\mathbf{J} - \mathbf{u} \times \mathbf{B} + \nabla\phi = 0 \quad \nabla \cdot \mathbf{J} = 0 \quad (3.2)$$

To complete the specification of the state equation, boundary conditions are required for the current density \mathbf{J} and the fluid velocity \mathbf{u} . For the former, we require

$$\mathbf{J} \cdot \mathbf{n} = \mathbf{J}_{\text{inj}} \cdot \mathbf{n} \quad \text{on } \partial\Omega_{\text{inj}} \cap \partial\Omega \quad (3.3)$$

$$\mathbf{J} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \setminus \partial\Omega_{\text{inj}} \quad (3.4)$$

with the injected current \mathbf{J}_{inj} specified below. For the fluid velocity, we impose Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{h} \quad \text{on } \partial\Omega. \quad (3.5)$$

From the description of our setup above it follows that the total magnetic field \mathbf{B} is a superposition of the field $\mathcal{B}(\mathbf{J})$ induced by the current \mathbf{J} inside the fluid domain, the fields $\mathcal{B}(\mathbf{J}_{\text{ext}})$ and $\mathcal{B}(\mathbf{J}_{\text{inj}})$ induced by the currents in the external conductors (whether or not attached to the fluid domain), and the magnetic field \mathbf{B}_{ext} associated with the permanent magnet, *i.e.*,

$$\mathbf{B} = \mathcal{B}(\mathbf{J}) + \mathcal{B}(\mathbf{J}_{\text{ext}}) + \mathcal{B}(\mathbf{J}_{\text{inj}}) + \mathbf{B}_{\text{ext}}. \quad (3.6)$$

We repeat that the external magnetic field \mathbf{B}_{ext} and the current fields \mathbf{J}_{ext} and \mathbf{J}_{inj} are assumed known except for their magnitude. For instance, in case of a smooth wire, the current field simply follows its shape. That is, we have

$$\mathbf{J}_{\text{ext}} = I_{\text{ext}} \cdot \tilde{\mathbf{J}}_{\text{ext}} \quad \mathbf{J}_{\text{inj}} = I_{\text{inj}} \cdot \tilde{\mathbf{J}}_{\text{inj}} \quad \mathbf{B}_{\text{ext}} = B_{\text{ext}} \cdot \tilde{\mathbf{B}}_{\text{ext}} \quad (3.7)$$

where

$$\mathbf{u} = (I_{\text{ext}}, I_{\text{inj}}, B_{\text{ext}}) \in \mathbb{R}^3 \quad (3.8)$$

is the vector of control variables. Herein, I_{ext} and I_{inj} denote the adjustable scalar current strengths and $\tilde{\mathbf{J}}_{\text{ext}}$ and $\tilde{\mathbf{J}}_{\text{inj}}$ are the given solenoidal current field distributions in the external conductors Ω_{ext} and Ω_{inj} , respectively. In practice, these currents must be maintained by an adjustable voltage source. Likewise, B_{ext} relates to the strength of the external magnetic field (associated to a permanent magnet) \mathbf{B}_{ext} . Note that the boundary conditions (3.3)–(3.4), together with Assumption 3.1(4) below, close the current loop and ensure that the total current $\mathbf{J} + \mathbf{J}_{\text{ext}} + \mathbf{J}_{\text{inj}}$ is solenoidal on \mathbb{R}^3 .

Note that our restriction to finite-dimensional controls is motivated with regard to practical applications. With the necessary changes to our theory, distributed current and magnetic fields $(\mathbf{J}_{\text{ext}}, \mathbf{J}_{\text{inj}}, \mathbf{B}_{\text{ext}}) \in \mathbf{L}_{\text{div}}^2(\Omega) \times (\mathbf{H}^1(\Omega_{\text{inj}}) \cap \mathbf{L}_{\text{div}}^2(\Omega_{\text{inj}})) \times \mathbf{V}^1(\mathbb{R}^3)$ can be used as controls if the norms in the objective below are adjusted accordingly. The most significant change will occur in the necessary optimality conditions (3.30), see Theorem 3.11 below, which will involve Poisson equations.

We recall that the primary concern of MHD flow control is to steer the fluid in a desired way. In addition, one may wish to also control the current in the fluid as well as the total magnetic field. Finally, control cost terms penalize excessive control action, and make the optimal control problem mathematically well-defined. Altogether, our problem **(P)** reads:

$$\begin{aligned} \text{Minimize } & \frac{\alpha_u}{2} \|\mathbf{u} - \mathbf{u}_d\|_{L^2(\Omega_{u,\text{obs}})}^2 + \frac{\alpha_B}{2} \|\mathbf{B} - \mathbf{B}_d\|_{L^2(\Omega_{B,\text{obs}})}^2 + \frac{\alpha_J}{2} \|\mathbf{J} - \mathbf{J}_d\|_{L^2(\Omega_{J,\text{obs}})}^2 \\ & + \frac{\gamma_{\text{ext}}}{2} |I_{\text{ext}}|^2 + \frac{\gamma_{\text{inj}}}{2} |I_{\text{inj}}|^2 + \frac{\gamma_B}{2} |B_{\text{ext}}|^2 \end{aligned} \quad (\mathbf{P})$$

subject to (3.1)–(3.6).

As described above, the objective reflects the goal of steering the fluid velocities and the magnetic and current fields towards the given desired fields \mathbf{u}_d , \mathbf{B}_d and \mathbf{J}_d , possibly only on subdomains $\Omega_{u,\text{obs}}$, $\Omega_{B,\text{obs}}$ and $\Omega_{J,\text{obs}}$ of interest. One may choose one or more of the weights α equal to zero if observation of the respective quantity is not desired. Note that due to the "state times control" terms $\mathbf{u} \times \mathbf{B}$ (through Ohm's Law (1.3)) and $\mathbf{J} \times \mathbf{B}$ (the Lorentz force), problem **(P)** is a particular type of bilinear control problem.

We begin with the analysis of the MHD system (3.1)–(3.6) and make the following assumption. The additional data concerning the control problem are specified in Assumption 3.9 below.

Assumption 3.1 (Problem data).

- (1) Let Ω , Ω_{inj} and Ω_{ext} denote bounded mutually disjoint domains with $C^{0,1}$ boundary such that Ω_{inj} and Ω have a part of their boundary of positive surface measure in common, see Figure 3.1.
- (2) Let ϱ (fluid density), η (fluid viscosity) and σ (fluid conductivity) be positive numbers. Likewise, the magnetic permeability μ is assumed constant in \mathbb{R}^3 .
- (3) Let the boundary velocity \mathbf{h} be given in $\mathbf{H}^{1/2}(\partial\Omega)$ such that $\int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n} = 0$.
- (4) Let $\tilde{\mathbf{J}}_{\text{ext}}$ and $\tilde{\mathbf{J}}_{\text{inj}}$ be given (current) fields in $\mathbf{L}^2_{\text{div}}(\Omega_{\text{ext}})$ and $\mathbf{H}^1(\Omega_{\text{inj}}) \cap \mathbf{L}^2_{\text{div}}(\Omega_{\text{inj}})$, respectively, such that $\int_{\partial\Omega \cap \partial\Omega_{\text{inj}}} \tilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n} = 0$.
- (5) Let $\tilde{\mathbf{B}}_{\text{ext}}$ be a given divergence-free (magnetic) field on \mathbb{R}^3 such that its restriction lies in $\mathbf{L}^{3+\varepsilon}(\Omega)$ for some $\varepsilon > 0$.

Let us briefly comment on the smoothness of the current shape function $\tilde{\mathbf{J}}_{\text{inj}}$, which belongs to the external conductor attached to the fluid region. The assumption $\tilde{\mathbf{J}}_{\text{inj}} \in \mathbf{H}^1(\Omega_{\text{inj}})$ implies that the normal trace $\tilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}$, when restricted to the intersection $\partial\Omega \cap \partial\Omega_{\text{inj}}$ and extended by zero, yields a function $j \in L^2(\partial\Omega)$, hence in particular $j \in H^{-1/2}(\partial\Omega)$ holds. The latter is needed to ensure the existence of a lifting \mathbf{J}_0 whose normal boundary values coincide with j , see Lemma 3.2 below. Note that $j \in H^{-1/2}(\partial\Omega)$ can in general not be achieved if merely $\tilde{\mathbf{J}}_{\text{inj}} \in \mathbf{L}^2_{\text{div}}(\Omega_{\text{inj}})$.

Next we turn to an appropriate choice of function spaces in which we seek solutions

$$y = (\mathbf{u}, p, \mathbf{J}, \phi) \tag{3.9}$$

to our problem (3.1)–(3.6). Following [20], we take

$$\begin{aligned} \mathbf{u} &\in \mathbf{H}^1(\Omega) & \mathbf{J} &\in \mathbf{L}^2(\Omega) \\ p &\in L^2(\Omega)/\mathbb{R} & \phi &\in H^1(\Omega)/\mathbb{R}. \end{aligned}$$

The elements of the quotient spaces are equivalence classes of functions which differ only by a constant. All operations carried out in these spaces will be described by their actions on individual functions, yielding the same result regardless of the representative.

In order to eliminate the boundary conditions for the velocity and current and to homogenize the problem, we introduce liftings \mathbf{u}_0 and \mathbf{J}_0 of the given boundary data such that

$$\begin{aligned} \mathbf{u}_0 &\in \mathbf{H}^1(\Omega), & \mathbf{u}_0|_{\partial\Omega} &= \mathbf{h}, & \nabla \cdot \mathbf{u}_0 &= 0 \\ \mathbf{J}_0 &\in \mathbf{L}^2(\Omega), & \mathbf{J}_0 \cdot \mathbf{n}|_{\partial\Omega} &= j, & \nabla \cdot \mathbf{J}_0 &= 0. \end{aligned}$$

with

$$j = \mathbf{J}_{\text{inj}} \cdot \mathbf{n} \text{ on } \partial\Omega \cap \partial\Omega_{\text{inj}} \quad \text{and} \quad j = 0 \text{ on } \partial\Omega \setminus \partial\Omega_{\text{inj}}. \tag{3.10}$$

Such a lifting exists according to the following lemma. Note that Assumption 3.1(4) implies that $\int_{\partial\Omega} j = 0$ as required in part (b).

Lemma 3.2 (Lifting).

Let β_i be the constants from the LBB condition (Lemma 2.1). Then we have:

- (a) For every $\mathbf{h} \in \mathbf{H}^{1/2}(\partial\Omega)$, there exists $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ such that $\mathbf{u}_0|_{\partial\Omega} = \mathbf{h}$ and $d_1(\mathbf{u}_0, q) = 0$ holds for all $q \in \mathbf{L}^2(\Omega)/\mathbb{R}$, i.e., $\nabla \cdot \mathbf{u}_0 = 0$. Moreover, the map $\mathbf{h} \mapsto \mathbf{u}_0$ can be chosen linearly and continuously, such that

$$\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \leq (1 + \beta_1^{-1}) \|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)}$$

is satisfied.

- (b) For every $j \in H^{-1/2}(\partial\Omega)$ which satisfies $\langle j, 1 \rangle_{\partial\Omega} = 0$, there exists $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$ such that $d_2(\mathbf{J}_0, \psi) = \langle j, \psi \rangle_{\partial\Omega}$ holds for all $\psi \in H^1(\Omega)/\mathbb{R}$, i.e., $\nabla \cdot \mathbf{J}_0 = 0$ and $\mathbf{J}_0 \cdot \mathbf{n} = j$. Moreover, the map $j \mapsto \mathbf{J}_0$ can be chosen linearly and continuously, such that

$$\|\mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} \leq \beta_2^{-1} \|j\|_{H^{-1/2}(\partial\Omega)}$$

is satisfied.

Proof. The claim is a consequence of the LBB condition, see [20] for details. Note that for functions $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$ such that $\nabla \cdot \mathbf{J}_0 \in \mathbf{L}^2(\Omega)$, the normal trace $\mathbf{J}_0 \cdot \mathbf{n}$ exists with values in $H^{-1/2}(\partial\Omega)$ [7]. \square

As a consequence, the fluid velocity and current can be written as

$$\mathbf{u} = \mathbf{u}_0 + \widehat{\mathbf{u}}, \quad \mathbf{J} = \mathbf{J}_0 + \widehat{\mathbf{J}},$$

where $\widehat{\mathbf{u}} \in \mathbf{H}_0^1(\Omega)$ and $\widehat{\mathbf{J}} \cdot \mathbf{n} = 0$ on $\partial\Omega$. It is important to note that in view of the current boundary condition (3.3)–(3.4), the lifting \mathbf{J}_0 depends on the control variable I_{inj} . We emphasize this dependence whenever appropriate by writing

$$\mathbf{J}_0 = \Lambda(I_{\text{inj}} \widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}) \quad (3.11)$$

meaning that \mathbf{J}_0 is the lifting, according to Lemma 3.2(b), of the function j in (3.10).

We now consider the homogenized state equation to find

$$\widehat{\mathbf{y}} = (\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi). \quad (3.12)$$

In its variational form, the homogenized system is given by

$$\begin{aligned} a_1(\widehat{\mathbf{u}} + \mathbf{u}_0, \mathbf{v}) - c(\widehat{\mathbf{J}} + \mathbf{J}_0, \mathbf{B}, \mathbf{v}) + b(\widehat{\mathbf{u}} + \mathbf{u}_0, \widehat{\mathbf{u}} + \mathbf{u}_0, \mathbf{v}) + d_1(\mathbf{v}, p) &= 0 \\ d_1(\widehat{\mathbf{u}}, q) &= 0 \\ a_2(\widehat{\mathbf{J}} + \mathbf{J}_0, \mathbf{K}) + c(\mathbf{K}, \mathbf{B}, \widehat{\mathbf{u}} + \mathbf{u}_0) + d_2(\mathbf{K}, \phi) &= 0 \\ d_2(\widehat{\mathbf{J}}, \psi) &= 0 \end{aligned} \quad (3.13)$$

for all $(\mathbf{v}, q, \mathbf{K}, \psi) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times \mathbf{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$, where we set

$$\mathbf{B} = \mathcal{B}(\Lambda(I_{\text{inj}} \widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})) + \mathcal{B}(\widehat{\mathbf{J}}) + \mathcal{B}(\mathbf{J}_{\text{ext}}) + \mathcal{B}(\mathbf{J}_{\text{inj}}) + \mathbf{B}_{\text{ext}} \quad (3.14)$$

as an abbreviation. The homogeneous solution $\widehat{\mathbf{y}}$ is sought in the space

$$\widehat{\mathbf{Y}} = \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times \mathbf{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}. \quad (3.15)$$

In its strong form, (3.13) corresponds to

$$\begin{aligned} -\eta \Delta \widehat{\mathbf{u}} + \rho(\widehat{\mathbf{u}} \cdot \nabla) \widehat{\mathbf{u}} + \rho(\widehat{\mathbf{u}} \cdot \nabla) \mathbf{u}_0 + \rho(\mathbf{u}_0 \cdot \nabla) \widehat{\mathbf{u}} + \nabla p \\ = \eta \Delta \mathbf{u}_0 - \rho(\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + \mathbf{J} \times \mathbf{B} + \mathbf{J}_0 \times \mathbf{B} \\ \sigma^{-1} \widehat{\mathbf{J}} - \widehat{\mathbf{u}} \times \mathbf{B} + \nabla \phi = -\sigma^{-1} \mathbf{J}_0 + \mathbf{u}_0 \times \mathbf{B} \end{aligned}$$

plus the incompressibility conditions $\nabla \cdot \widehat{\mathbf{u}} = 0$ and $\nabla \cdot \widehat{\mathbf{J}} = 0$ and boundary conditions $\widehat{\mathbf{J}} \cdot \mathbf{n} = 0$ and $\widehat{\mathbf{u}} = 0$ on $\partial\Omega$. We note that the velocity boundary condition is incorporated in the space $\mathbf{H}_0^1(\Omega)$, whereas the boundary condition for the current is of variational type.

We now comment on the solvability of (3.13)–(3.14) and thus of the original system (3.1)–(3.6). As was observed in [20] for the MHD system without \mathbf{J}_{ext} and \mathbf{B}_{ext} , the existence of a solution seems to be contingent upon the smallness of the liftings \mathbf{u}_0 and \mathbf{J}_0 , *i.e.*, smallness of the boundary data \mathbf{h} and $\mathbf{J}_{\text{inj}} \cdot \mathbf{n}$. The proof given there uses the Ladyzhenskaya–Babuška–Brezzi theory [7, Ch. IV, Theorem 1.2], and it is based on a limiting process of Galerkin approximations. Applying this technique to the present situation likewise yields solvability provided that the data \mathbf{h} and $\mathbf{J}_{\text{inj}} \cdot \mathbf{n}$ are sufficiently small. Under stronger assumptions involving also the remaining controls \mathbf{J}_{ext} and \mathbf{B}_{ext} , uniqueness of the solution can be proved using [7, Ch. IV, Theorem 1.3]. In any case, the required bounds on the data seem not exactly tangible since they involve the embedding constants of $H^1(\Omega) \hookrightarrow L^p(\Omega)$ and the constant in the Poincaré inequality as well as the norms of the lifting operator Λ and the Biot-Savart operator \mathcal{B} . This is the reason that we refrain from stating the exact conditions here. It is worth noting that it is not necessary to impose a priori bounds on the control variables to prove the existence of Lagrange multipliers for the optimal control problem below.

We provide here an alternative existence proof based on the Leray-Schauder fixed point theorem, see, for instance [6, p. 222]. Let us define the operator $A : \widehat{Y} \rightarrow \widehat{Y}'$ by its components:

$$\begin{aligned} A^1(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)(\mathbf{v}) &= a_1(\delta \mathbf{u}, \mathbf{v}) + d_1(\mathbf{v}, \delta p) \\ A^2(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)(q) &= d_1(\delta \mathbf{u}, q) \\ A^3(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)(\mathbf{K}) &= a_2(\delta \mathbf{J}, \mathbf{K}) + d_2(\mathbf{K}, \delta \phi) \\ A^4(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)(\psi) &= d_2(\delta \mathbf{J}, \psi). \end{aligned}$$

We observe that A is an isomorphism since $(A_1, A_2)(\cdot, \cdot, 0, 0)$ defined between $\mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times \{0\} \times \{0\}$ and its dual is an isomorphism, and so is $(A_3, A_4)(0, 0, \cdot, \cdot)$, defined between $\{0\} \times \{0\} \times \mathbf{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ and its dual, see [20].

Proposition 3.3 (State equation).

Assume that Assumption 3.1 holds and that $\|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}$ and $\|\mathbf{J}_0\|_{\mathbf{L}^2(\Omega)}$ are sufficiently small. Then the homogenized state equation (3.13)–(3.14) and hence the original system (3.1)–(3.6) possesses at least one variational solution. Every such solution satisfies the a priori bound

$$\begin{aligned} \|\mathbf{u}\|^2 + \|\mathbf{J}\|^2 &\leq c_1 \|\mathbf{J}_0\|_{\mathbf{L}^2(\Omega)}^2 \left(1 + \|\mathbf{J}_0\|_{\mathbf{L}^2(\Omega)}^2 + |u|^2\right) \\ &\quad + c_2 \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}^2 \left(1 + \|\mathbf{J}_0\|_{\mathbf{L}^2(\Omega)}^2 + |u|^2\right). \end{aligned} \quad (3.16)$$

Moreover, we have the bound

$$\|p\|_{L^2(\Omega)/\mathbb{R}} + \|\phi\|_{H^1(\Omega)/\mathbb{R}} \leq c_3 \left(\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}^2 + \|\mathbf{J}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{J}\|_{\mathbf{L}^2(\Omega)}^2 \right).$$

Proof. Let us define $T : \widehat{Y} \rightarrow \widehat{Y}$ according to

$$(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi) = T(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)$$

if and only if

$$A(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi) = R(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi) \quad (3.17)$$

holds in \widehat{Y}' . That is, T is the solution operator of a linear PDE, which depends nonlinearly on the data $(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)$. Defining the components of R as

$$R^1(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)(\mathbf{v}) = c(\widehat{\mathbf{J}} + \mathbf{J}_0, \mathcal{B}(\mathbf{J}_0) + \mathcal{B}(\widehat{\mathbf{J}}) + \mathcal{B}(\mathbf{J}_{\text{ext}}) + \mathcal{B}(\mathbf{J}_{\text{inj}}) + \mathbf{B}_{\text{ext}}, \mathbf{v}) \\ - a_1(\mathbf{u}_0, \mathbf{v}) - b(\widehat{\mathbf{u}} + \mathbf{u}_0, \widehat{\mathbf{u}} + \mathbf{u}_0, \mathbf{v})$$

$$R^2(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)(q) = 0$$

$$R^3(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)(\mathbf{K}) = -c(\mathbf{K}, \mathcal{B}(\mathbf{J}_0) + \mathcal{B}(\widehat{\mathbf{J}}) + \mathcal{B}(\mathbf{J}_{\text{ext}}) + \mathcal{B}(\mathbf{J}_{\text{inj}}) + \mathbf{B}_{\text{ext}}, \widehat{\mathbf{u}} + \mathbf{u}_0) \\ - a_2(\mathbf{J}_0, \mathbf{K})$$

$$R^4(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)(\psi) = 0,$$

we easily verify that the solutions to the homogenized problem (3.13)–(3.14) are exactly the fixed points of T . In view of Proposition 3.6 below, $A : \widehat{Y} \rightarrow \widehat{Y}'$ is an isomorphism, and hence T is well-defined from \widehat{Y} to itself. We now confirm that T is compact. To this end, we consider a bounded and weakly convergent sequence $(\widehat{\mathbf{u}}^n, p^n, \widehat{\mathbf{J}}^n, \phi^n) \rightharpoonup (\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)$ in \widehat{Y} . Since the norm in \widehat{Y}' of the right hand side in (3.17) is a quadratic polynomial in the norms of $\widehat{\mathbf{u}}$, and $\widehat{\mathbf{J}}$, the sequence $(\delta \mathbf{u}^n, \delta p^n, \delta \mathbf{J}^n, \delta \phi^n) := T(\widehat{\mathbf{u}}^n, p^n, \widehat{\mathbf{J}}^n, \phi^n)$ is bounded in \widehat{Y} and thus possesses a weakly convergent subsequence in \widehat{Y} , *i.e.*, $(\delta \mathbf{u}^n, \delta p^n, \delta \mathbf{J}^n, \delta \phi^n) \rightharpoonup (\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)$. Using Lemma 2.3, one confirms that the weak limit $(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)$ satisfies $(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi) = T(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)$, *i.e.*, T is weakly continuous. The difference $(\delta \mathbf{u}^n, \delta p^n, \delta \mathbf{J}^n, \delta \phi^n) - (\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)$ satisfies (3.17) with right hand side

$$R(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi) - R(\widehat{\mathbf{u}}^n, p^n, \widehat{\mathbf{J}}^n, \phi^n),$$

which converges to zero strongly in \widehat{Y}' , as a straightforward application of Lemmas 2.3, 2.4 and 2.6 shows. Hence T is indeed compact.

Now let $(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)$ be a fixed point of $s \cdot T$ for any $s \in [0, 1]$, *i.e.*, $(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)$ satisfies (3.17) with the right hand side multiplied by s . Testing this system with $(\widehat{\mathbf{u}}, p, \widehat{\mathbf{J}}, \phi)$, we obtain

$$\eta \|\nabla \widehat{\mathbf{u}}\|_{L^2(\Omega)}^2 + \sigma^{-1} \|\widehat{\mathbf{J}}\|_{L^2(\Omega)}^2 = s \left(c(\mathbf{J}_0, \mathbf{B}, \widehat{\mathbf{u}}) - c(\widehat{\mathbf{J}}, \mathbf{B}, \mathbf{u}_0) - b(\widehat{\mathbf{u}}, \mathbf{u}_0, \widehat{\mathbf{u}}) \right. \\ \left. - a_1(\mathbf{u}_0, \widehat{\mathbf{u}}) - a_2(\mathbf{J}_0, \widehat{\mathbf{J}}) \right)$$

with \mathbf{B} according to (3.14). Inserting Poincaré's inequality $\|\widehat{\mathbf{u}}\|_{L^2(\Omega)} \leq c_P \|\nabla \widehat{\mathbf{u}}\|_{L^2(\Omega)}$, one obtains

$$\frac{\eta}{1 + c_P^2} \|\widehat{\mathbf{u}}\|_{H^1(\Omega)}^2 + \sigma^{-1} \|\widehat{\mathbf{J}}\|_{L^2(\Omega)}^2 \leq s \left(c(\mathbf{J}_0, \mathbf{B}, \widehat{\mathbf{u}}) - c(\widehat{\mathbf{J}}, \mathbf{B}, \mathbf{u}_0) - b(\widehat{\mathbf{u}}, \mathbf{u}_0, \widehat{\mathbf{u}}) \right. \\ \left. - a_1(\mathbf{u}_0, \widehat{\mathbf{u}}) - a_2(\mathbf{J}_0, \widehat{\mathbf{J}}) \right). \quad (3.18)$$

The application of the Leray-Schauder fixed point theorem requires that the left hand side be a priori bounded uniformly in $s \in [0, 1]$. The bound may depend on the controls $(I_{\text{ext}}, I_{\text{inj}}, B_{\text{ext}})$ and the boundary data \mathbf{h} . We observe that the right hand side in (3.18) is bounded above by

$$\|\mathbf{J}_0\|_{L^2(\Omega)} \|\mathcal{B}(\widehat{\mathbf{J}})\|_{L^3(\Omega)} \|\widehat{\mathbf{u}}\|_{L^6(\Omega)} + \|\widehat{\mathbf{J}}\|_{L^2(\Omega)} \|\mathcal{B}(\widehat{\mathbf{J}})\|_{L^3(\Omega)} \|\mathbf{u}_0\|_{L^6(\Omega)} \\ + \|\nabla \mathbf{u}_0\|_{L^2(\Omega)} \|\widehat{\mathbf{u}}\|_{L^4(\Omega)} \quad (3.19)$$

plus a number of terms which are at most linear in $\|\widehat{\mathbf{u}}\|$ and $\|\mathbf{J}\|$. The latter can be treated using Young's inequality according to the pattern $c\|\widehat{\mathbf{u}}\| \leq \varepsilon\|\widehat{\mathbf{u}}\|^2 + c/(4\varepsilon)$ and $\varepsilon\|\widehat{\mathbf{u}}\|^2$ can then be absorbed in the left hand side of (3.18) for sufficiently small $\varepsilon > 0$. However, in order that the terms in (3.19) can likewise be absorbed in the left hand side of (3.18), the coefficients $\|\mathbf{J}_0\|_{L^2(\Omega)}$, $\|\mathbf{u}_0\|_{L^6(\Omega)}$ and $\|\nabla \mathbf{u}_0\|_{L^2(\Omega)}$

must be sufficiently small. In this case, $\|\widehat{\mathbf{u}}\|_{\mathbf{H}^1(\Omega)}$ and $\|\widehat{\mathbf{J}}\|_{\mathbf{L}^2(\Omega)}$ are indeed a priori bounded by the right hand side in (3.16). In view of Lemma 3.2, the same bound holds for the inhomogeneous solution \mathbf{u} and \mathbf{J} . Finally, the bounds for the pressure p and potential ϕ follow from Lemma 2.2. Hence we conclude the applicability of the Leray-Schauder Theorem which yields the existence of a fixed point of T . \square

Remark 3.4. From (3.18) we infer that the larger the viscosity η of the fluid and the smaller its conductivity σ , the larger the liftings \mathbf{u}_0 and \mathbf{J}_0 in Proposition 3.3 and thus the larger the boundary data \mathbf{h} and the control I_{inj} are allowed to become.

The variational form of the state equation (3.13) gives rise to the definition of the PDE constraint operator

$$e : \widehat{Y} \times \mathbb{R}^3 \rightarrow \widehat{Y}'. \quad (3.20)$$

The components $e^1(\widehat{y}, u)(\mathbf{v}), \dots, e^4(\widehat{y}, u)(\psi)$ are defined through the left hand sides of (3.13). This concise form of the MHD system

$$e(\widehat{y}, u) = 0 \quad \text{in } \widehat{Y}' \quad (3.21)$$

will be used below to argue existence of the Lagrange multipliers in the optimality system, based on the following results on the linearization of e whose proofs are only given as necessary.

Lemma 3.5 (Linearized PDE constraint).

The operator e is infinitely Fréchet differentiable. Its first order partial derivative with respect to the state variables in the direction of $\delta y = (\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)$ is given by

$$\begin{aligned} e_y^1(\widehat{y}, u)(\delta y)(\mathbf{v}) &= a_1(\delta \mathbf{u}, \mathbf{v}) - c(\delta \mathbf{J}, \mathbf{B}, \mathbf{v}) - c(\widehat{\mathbf{J}} + \Lambda(I_{\text{inj}} \widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}), \mathcal{B}(\delta \mathbf{J}), \mathbf{v}) \\ &\quad + b(\delta \mathbf{u}, \widehat{\mathbf{u}} + \mathbf{u}_0, \mathbf{v}) + b(\widehat{\mathbf{u}} + \mathbf{u}_0, \delta \mathbf{u}, \mathbf{v}) + d_1(\mathbf{v}, \delta p) \\ e_y^2(\widehat{y}, u)(\delta y)(q) &= d_1(\delta \mathbf{u}, q) \\ e_y^3(\widehat{y}, u)(\delta y)(\mathbf{K}) &= a_2(\delta \mathbf{J}, \mathbf{K}) + c(\mathbf{K}, \mathbf{B}, \delta \mathbf{u}) + c(\mathbf{K}, \mathcal{B}(\delta \mathbf{J}), \widehat{\mathbf{u}} + \mathbf{u}_0) + d_2(\mathbf{K}, \delta \phi) \\ e_y^4(\widehat{y}, u)(\delta y)(\psi) &= d_2(\delta \mathbf{J}, \psi) \end{aligned} \quad (3.22)$$

where we have set again $\mathbf{B} = \mathcal{B}(\Lambda(I_{\text{inj}} \widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})) + \mathcal{B}(\widehat{\mathbf{J}}) + \mathcal{B}(\mathbf{J}_{\text{ext}}) + \mathcal{B}(\mathbf{J}_{\text{inj}}) + \mathbf{B}_{\text{ext}}$. As for the control variables, the first order derivative in the direction of $\delta u = (\delta I_{\text{ext}}, \delta I_{\text{inj}}, \delta B_{\text{ext}})$ is

$$\begin{aligned} e_u^1(\widehat{y}, u)(\delta u)(\mathbf{v}) &= -c(\delta I_{\text{inj}} \cdot \Lambda(\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}), \mathbf{B}, \mathbf{v}) \\ &\quad - c(\widehat{\mathbf{J}} + \Lambda(I_{\text{inj}} \widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}), \delta I_{\text{inj}} \cdot \mathcal{B}(\Lambda(\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})) + \delta I_{\text{ext}} \cdot \mathcal{B}(\widetilde{\mathbf{J}}_{\text{ext}}) + \delta I_{\text{inj}} \cdot \mathcal{B}(\widetilde{\mathbf{J}}_{\text{inj}}) + \delta B_{\text{ext}} \cdot \widetilde{\mathbf{B}}_{\text{ext}}, \mathbf{v}) \\ e_u^2(\widehat{y}, u)(\delta u)(q) &= 0 \\ e_u^3(\widehat{y}, u)(\delta u)(\mathbf{K}) &= a_2(\delta I_{\text{inj}} \cdot \Lambda(\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}), \mathbf{K}) \\ &\quad + c(\mathbf{K}, \delta I_{\text{inj}} \cdot \mathcal{B}(\Lambda(\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})) + \delta I_{\text{ext}} \cdot \mathcal{B}(\widetilde{\mathbf{J}}_{\text{ext}}) + \delta I_{\text{inj}} \cdot \mathcal{B}(\widetilde{\mathbf{J}}_{\text{inj}}) + \delta B_{\text{ext}} \cdot \widetilde{\mathbf{B}}_{\text{ext}}, \widehat{\mathbf{u}} + \mathbf{u}_0) \\ e_u^4(\widehat{y}, u)(\delta u)(\psi) &= 0 \end{aligned} \quad (3.23)$$

We note that in its strong form, the system $e_y(\widehat{y}, u)(\delta y) = (\bar{e}, \bar{f}, \bar{g}, \bar{h}) \in \widehat{Y}'$ corresponds to

$$\begin{aligned}
& -\eta\Delta\delta\mathbf{u} + \rho(\delta\mathbf{u} \cdot \nabla)(\widehat{\mathbf{u}} + \mathbf{u}_0) + \rho((\widehat{\mathbf{u}} + \mathbf{u}_0) \cdot \nabla)\delta\mathbf{u} + \nabla\delta p \\
& \qquad \qquad \qquad = \delta\mathbf{J} \times \mathbf{B} + (\widehat{\mathbf{J}} + \mathbf{J}_0) \times \mathcal{B}(\delta\mathbf{J}) + \bar{e} \\
& \qquad \qquad \qquad \nabla \cdot \delta\mathbf{u} = \bar{f} \\
& \sigma^{-1}\delta\mathbf{J} - \delta\mathbf{u} \times \mathbf{B} - (\widehat{\mathbf{u}} + \mathbf{u}_0) \times \mathcal{B}(\delta\mathbf{J}) + \nabla\delta\phi = \bar{g} \\
& \qquad \qquad \qquad \nabla \cdot \mathbf{J} = \bar{h}.
\end{aligned}$$

The first and third component of $e_u(\widehat{y}, u)\delta u$ must be read as

$$\begin{aligned}
& -\delta I_{\text{inj}} \cdot \Lambda(\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}) \times \mathbf{B} - (\widehat{\mathbf{J}} + \Lambda(I_{\text{inj}}\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})) \\
& \qquad \times \left(\delta I_{\text{inj}} \cdot \mathcal{B}(\Lambda(\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})) + \delta I_{\text{ext}} \cdot \mathcal{B}(\widetilde{\mathbf{J}}_{\text{ext}}) + \delta I_{\text{inj}} \cdot \mathcal{B}(\widetilde{\mathbf{J}}_{\text{inj}}) + \delta B_{\text{ext}} \cdot \widetilde{\mathbf{B}}_{\text{ext}} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \sigma^{-1}\delta I_{\text{inj}} \cdot \Lambda(\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}) - (\widehat{\mathbf{u}} + \mathbf{u}_0) \times \\
& \qquad \left(\delta I_{\text{inj}} \cdot \mathcal{B}(\Lambda(\widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})) + \delta I_{\text{ext}} \cdot \mathcal{B}(\widetilde{\mathbf{J}}_{\text{ext}}) + \delta I_{\text{inj}} \cdot \mathcal{B}(\widetilde{\mathbf{J}}_{\text{inj}}) + \delta B_{\text{ext}} \cdot \widetilde{\mathbf{B}}_{\text{ext}} \right),
\end{aligned}$$

respectively.

In preparation of the following proposition, let us define the operator $C : \widehat{Y} \rightarrow \widehat{Y}'$ by its components:

$$\begin{aligned}
C^1(\widehat{y}, u)(\delta\mathbf{u}, \delta p, \delta\mathbf{J}, \delta\phi)(\mathbf{v}) &= -c(\delta\mathbf{J}, \mathbf{B}, \mathbf{v}) - c(\widehat{\mathbf{J}} + \mathbf{J}_0, \mathcal{B}(\delta\mathbf{J}), \mathbf{v}) + b(\delta\mathbf{u}, \widehat{\mathbf{u}} + \mathbf{u}_0, \mathbf{v}) \\
& \qquad \qquad \qquad + b(\widehat{\mathbf{u}} + \mathbf{u}_0, \delta\mathbf{u}, \mathbf{v})
\end{aligned}$$

$$C^2(\widehat{y}, u)(\delta\mathbf{u}, \delta p, \delta\mathbf{J}, \delta\phi)(q) = 0$$

$$C^3(\widehat{y}, u)(\delta\mathbf{u}, \delta p, \delta\mathbf{J}, \delta\phi)(\mathbf{K}) = c(\mathbf{K}, \mathbf{B}, \delta\mathbf{u}) + c(\mathbf{K}, \mathcal{B}(\delta\mathbf{J}), \widehat{\mathbf{u}} + \mathbf{u}_0)$$

$$C^4(\widehat{y}, u)(\delta\mathbf{u}, \delta p, \delta\mathbf{J}, \delta\phi)(\psi) = 0.$$

The quantities \mathbf{u}_0 and \mathbf{J}_0 are the liftings according to Lemma 3.2 for given arbitrary but fixed controls u and boundary data \mathbf{h} .

Proposition 3.6 (Linearized state equation).

For any given $\widehat{y} \in \widehat{Y}$, $u \in \mathbb{R}^3$ and $\mathbf{h} \in \mathbf{H}^{1/2}(\partial\Omega)$, the linearization with respect to the state variables of the operator e can be decomposed as

$$e_y(\widehat{y}, u) = A + C(\widehat{y}, u) \tag{3.24}$$

where $A : \widehat{Y} \rightarrow \widehat{Y}'$ is an isomorphism, independent of (\widehat{y}, u) , and $C(\widehat{y}, u) : \widehat{Y} \rightarrow \widehat{Y}'$ is a compact linear operator.

Proof. The isomorphism property of A has been noted previously, see the definition of A . As for compactness of C , we recall that $\mathbf{B} \in \mathbf{L}^{3+\varepsilon}(\Omega)$ (see Lemma 2.4 and Assumption 3.1) and infer from Lemma 2.6 that $\delta\mathbf{J} \mapsto \delta\mathbf{J} \times \mathbf{B}$ is compact from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}^{-1}(\Omega)$. Hence $\delta\mathbf{J} \mapsto c(\delta\mathbf{J}, \mathbf{B}, \cdot)$ is compact from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}^{-1}(\Omega)$. Similarly, by Lemmas 2.4 and 2.6, $\delta\mathbf{J} \mapsto \mathbf{J} \times \mathcal{B}(\delta\mathbf{J})$ is compact from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}^{-1}(\Omega)$, and hence $\delta\mathbf{J} \mapsto c(\mathbf{J}, \mathcal{B}(\delta\mathbf{J}), \cdot)$ is compact from $\mathbf{L}^2(\Omega)$ to $\mathbf{H}^{-1}(\Omega)$. In addition, $\delta\mathbf{u} \mapsto b(\delta\mathbf{u}, \widehat{\mathbf{u}} + \mathbf{u}_0, \cdot)$ and $\delta\mathbf{u} \mapsto b(\widehat{\mathbf{u}} + \mathbf{u}_0, \delta\mathbf{u}, \cdot)$ are continuous from $\mathbf{H}_0^1(\Omega)$ to $\mathbf{L}^{3/2}(\Omega)$ which embeds compactly into $\mathbf{W}^{-1, 3-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, and thus into $\mathbf{H}^{-1}(\Omega)$. This completes the proof of compactness for C^1 . As for C^3 , we let $p = 2(3+\varepsilon)/(1+\varepsilon) < 6$ and observe that $\delta\mathbf{u} \mapsto c(\cdot, \mathbf{B}, \delta\mathbf{u})$ is continuous from $\mathbf{L}^p(\Omega)$ to $\mathbf{L}^2(\Omega)$. In view of the compact embedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$, this map is compact from $\mathbf{H}_0^1(\Omega)$ to $\mathbf{L}^2(\Omega)$. Finally, since $\delta\mathbf{J} \mapsto \mathcal{B}(\delta\mathbf{J})$ is continuous from $\mathbf{L}^2(\Omega)$ to $V^1(\Omega)$ which embeds compactly into $\mathbf{L}^3(\Omega)$, the map $\delta\mathbf{J} \mapsto c(\cdot, \mathcal{B}(\delta\mathbf{J}), \widehat{\mathbf{u}} + \mathbf{u}_0)$ is compact from $\mathbf{L}^2(\Omega)$ to $\mathbf{L}^2(\Omega)$, which completes the proof. \square

The previous proposition allows us to draw the following conclusions about the properties of the linearized state equation operator $e_y(\widehat{y}, u)$:

Proposition 3.7 (Bounded invertibility of $e_y(\widehat{y}, u)$).

Except for a countable set of (η, σ) -values, the operator $e_y(\widehat{y}, u)$ is an isomorphism. Moreover, $e_y(\widehat{y}, u)$ is an isomorphism whenever η is sufficiently large and σ is sufficiently small.

Proof. For $(\bar{e}, \bar{f}, \bar{g}, \bar{h}) \in \widehat{Y}'$, consider the equation

$$(A + C)(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi) = (\bar{e}, \bar{f}, \bar{g}, \bar{h}) \quad (3.25)$$

and define $\mathcal{A} : \widehat{Y} \rightarrow \widehat{Y}'$ through its coordinates

$$\begin{aligned} \mathcal{A}^1(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)(\mathbf{v}) &= \int_{\Omega} \nabla \delta \mathbf{u} : \nabla \mathbf{v} - \int_{\Omega} (\nabla \cdot \mathbf{v}) \mathcal{P}(\delta p) \\ \mathcal{A}^2 &= A^2 \\ \mathcal{A}^3(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi)(\mathbf{K}) &= \int_{\Omega} \delta \mathbf{J} \cdot \mathbf{K} + \int_{\Omega} \mathbf{K} \cdot \nabla(\delta \phi) \\ \mathcal{A}^4 &= A^4 \end{aligned}$$

where $(\mathbf{v}, \mathbf{K}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$, *i.e.*, \mathcal{A} arises from A by setting $\eta = \sigma = 1$. Multiplying the first two equations in (3.25) by η^{-1} and the last two by σ , we find that the following equation is equivalent to (3.25):

$$(\mathcal{A} + \eta^{-1}\mathcal{C}_1 + \sigma\mathcal{C}_2)(\delta \mathbf{u}, \widehat{\delta p}, \delta \mathbf{J}, \widehat{\delta \phi}) = (\eta^{-1}\bar{e}, \bar{f}, \sigma\bar{g}, \bar{h}), \quad (3.26)$$

where $\widehat{\delta p} = \eta^{-1}\delta p$ and $\widehat{\delta \phi} = \sigma\delta \phi$ and $\mathcal{C}_1 = (C_1, C_2, 0, 0)^\top$, $\mathcal{C}_2 = (0, 0, C_3, C_4)^\top$. From the proof of Proposition 3.6 we have that $\mathcal{A} : \widehat{Y} \rightarrow \widehat{Y}'$ is an isomorphism and that \mathcal{C}_1 and \mathcal{C}_2 are compact operators from \widehat{Y} to \widehat{Y}' . Moreover, (3.26) is equivalent to

$$(I + \eta^{-1}\mathcal{K}_1 + \sigma\mathcal{K}_2)(\delta \mathbf{u}, \widehat{\delta p}, \delta \mathbf{J}, \widehat{\delta \phi}) = (e, f, g, h) \quad (3.27)$$

where $(e, f, g, h) = \mathcal{A}^{-1}(\eta^{-1}\bar{e}, \bar{f}, \sigma\bar{g}, \bar{h})$, and

$$\mathcal{K}_1 = \mathcal{A}^{-1}\mathcal{C}_1, \quad \mathcal{K}_2 = \mathcal{A}^{-1}\mathcal{C}_2.$$

Hence, \mathcal{K}_1 and \mathcal{K}_2 are compact operators in \widehat{Y} . Therefore the spectrum of \mathcal{K}_1 , denoted by $\sum(\mathcal{K}_1)$, consists of 0 and at most countably many eigenvalues, with 0 being the only possible accumulation point. For $-\eta \notin \sum(\mathcal{K}_1)$ we have

$$(I + \sigma(I + \eta^{-1}\mathcal{K}_1)^{-1}\mathcal{K}_2)(\delta \mathbf{u}, \widehat{\delta p}, \delta \mathbf{J}, \widehat{\delta \phi}) = (I + \eta^{-1}\mathcal{K}_1)^{-1}(e, f, g, h). \quad (3.28)$$

Since $(I + \eta^{-1}\mathcal{K}_1)$ has a continuous inverse, we find that (3.28), and hence (3.25) are solvable if $-\sigma^{-1} \notin \sum((I + \eta^{-1}\mathcal{K}_1)^{-1}\mathcal{K}_2)$. Since the set of points of $\{(-\eta, -\sigma^{-1}) : -\eta \in \sum(\mathcal{K}_1), -\sigma^{-1} \in \sum((I + \eta^{-1}\mathcal{K}_1)^{-1}\mathcal{K}_2)\}$ is countable in \mathbb{R}^2 , the first claim follows. The second claim is a consequence of a Neumann series argument applied to (3.27). \square

In the unlikely event that $e_y(\widehat{y}, u)$ is not surjective, the following proposition states a condition that $e_x(\widehat{y}, u)$ still is. Note that $e_u(\widehat{y}, u) \delta u$ can be written in the form

$$e_u(\widehat{y}, u) \delta u = \delta I_{\text{inj}} \psi_1 + \delta I_{\text{ext}} \psi_2 + \delta B_{\text{ext}} \psi_3$$

with $\psi_i \in \widehat{Y}'$.

Proposition 3.8 (Surjectivity of $e_x(\widehat{y}, u)$).

If $\text{span}\{\psi_1, \psi_2, \psi_3\} \supseteq \ker(A + C^)$ then $e_x(\widehat{y}, u)$ is surjective.*

Proof. Since $A : \widehat{Y} \rightarrow \widehat{Y}'$ is an isomorphism and C is compact, the Fredholm alternative implies that the range of $A + C$ is closed and $R(A + C) = \ker(A + C^*)^\perp$, with $\dim \ker(A + C^*) =: L < \infty$. Let $\{\omega_i\}_{i=1}^L$ be a basis for $\ker(A + C^*)$, orthonormalized such that $\langle \omega_i, \omega_j \rangle_{\widehat{Y}, \widehat{Y}'} = \delta_{ij}$. For arbitrary $\widehat{f} \in \widehat{Y}'$, define $\widetilde{f} = \widehat{f} - \sum_{i=1}^L \langle \widehat{f}, \omega_i \rangle_{\widehat{Y}, \widehat{Y}'} \cdot \omega_i$. Then $\widetilde{f} \in \ker(A + C^*)^\perp$ and there exists $x_{\widetilde{f}}$ such that $(A + C)x_{\widetilde{f}} = \widetilde{f}$. By assumption there exist $(\delta I_{\text{inj}}, \delta I_{\text{ext}}, \delta \mathbf{B}_{\text{ext}})$ such that $e_u(\widehat{y}, u)\delta u = \widehat{f} - \widetilde{f}$ and the claim follows. \square

Having established these properties for the stationary MHD system and its linearization, we now return to the optimal control problem. We recall that we aim to

$$\begin{aligned} \text{minimize } f(\widehat{y}, u) &= \frac{\alpha_u}{2} \|\mathbf{u} - \mathbf{u}_d\|_{L^2(\Omega_{u,\text{obs}})}^2 + \frac{\alpha_B}{2} \|\mathbf{B} - \mathbf{B}_d\|_{L^2(\Omega_{B,\text{obs}})}^2 \\ &\quad + \frac{\alpha_J}{2} \|\mathbf{J} - \mathbf{J}_d\|_{L^2(\Omega_{J,\text{obs}})}^2 + \frac{\gamma_{\text{ext}}}{2} |I_{\text{ext}}|^2 + \frac{\gamma_{\text{inj}}}{2} |I_{\text{inj}}|^2 + \frac{\gamma_B}{2} |B_{\text{ext}}|^2 \quad (\mathbf{P}) \\ \text{over } \widehat{y} &\in \widehat{Y} \text{ and } u \in \mathbb{R}^3 \\ \text{subject to } e(\widehat{y}, u) &= 0, \end{aligned}$$

where $\mathbf{u} = \widehat{\mathbf{u}} + \mathbf{u}_0$ and $\mathbf{J} = \widehat{\mathbf{J}} + \mathbf{J}_0$, and \mathbf{B} is defined in (3.6). Recall also that the lifting \mathbf{J}_0 and hence \mathbf{B} depend on the control I_{inj} . In order to ensure well-posedness of problem (P), we need the following assumption on the problem data:

Assumption 3.9 (Control problem data).

- (1) Let the weights $(\alpha_u, \alpha_B, \alpha_J)$ be non-negative numbers and let $(\gamma_{\text{ext}}, \gamma_{\text{inj}}, \gamma_B)$ be positive. In addition, let \mathbf{u}_d , \mathbf{B}_d and \mathbf{J}_d be \mathbf{L}^2 functions in their respective domains of definition, $\Omega_{u,\text{obs}}$, $\Omega_{B,\text{obs}}$ and $\Omega_{J,\text{obs}}$, where $\Omega_{u,\text{obs}}$ and $\Omega_{J,\text{obs}}$ are subsets of the region Ω occupied by the fluid.
- (2) Assume that the boundary data $\|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)}$ is sufficiently small and that for some fixed $r > 0$ and all controls in the set

$$U_{\text{ad}} = \{(I_{\text{ext}}, I_{\text{inj}}, B_{\text{ext}}) \in \mathbb{R}^3 : |I_{\text{inj}}| \leq r\},$$

the liftings \mathbf{u}_0 and $\mathbf{J}_0 = \Lambda(I_{\text{inj}} \widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})$ allow the existence of a solution to the stationary MHD system, according to Proposition 3.3.

Proposition 3.10 (Existence of a global minimum).

Under Assumptions 3.1 and 3.9, problem (P) possesses at least one global optimal solution in $\widehat{Y} \times U_{\text{ad}}$.

Proof. The proof follows along the usual lines. We set $m = \inf f(\widehat{y}, u)$ where the infimum extends over all state/control pairs $(\widehat{y}, u) \in \widehat{Y} \times U_{\text{ad}}$ which satisfy the state equation (admissible pairs). Note that m is non-negative and finite since f is non-negative and the set of admissible pairs is non-empty (Assumption 3.9(2) and Proposition 3.3). Now if $\{(\widehat{y}^n, u^n)\}$ is a minimizing sequence, we can infer from the cost functional that the controls u^n are bounded in \mathbb{R}^3 . By the a priori estimate (3.16), \widehat{y}^n is bounded in \widehat{Y} . We extract weakly convergent subsequences, still denoted by index n , such that

$$\begin{aligned} \widehat{\mathbf{u}}^n &\rightharpoonup \widehat{\mathbf{u}} \quad \text{in } \mathbf{H}_0^1(\Omega) & p^n &\rightharpoonup p \quad \text{in } L^2(\Omega)/\mathbb{R} \\ \widehat{\mathbf{J}}^n &\rightharpoonup \widehat{\mathbf{J}} \quad \text{in } \mathbf{L}^2(\Omega) & \phi^n &\rightharpoonup \phi \quad \text{in } H^1(\Omega)/\mathbb{R} \\ u^n &\rightarrow u \quad \text{in } \mathbb{R}^3. \end{aligned}$$

Note that the lifting \mathbf{u}_0 is independent of n and that $\mathbf{J}_0^n = \Lambda(I_{\text{inj}}^n \widetilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})$ converges strongly to some \mathbf{J}_0 in $\mathbf{L}^2(\Omega)$. In order to pass to the limit in $e(\widehat{y}^n, u^n)$, we consider

the individual terms in (3.13). For the terms involving the bilinear forms a_i and d_i , the convergence is evident. In addition, $b(\hat{\mathbf{u}}^n, \hat{\mathbf{u}}^n, \mathbf{v}) \rightarrow b(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v})$ is known from the theory of Navier-Stokes optimal control problems, see [7, Ch. IV, Theorem 2.1]. The convergence of all terms involving the trilinear form c follows from Lemmas 2.3 and 2.6. Consequently, the weak limit $(\hat{\mathbf{y}}, u)$ satisfies the state equation $e(\hat{\mathbf{y}}, u) = 0$ and hence the weak limit $(\hat{\mathbf{u}} + \mathbf{u}_0, p, \hat{\mathbf{J}} + \mathbf{J}_0, \phi)$ satisfies our inhomogeneous MHD system. The claim now follows from weak lower semicontinuity of the objective, by which

$$m \leq f(\hat{\mathbf{y}}, u) \leq \liminf_{n \rightarrow \infty} f(\hat{\mathbf{y}}^n, u^n) = m.$$

□

Theorem 3.11 (Optimality system).

Let Assumptions 3.1 and 3.9(1) hold and let the state $\hat{\mathbf{y}} = (\hat{\mathbf{u}}, p, \hat{\mathbf{J}}, \phi) \in \hat{Y}$ and control $u = (I_{\text{ext}}, I_{\text{inj}}, B_{\text{ext}}) \in \mathbb{R}^3$ constitute a local optimal pair for problem (P). In addition, let $e_x(\hat{\mathbf{y}}, u)$ be surjective. Then there exists a unique Lagrange multiplier

$$\lambda = (\mathbf{v}, q, \mathbf{K}, \psi) \in \hat{Y}$$

which satisfies the adjoint equations

$$\begin{aligned} a_1(\delta \mathbf{u}, \mathbf{v}) + b(\delta \mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \delta \mathbf{u}, \mathbf{v}) + d_1(\delta \mathbf{u}, q) + c(\mathbf{K}, \mathbf{B}, \delta \mathbf{u}) \\ + \alpha_u \int_{\Omega_{\mathbf{u}, \text{obs}}} (\mathbf{u} - \mathbf{u}_d) \cdot \delta \mathbf{u} = 0 \end{aligned} \quad (3.29a)$$

$$d_1(\mathbf{v}, \delta p) = 0 \quad (3.29b)$$

$$\begin{aligned} -c(\delta \mathbf{J}, \mathbf{B}, \mathbf{v}) - c(\mathbf{J}, \mathcal{B}(\delta \mathbf{J}), \mathbf{v}) + a_2(\delta \mathbf{J}, \mathbf{K}) + c(\mathbf{K}, \mathcal{B}(\delta \mathbf{J}), \mathbf{u}) \\ + d_2(\delta \mathbf{J}, \psi) + \alpha_B \int_{\Omega_{\mathbf{B}, \text{obs}}} (\mathbf{B} - \mathbf{B}_d) \cdot \mathcal{B}(\delta \mathbf{J}) + \alpha_J \int_{\Omega_{\mathbf{J}, \text{obs}}} (\mathbf{J} - \mathbf{J}_d) \cdot \delta \mathbf{J} = 0 \end{aligned} \quad (3.29c)$$

$$d_2(\mathbf{K}, \delta \phi) = 0 \quad (3.29d)$$

for all $(\delta \mathbf{u}, \delta p, \delta \mathbf{J}, \delta \phi) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times \mathbf{L}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$, and which satisfy the three scalar optimality conditions

$$\begin{aligned} -c(\mathbf{J}, \mathcal{B}(\tilde{\mathbf{J}}_{\text{ext}}), \mathbf{v}) + c(\mathbf{K}, \mathcal{B}(\tilde{\mathbf{J}}_{\text{ext}}), \mathbf{u}) + \gamma_{\text{ext}} I_{\text{ext}} + \alpha_B \int_{\Omega_{\mathbf{B}, \text{obs}}} (\mathbf{B} - \mathbf{B}_d) \cdot \mathcal{B}(\tilde{\mathbf{J}}_{\text{ext}}) = 0 \end{aligned} \quad (3.30a)$$

$$\begin{aligned} -c(\mathbf{J}, \mathcal{B}(\tilde{\mathbf{J}}_{\text{inj}}), \mathbf{v}) + c(\mathbf{K}, \mathcal{B}(\tilde{\mathbf{J}}_{\text{inj}}), \mathbf{u}) + \gamma_{\text{inj}} I_{\text{inj}} \\ + \alpha_B \int_{\Omega_{\mathbf{B}, \text{obs}}} (\mathbf{B} - \mathbf{B}_d) \cdot \mathcal{B}(\tilde{\mathbf{J}}_{\text{inj}}) - \langle \tilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}, \psi \rangle_{\partial \Omega \cap \partial \Omega_{\text{inj}}} = 0 \end{aligned} \quad (3.30b)$$

$$\begin{aligned} -c(\mathbf{J}, \tilde{\mathbf{B}}_{\text{ext}}, \mathbf{v}) + c(\mathbf{K}, \tilde{\mathbf{B}}_{\text{ext}}, \mathbf{u}) + \gamma_B B_{\text{ext}} + \alpha_B \int_{\Omega_{\mathbf{B}, \text{obs}}} (\mathbf{B} - \mathbf{B}_d) \cdot \tilde{\mathbf{B}}_{\text{ext}} = 0. \end{aligned} \quad (3.30c)$$

Proof. Our proof relies on a classical abstract multiplier result, see, e.g., Maurer and Zowe [18]. Since f is Fréchet differentiable and e is continuously Fréchet differentiable and e_x is assumed surjective at $(\hat{\mathbf{y}}, u)$, it follows that there exists a Lagrange multiplier $\lambda \in \hat{Y}$ which satisfies

$$f_x(\hat{\mathbf{y}}, u)(\delta y, \delta u) + \langle \lambda, e_x(\hat{\mathbf{y}}, u)(\delta y, \delta u) \rangle = 0 \quad (3.31)$$

for all $\delta y \in \hat{Y}$ and all $\delta u \in \mathbb{R}^3$. Above, we have used the notation $f_x = (f_y, f_u)$ and $e_x = (e_y, e_u)$ and the duality holds in $\hat{Y} \times \hat{Y}'$. It is now straightforward to verify that (3.31) is nothing else than (3.29)–(3.30). □

The elements of the Lagrange multiplier λ are termed the adjoint velocity \mathbf{v} , the adjoint pressure q , the adjoint current density \mathbf{K} , and the adjoint potential ψ , respectively, all defined on Ω .

In order to improve our understanding of the adjoint system (3.29), we also paraphrase it in its strong form. Exploiting the self-adjointness of the linear Biot-Savart operator \mathcal{B} , we obtain the dual state equations on Ω

$$\varrho(\nabla \mathbf{u})^\top \mathbf{v} - \varrho(\mathbf{u} \cdot \nabla) \mathbf{v} - \eta \Delta \mathbf{v} + \nabla q - \mathbf{B} \times \mathbf{K} = -\alpha_u \chi_{\Omega_{u,\text{obs}}}(\mathbf{u} - \mathbf{u}_d) \quad (3.32)$$

$$\begin{aligned} \sigma^{-1} \mathbf{K} - \mathcal{B}(\mathbf{K} \times \mathbf{u} + \mathbf{v} \times \mathbf{J}) + \nabla \psi - \mathbf{B} \times \mathbf{v} \\ = -\alpha_J \chi_{\Omega_{J,\text{obs}}}(\mathbf{J} - \mathbf{J}_d) - \alpha_B \chi_{\Omega_{B,\text{obs}} \cap \Omega} \mathcal{B}(\mathbf{B} - \mathbf{B}_d) \end{aligned} \quad (3.33)$$

plus incompressibility conditions

$$\nabla \cdot \mathbf{v} = 0 \quad \text{on } \Omega \qquad \nabla \cdot \mathbf{K} = 0 \quad \text{on } \Omega \quad (3.34)$$

and boundary conditions

$$\mathbf{v} = 0 \quad \text{on } \partial\Omega \qquad \mathbf{K} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \quad (3.35)$$

The symbol χ_A denotes the characteristic function of a set A . Note that the controls I_{ext} , I_{inj} and B_{ext} appear (hidden within \mathbf{B}) in the adjoint equations (3.29a) and (3.29c), or in (3.32)–(3.33) for that matter. The reason is that the state equation (3.1)–(3.2) contains the "state times control" terms $\mathbf{u} \times \mathbf{B}$ (through Ohm's Law (1.3)) and $\mathbf{J} \times \mathbf{B}$ (the Lorentz force).

Let us now turn to a condition which ensures the strict local optimality of a given point (y, u, λ) satisfying the first order optimality condition set forth in Theorem 3.11. To this end, we define the Lagrangian function for problem (\mathbf{P}) as

$$\begin{aligned} \mathcal{L}(\hat{y}, u, \lambda) &= \frac{\alpha_u}{2} \|\mathbf{u} - \mathbf{u}_d\|_{L^2(\Omega_{u,\text{obs}})}^2 + \frac{\alpha_B}{2} \|\mathbf{B} - \mathbf{B}_d\|_{L^2(\Omega_{B,\text{obs}})}^2 + \frac{\alpha_J}{2} \|\mathbf{J} - \mathbf{J}_d\|_{L^2(\Omega_{J,\text{obs}})}^2 \\ &+ \frac{\gamma_{\text{ext}}}{2} |I_{\text{ext}}|^2 + \frac{\gamma_{\text{inj}}}{2} |I_{\text{inj}}|^2 + \frac{\gamma_B}{2} |B_{\text{ext}}|^2 + a_1(\hat{\mathbf{u}} + \mathbf{u}_0, \mathbf{v}) - c(\hat{\mathbf{J}} + \mathbf{J}_0, \mathbf{B}, \mathbf{v}) \\ &+ b(\hat{\mathbf{u}} + \mathbf{u}_0, \hat{\mathbf{u}} + \mathbf{u}_0, \mathbf{v}) + d_1(\mathbf{v}, p) - d_1(\hat{\mathbf{u}}, q) + a_2(\hat{\mathbf{J}} + \mathbf{J}_0, \mathbf{K}) \\ &+ c(\mathbf{K}, \mathbf{B}, \hat{\mathbf{u}} + \mathbf{u}_0) + d_2(\mathbf{K}, \phi) - d_2(\hat{\mathbf{J}}, \psi) + \langle I_{\text{inj}} \tilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n}, \psi \rangle_{\partial\Omega \cap \partial\Omega_{\text{inj}}}, \end{aligned}$$

where again $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_0$ and $\mathbf{J} = \hat{\mathbf{J}} + \Lambda(I_{\text{inj}} \tilde{\mathbf{J}}_{\text{inj}} \cdot \mathbf{n})$. Moreover, \mathbf{u}_0 is the fixed lifting of the velocity boundary data \mathbf{h} from Lemma 3.2, and \mathbf{B} is still taken according to (3.14). It is readily checked that the following quadratic form is the Hessian of \mathcal{L} with respect to the state/control pair:

$$\begin{aligned} \mathcal{L}''(\hat{y}, u, \lambda)[(\delta y, \delta u)]^2 &= \alpha_u \|\delta \mathbf{u}\|_{L^2(\Omega_{u,\text{obs}})}^2 + \alpha_J \|\delta \mathbf{J}\|_{L^2(\Omega_{J,\text{obs}})}^2 + \alpha_B \|\delta \mathbf{B}\|_{L^2(\Omega_{B,\text{obs}})}^2 \\ &+ \gamma_{\text{ext}} |\delta I_{\text{ext}}|^2 + \gamma_{\text{inj}} |\delta I_{\text{inj}}|^2 + \gamma_B |\delta B_{\text{ext}}|^2 + 2b(\delta \mathbf{u}, \delta \mathbf{u}, \mathbf{v}) - 2c(\delta \mathbf{J}, \delta \mathbf{B}, \mathbf{v}) + 2c(\mathbf{K}, \delta \mathbf{B}, \delta \mathbf{u}) \end{aligned}$$

with the abbreviation

$$\delta \mathcal{B} = \mathcal{B}(\delta \mathbf{J}) + \delta I_{\text{inj}} \mathcal{B}(\tilde{\mathbf{J}}_{\text{inj}}) + \delta I_{\text{ext}} \mathcal{B}(\tilde{\mathbf{J}}_{\text{ext}}) + \delta B_{\text{ext}} \tilde{\mathbf{B}}_{\text{ext}}.$$

Proposition 3.12 (Second order sufficient conditions).

Suppose that (\hat{y}, u, λ) satisfies the optimality system (3.13)–(3.14), (3.29)–(3.30), and that $e_y(\hat{y}, u)$ is boundedly invertible. If moreover

$$\alpha_u \|\mathbf{u} - \mathbf{u}_d\|_{L^2(\Omega_{u,\text{obs}})}^2 + \alpha_B \|\mathbf{B} - \mathbf{B}_d\|_{L^2(\Omega_{B,\text{obs}})}^2 + \alpha_J \|\mathbf{J} - \mathbf{J}_d\|_{L^2(\Omega_{J,\text{obs}})}^2$$

is sufficiently small, then there exists a neighborhood \mathcal{U} of (\hat{y}, u) and $\kappa > 0$ such that

$$f(\bar{y}, \bar{u}) \geq f(\hat{y}, u) + \kappa \left(\|\bar{u} - u\|^2 + \|\bar{y} - \hat{y}\|_{\mathcal{Y}}^2 \right)$$

holds for all $(\bar{y}, \bar{u}) \in \mathcal{U}$ satisfying the state equation. In particular, (\hat{y}, u) is a strict local optimum for (\mathbf{P}) .

Proof. We shall argue that there exists $\rho > 0$ such that the coercivity condition

$$\mathcal{L}''(\hat{y}, u, \lambda)[(\delta y, \delta u)]^2 \geq \rho \left(\|\delta y\|_{\hat{Y}}^2 + |\delta u|^2 \right) \quad (3.36)$$

holds for all $(\delta y, \delta u) \in \hat{Y} \times \mathbb{R}^3$ which satisfy the linear MHD system (see Lemma 3.5)

$$e_y(\hat{y}, u) \delta y + e_u(\hat{y}, u) \delta u = 0. \quad (3.37)$$

The claim then follows from a Taylor series expansion of \mathcal{L} at (\hat{y}, u, λ) , see, e.g., [18, Theorem 5.6]. Since $e_y(\hat{y}, u)$ is surjective, $e_y^*(\hat{y}, u) : \hat{Y} \rightarrow \hat{Y}'$ has closed range and is continuously invertible on its range [2]. Hence in view of (3.32)–(3.35), there exists $\kappa_1 > 0$ such that

$$\begin{aligned} & \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{K}\|_{\mathbf{L}^2(\Omega)} \\ & \leq \kappa_1 \left(\alpha_u \|\mathbf{u} - \mathbf{u}_d\|_{\mathbf{L}^2(\Omega_{u,\text{obs}})}^2 + \alpha_B \|\mathbf{B} - \mathbf{B}_d\|_{\mathbf{L}^2(\Omega_{B,\text{obs}})}^2 + \alpha_J \|\mathbf{J} - \mathbf{J}_d\|_{\mathbf{L}^2(\Omega_{J,\text{obs}})}^2 \right) \end{aligned}$$

holds. From (3.37) and the bounded invertibility of $e_y(\hat{y}, u)$ we have

$$\|\delta \mathbf{J}\|_{\mathbf{L}^2(\Omega)} + \|\delta \mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq 2 \|\delta y\|_{\hat{Y}} \leq \kappa_2 |\delta u| \quad (3.38)$$

for a constant $\kappa_2 > 0$ independent of $\delta u \in \mathbb{R}^3$. Hence

$$\begin{aligned} & |b(\delta \mathbf{u}, \delta \mathbf{u}, \mathbf{v})| \leq \kappa_3 |\delta u|^2 \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \\ & \leq \kappa_1 \kappa_3 |\delta u|^2 \left(\alpha_u \|\mathbf{u} - \mathbf{u}_d\|_{\mathbf{L}^2(\Omega_{u,\text{obs}})}^2 + \alpha_B \|\mathbf{B} - \mathbf{B}_d\|_{\mathbf{L}^2(\Omega_{B,\text{obs}})}^2 + \alpha_J \|\mathbf{J} - \mathbf{J}_d\|_{\mathbf{L}^2(\Omega_{J,\text{obs}})}^2 \right). \end{aligned} \quad (3.39)$$

Further there exists κ_4 independent of δu such that

$$\begin{aligned} & |c(\delta \mathbf{J}, \delta \mathbf{B}, \mathbf{v})| + |c(\mathbf{K}, \delta \mathbf{B}, \delta \mathbf{u})| \\ & \leq \kappa_4 \left(\|\delta \mathbf{J}\|_{\mathbf{L}^2(\Omega)} + |\delta u| \right) \left(\|\delta \mathbf{J}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{K}\|_{\mathbf{L}^2(\Omega)} \|\delta \mathbf{u}\|_{\mathbf{H}^1(\Omega)} \right) \\ & \leq C \left(\alpha_u \|\mathbf{u} - \mathbf{u}_d\|_{\mathbf{L}^2(\Omega_{u,\text{obs}})}^2 + \alpha_B \|\mathbf{B} - \mathbf{B}_d\|_{\mathbf{L}^2(\Omega_{B,\text{obs}})}^2 + \alpha_J \|\mathbf{J} - \mathbf{J}_d\|_{\mathbf{L}^2(\Omega_{J,\text{obs}})}^2 \right), \end{aligned}$$

where $C = \kappa_1 \kappa_2 (\kappa_2 + 1) \kappa_4 |\delta u|^2$. This last estimate, together with (3.38) and (3.39) implies (3.36). \square

4. AN OPERATOR SPLITTING SCHEME

In this section, we address an operator splitting scheme for the numerical solution of the MHD state equation (3.13)–(3.14). A different iterative scheme has been proposed in [20], which lags the first or second arguments in the trilinear forms b and c , respectively. Like Newton's method, this scheme requires the repeated solution of *coupled* linear systems for (\mathbf{u}, \mathbf{J}) and their respective pressure and potential fields. In contrast, our approach is based on the hypothesis that one wants to put to use existing and validated solvers for the Navier-Stokes equations and for div-curl systems. To this end, we propose the following iterative method to compute a solution of the MHD system for given controls $u = (I_{\text{ext}}, I_{\text{inj}}, B_{\text{ext}}) \in \mathbb{R}^3$ and given velocity boundary data \mathbf{h} . As before, \mathbf{u}_0 and \mathbf{J}_0 denote the liftings according to Lemma 3.2.

Algorithm 4.1 (Operator splitting scheme).

- (1) Choose an initial guess $\hat{\mathbf{J}}^0 \in \mathbf{L}_{\text{div}}^2(\Omega)$, set $n = 0$.

(2) Solve the div-curl system for $\mathbf{B}^{n+1} \in \mathbf{V}^1(\mathbb{R}^3)$

$$\nabla \cdot \mathbf{B}^{n+1} = 0 \quad \nabla \times (\mu^{-1} \mathbf{B}^{n+1}) = \widehat{\mathbf{J}}^n$$

with the interface condition $[\mathbf{B}^{n+1}]_{\partial\Omega} = 0$.

(3) Solve the Navier-Stokes system with Lorentz force for $\widehat{\mathbf{u}}^{n+1} \in \mathbf{H}_0^1(\Omega)$ and $p^{n+1} \in L^2(\Omega)/\mathbb{R}$

$$\begin{aligned} -\eta \Delta \widehat{\mathbf{u}}^{n+1} + \rho(\widehat{\mathbf{u}}^{n+1} \cdot \nabla) \widehat{\mathbf{u}}^{n+1} + \rho(\widehat{\mathbf{u}}^{n+1} \cdot \nabla) \mathbf{u}_0 + \rho(\mathbf{u}_0 \cdot \nabla) \widehat{\mathbf{u}}^{n+1} + \nabla p^{n+1} \\ = \eta \Delta \mathbf{u}_0 - \rho(\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + (\widehat{\mathbf{J}}^n + \mathbf{J}_0) \times (\mathbf{B}^{n+1} + \mathbf{B}_0) \\ \nabla \cdot \widehat{\mathbf{u}}^{n+1} = 0 \end{aligned}$$

and homogeneous Dirichlet boundary data on $\partial\Omega$.

(4) Solve for $\widehat{\mathbf{J}}^{n+1} \in \mathbf{L}_{\text{div}}^2(\Omega)$ and $\phi^{n+1} \in H^1(\Omega)/\mathbb{R}$

$$\begin{aligned} \sigma^{-1} \widehat{\mathbf{J}}^{n+1} + \nabla \phi^{n+1} &= (\widehat{\mathbf{u}}^{n+1} + \mathbf{u}_0) \times (\mathbf{B}^{n+1} + \mathbf{B}_0) - \sigma^{-1} \mathbf{J}_0 \\ \nabla \cdot \widehat{\mathbf{J}}^{n+1} &= 0 \end{aligned}$$

with boundary condition $\widehat{\mathbf{J}}^{n+1} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

(5) Unless $\|\widehat{\mathbf{J}}^{n+1} - \widehat{\mathbf{J}}^n\|_{L^2(\Omega)}$ is sufficiently small, increase n and go to (2).

Note that the solution to step (2) is given by the Biot-Savart operator $\mathcal{B}(\widehat{\mathbf{J}}^n)$. In steps (3) and (4), $\mathbf{B}_0 = \mathcal{B}(\mathbf{J}_0) + \mathcal{B}(\mathbf{J}_{\text{ext}}) + \mathcal{B}(\mathbf{J}_{\text{inj}}) + \mathbf{B}_{\text{ext}}$ collects the constant contributions to the total magnetic field. Obviously, instead of computing the liftings \mathbf{u}_0 and \mathbf{J}_0 and repeatedly solving homogeneous problems in steps (3) and (4), one may directly address the inhomogeneous ones with unknowns $\widehat{\mathbf{u}}^{n+1} + \mathbf{u}_0$ and $\widehat{\mathbf{J}}^{n+1} + \mathbf{J}_0$. The same applies to the div-curl system in step (2), which yields $\mathbf{B}^{n+1} + \mathbf{B}_0 - \mathbf{B}_{\text{ext}}$ if $\widehat{\mathbf{J}}^n$ is replaced by $\widehat{\mathbf{J}}^n + \mathbf{J}_0 + \mathbf{J}_{\text{ext}} + \mathbf{J}_{\text{inj}}$.

Remark 4.2 (Alternative form of step (4)).

Note that step (4) in Algorithm 4.1 above is equivalent to the solution of the div-curl system on Ω

$$\begin{aligned} \nabla \cdot \widehat{\mathbf{J}}^{n+1} &= 0 \\ \nabla \times (\sigma^{-1} \widehat{\mathbf{J}}^{n+1}) &= \nabla \times \left[(\widehat{\mathbf{u}}^{n+1} + \mathbf{u}_0) \times (\mathbf{B}^{n+1} + \mathbf{B}_0) \right] - \nabla \times (\sigma^{-1} \mathbf{J}_0) \end{aligned}$$

with boundary condition $\widehat{\mathbf{J}}^{n+1} \cdot \mathbf{n} = 0$ on $\partial\Omega$, provided that the right hand side is in $\mathbf{L}^2(\Omega)$. This can be guaranteed if \mathbf{J}_0 , \mathbf{B}_{ext} and $\partial\Omega$ are smooth enough.

For the proposed scheme, we have the following conditional convergence result:

Proposition 4.3 (Convergence of the operator splitting scheme).

Let $u \in \mathbb{R}^3$ and $\mathbf{h} \in \mathbf{H}^{1/2}(\partial\Omega)$ be a given control vector and boundary data and suppose that

- (a) η is sufficiently large, or $|u|$ and $\|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)}$ are sufficiently small; and that
- (b) σ is sufficiently small, or μ and $|B_{\text{ext}}|$ are sufficiently small.

Then there exists $\rho_J > 0$ such that whenever the initial iterate $\widehat{\mathbf{J}}^0 \in \mathbf{L}_{\text{div}}^2(\Omega)$ satisfies $\|\widehat{\mathbf{J}}^0 + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} < \rho_J$, then the iterates $(\widehat{\mathbf{J}}^n, \widehat{\mathbf{u}}^n)$ of Algorithm 4.1 converge in $\mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Omega)$ to the necessarily unique solution of (3.13)–(3.14) which satisfies $\|\widehat{\mathbf{J}} + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} \leq \rho_J$.

Proof. The proof uses the Banach fixed point theorem. This choice is due to the fact that the nonlinearities in (3.13)–(3.14) are not of strictly monotone or energy preservation type, so that techniques analogous to those developed for decomposition methods, e.g., in [8], cannot be used. Let $T : \mathbf{L}_{\text{div}}^2(\Omega) \rightarrow \mathbf{L}_{\text{div}}^2(\Omega)$ denote the operator which assigns to $\widehat{\mathbf{J}}^n$ the value $\widehat{\mathbf{J}}^{n+1}$ defined by steps (2)–(4) of Algorithm 4.1. Let us denote by ρ_I a common bound for the inhomogeneities $\tilde{u} = (I_{\text{ext}}, I_{\text{inj}})$ and \mathbf{h} , i.e., $|\tilde{u}| \leq \rho_I$ and $\|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)} \leq \rho_I$. Given the solenoidal current field $\widehat{\mathbf{J}}^n$, we infer from Lemma 2.4 the existence of \mathbf{B}^{n+1} satisfying the equations in step (2) and the a priori estimate

$$\|\mathbf{B}^{n+1} + \mathbf{B}_0\| \leq c_1 \mu \left(\|\widehat{\mathbf{J}}^n + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} + |\tilde{u}| \right) + c_1 |\mathbf{B}_{\text{ext}}|.$$

Here and below, the constants c_i are independent of μ, η, σ , iteration index n and controls u . Let us further assume that $\|\widehat{\mathbf{J}}^n + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} \leq \rho_J$. Then we have

$$\|\mathbf{B}^{n+1} + \mathbf{B}_0\|_{\mathbf{L}^3(\Omega)} \leq c_1 (\mu \rho_J + \mu \rho_I + |\mathbf{B}_{\text{ext}}|). \quad (4.1)$$

Standard estimates for the Navier-Stokes equations in step (3) imply that

$$\begin{aligned} \|\widehat{\mathbf{u}}^{n+1} + \mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} &\leq \mu \eta^{-1} c_2 \left(\|\widehat{\mathbf{J}}^n + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)}^2 + |\tilde{u}|^2 \right) \\ &\quad + c_2 \|\widehat{\mathbf{J}}^n + \mathbf{J}_0\| |\mathbf{B}_{\text{ext}}| + c_2 \left(\|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)}^2 \right) \\ &\leq c_2 \left(\mu \eta^{-1} (\rho_J^2 + \rho_I^2) + \rho_J |\mathbf{B}_{\text{ext}}| + \rho_I + \rho_I^2 \right). \end{aligned} \quad (4.2)$$

By Lemma 2.1 and a direct computation (or [7, Ch. I, Cor. 4.1]), the system

$$\sigma^{-1} \widehat{\mathbf{J}} + \nabla \phi = f \text{ on } \Omega \quad \nabla \cdot \widehat{\mathbf{J}} = 0 \text{ on } \Omega$$

with given $f \in \mathbf{L}^2(\Omega)$ has a unique solution $\mathbf{J} \in \mathbf{L}_{\text{div}}^2(\Omega)$ and $\phi \in H^1(\Omega)/\mathbb{R}$ which satisfies $\|\mathbf{J}\|_{\mathbf{L}^2(\Omega)} \leq \sigma \|f\|_{\mathbf{L}^2(\Omega)}$. Hence we conclude from step (4) that

$$\begin{aligned} \|\widehat{\mathbf{J}}^{n+1} + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} &\leq \sigma c'_3 \left(\mu \eta^{-1} \left(\|\widehat{\mathbf{J}}^n + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)}^2 + |\tilde{u}|^2 \right) + \|\widehat{\mathbf{J}}^n + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} |\mathbf{B}_{\text{ext}}| \right. \\ &\quad \left. + \|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)} + \|\mathbf{h}\|_{\mathbf{H}^{1/2}(\partial\Omega)}^2 \right) \left(\mu \left(\|\widehat{\mathbf{J}}^n + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} + |\tilde{u}| \right) + |\mathbf{B}_{\text{ext}}| \right) + c'_3 |I_{\text{inj}}| \\ &\leq \sigma c'_3 \left(\mu \eta^{-1} (\rho_J^2 + \rho_I^2) + \rho_I |\mathbf{B}_{\text{ext}}| + \rho_I + \rho_I^2 \right) \left(\mu (\rho_J + \rho_I) + |\mathbf{B}_{\text{ext}}| \right) + c'_3 |I_{\text{inj}}| \\ &\leq \sigma c_3 \left(\mu^2 \eta^{-1} (\rho_J^3 + \rho_I^3) + |\mathbf{B}_{\text{ext}}| \left(\mu \rho_J (\rho_J + \rho_I) + \mu \eta^{-1} (\rho_J^2 + \rho_I^2) + \rho_J |\mathbf{B}_{\text{ext}}| \right) \right. \\ &\quad \left. + (\rho_I + \rho_I^2) (\mu (\rho_J + \rho_I) + |\mathbf{B}_{\text{ext}}|) \right) + c_3 |I_{\text{inj}}|. \end{aligned} \quad (4.3)$$

Concerning the initialization, note that $\widehat{\mathbf{J}}^0$ can be taken zero. Then $\|\widehat{\mathbf{J}}^0 + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)}$ is bounded by $c_3 |I_{\text{inj}}|$. Choosing $\rho_J := 2c_3 |I_{\text{inj}}|$ and assuming that $\|\widehat{\mathbf{J}}^n + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} \leq \rho_J$, then we obtain from (4.3) that $\|\widehat{\mathbf{J}}^{n+1} + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} \leq \rho_J$, provided that, for instance, σ is sufficiently small, or μ and $|\mathbf{B}_{\text{ext}}|$ are sufficiently small. From (4.1) and (4.2) follows the existence of constants ρ_B and ρ_u independent of n such that

$$\|\mathbf{B}^n + \mathbf{B}_0\|_{\mathbf{L}^3(\Omega)} \leq \rho_B \quad \text{and} \quad \|\widehat{\mathbf{u}}^n + \mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \leq \rho_u \quad (4.4)$$

for all n . To prove that T is a contraction, let $\widehat{\mathbf{J}}_i \in \mathbf{L}_{\text{div}}^2(\Omega)$, $i = 1, 2$, and let \mathbf{K}_i be their images under T . Further let \mathbf{B}_i and $\widehat{\mathbf{u}}_i$ denote the associated magnetic and velocity fields according to Algorithm 4.1. Then

$$\|\mathbf{B}_1 - \mathbf{B}_2\|_{\mathbf{H}^1(\Omega)} \leq \mu c_4 \|\widehat{\mathbf{J}}_1 - \widehat{\mathbf{J}}_2\|_{\mathbf{L}^2(\Omega)}. \quad (4.5)$$

Here and below, the constants are also independent of $\widehat{\mathbf{J}}_1$ and $\widehat{\mathbf{J}}_2$. Moreover, $U = \widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2 \in \mathbf{H}_0^1(\Omega)$ satisfies

$$\begin{aligned} -\eta \Delta U + \rho(U \cdot \nabla)(\widehat{\mathbf{u}}_1 + \mathbf{u}_0) + \rho((\widehat{\mathbf{u}}_2 + \mathbf{u}_0) \cdot \nabla)U + \nabla P \\ = (\widehat{\mathbf{J}}_1 - \widehat{\mathbf{J}}_2) \times (\mathbf{B}_1 + \mathbf{B}_2) + (\widehat{\mathbf{J}}_2 + \mathbf{J}_0) \times (\mathbf{B}_1 - \mathbf{B}_2) \end{aligned}$$

for some $P \in L^2(\Omega)/\mathbb{R}$. This implies that

$$\begin{aligned} \eta \|\nabla U\|_{\mathbf{L}^2(\Omega)}^2 &\leq c_5 \left(\rho_u \|\nabla U\|_{\mathbf{L}^2(\Omega)}^2 + \rho_B \|\nabla U\|_{\mathbf{L}^2(\Omega)} \|\widehat{\mathbf{J}}_1 - \widehat{\mathbf{J}}_2\|_{\mathbf{L}^2(\Omega)} \right. \\ &\quad \left. + \rho_J \|\nabla U\|_{\mathbf{L}^2(\Omega)} \|\mathbf{B}_1 - \mathbf{B}_2\|_{\mathbf{L}^3(\Omega)} \right) \\ &\leq c_5 \left(\rho_u \|\nabla U\|_{\mathbf{L}^2(\Omega)}^2 + (\rho_B + \mu c_4 \rho_J) \|\nabla U\|_{\mathbf{L}^2(\Omega)} \|\widehat{\mathbf{J}}_1 - \widehat{\mathbf{J}}_2\|_{\mathbf{L}^2(\Omega)} \right) \\ &\leq c_5 \left(\rho_u \|\nabla U\|_{\mathbf{L}^2(\Omega)}^2 + \eta/2 \|\nabla U\|_{\mathbf{L}^2(\Omega)}^2 + (\rho_B + \mu c_4 \rho_J)^2 / (2\eta) \|\widehat{\mathbf{J}}_1 - \widehat{\mathbf{J}}_2\|_{\mathbf{L}^2(\Omega)}^2 \right). \end{aligned}$$

Hence if η is sufficiently large, or ρ_u is sufficiently small (which can be achieved by μ , $|B_{\text{ext}}|$ and ρ_I sufficiently small), we have

$$\|\nabla U\|_{\mathbf{L}^2(\Omega)} \leq c_6 \eta^{-1} (\rho_B + \mu c_4 \rho_J) \|\widehat{\mathbf{J}}_1 - \widehat{\mathbf{J}}_2\|_{\mathbf{L}^2(\Omega)}. \quad (4.6)$$

Finally, we obtain from step (4)

$$\begin{aligned} \|\mathbf{K}_1 - \mathbf{K}_2\| &\leq \sigma c_7 \left(\|U\|_{\mathbf{H}^1(\Omega)} \|\mathbf{B}_1 + \mathbf{B}_0\|_{\mathbf{L}^3(\Omega)} + \|\widehat{\mathbf{u}}_2 + \mathbf{u}_0\|_{\mathbf{H}^1(\Omega)} \|\mathbf{B}_1 - \mathbf{B}_2\|_{\mathbf{L}^3(\Omega)} \right) \\ &\leq \sigma c_7 \left(\eta^{-1} \rho_B (\rho_B + \mu \rho_J) + \mu c_4 \rho_u \right) \|\widehat{\mathbf{J}}_1 - \widehat{\mathbf{J}}_2\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \quad (4.7)$$

Hence we conclude that if σ is sufficiently small, or if μ is sufficiently small and η sufficiently large, then T is a contraction on the ball $\{\mathbf{J} : \|\widehat{\mathbf{J}} + \mathbf{J}_0\|_{\mathbf{L}^2(\Omega)} < \rho_J\}$. \square

5. CONCLUSION AND OUTLOOK

In this paper, we have presented and analyzed a practical optimal control problem for the stationary MHD system. We have given a new proof of existence concerning the state equation and derived necessary and sufficient conditions for local optimal solutions. In addition, we presented an iterative scheme for the numerical solution of the MHD state equations which is tailored to make use of existing Navier-Stokes and div-curl solvers. We believe that in the face of industrial MHD applications, there is ample room to extend our results in several directions. Of particular interest are the cases of instationary MHD flows, unknown current densities in external conductors, flows with thermal coupling and Ohmic heating, and the case of material-dependent magnetic permeability. All of the above present additional technical difficulties which are the subject of future investigations. Finally, devising an efficient numerical algorithm to solve optimal control problems involving MHD flows presents another challenging task.

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