On the X-ray transform of planar symmetric 2-tensors
ON THE X-RAY TRANSFORM OF PLANAR SYMMETRIC
2-TENSORS

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ABSTRACT. In this paper we study the attenuated X-ray transform of 2-tensors supported in strictly convex bounded subsets in the Euclidean plane. We characterize its range and reconstruct all possible 2-tensors yielding identical X-ray data. The characterization is in terms of a Hilbert-transform associated with $A$-analytic maps in the sense of Bukhgeim.

1. INTRODUCTION

This paper concerns the range characterization of the attenuated X-ray transform of symmetric 2-tensors in the plane. Range characterization of the non-attenuated X-ray transform of functions (0-tensors) in the Euclidean space has been long known [10, 11, 19], whereas in the case of a constant attenuation some range conditions can be inferred from [17, 1, 2]. For a varying attenuation the two dimensional case has been particularly interesting with inversion formulas requiring new analytical tools: the theory of $A$-analytic maps originally employed in [3], and ideas from inverse scattering in [24]. Constraints on the range for the two dimensional X-ray transform of functions were given in [25, 4], and a range characterization based on Bukhgeim’s theory of $A$-analytic maps was given in [30].

Inversion of the X-ray transform of higher order tensors has been formulated directly in the setting of Riemmanian manifolds with boundary [32]. The case of 2-tensors appears in the linearization of the boundary rigidity problem. It is easy to see that injectivity can hold only in some restricted class: e.g., the class of solenoidal tensors. For two dimensional simple manifolds with boundary, injectivity with in the solenoidal tensor fields has been establish fairly recent: in the non-attenuated case for 0- and 1-tensors we mention the breakthrough result in [29], and in the attenuated case in [34]; see also [13] for a more general weighted transform. Inversion for the attenuated X-ray transform for solenoidal tensors of rank two and higher can be found in [27], with a range characterization in [28]. In the Euclidean

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case we mention an earlier inversion of the attenuated X-ray transform of solenoidal tensors in [16]; however this work does not address range characterization.

Different from the recent characterization in terms of the scattering relation in [28], in this paper the range conditions are in terms of the Hilbert-transform for $A$-analytic maps introduced in [30, 31]. Our characterization can be understood as an explicit description of the scattering relation in particularized to the Euclidean setting. In the sufficiency part we reconstruct all possible 2-tensors yielding identical X-ray data; see (30) for the non-attenuated case and (82) for the attenuated case.

For a real symmetric 2-tensor $F \in L^1(\mathbb{R}^2; \mathbb{R}^{2\times2})$,

$$F(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{12}(x) & f_{22}(x) \end{pmatrix}, \quad x \in \mathbb{R}^2,$$

and a real valued function $a \in L^1(\mathbb{R}^2)$, the $a$-attenuated X-ray transform of $F$ is defined by

$$X_a F(x, \theta) := \int_{-\infty}^{\infty} \langle F(x + t\theta) \theta, \theta \rangle \exp \left\{ - \int_t^\infty a(x + s\theta) \, ds \right\} \, dt,$$

where $\theta$ is a direction in the unit sphere $S^1$, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^2$. For the non attenuated case $a \equiv 0$ we use the notation $X F$.

In this paper, we consider $F$ be defined on a strictly convex bounded set $\Omega \subset \mathbb{R}^2$ with vanishing trace at the boundary $\Gamma$; further regularity and the order of vanishing will be specified in the theorems. In the attenuated case we assume $a > 0$ in $\Omega$.

For any $(x, \theta) \in \bar{\Omega} \times S^1$ let $\tau(x, \theta)$ be length of the chord in the direction of $\theta$ passing through $x$. Let also consider the incoming $(-)$, respectively outgoing $(+)$ submanifolds of the unit bundle restricted to the boundary

$$\Gamma_\pm := \{(x, \theta) \in \Gamma \times S^1 : \pm \theta \cdot n(x) > 0\},$$

and the variety

$$\Gamma_0 := \{(x, \theta) \in \Gamma \times S^1 : \theta \cdot n(x) = 0\},$$

where $n(x)$ denotes outer normal.

The $a$-attenuated X-ray transform of $F$ is realized as a function on $\Gamma_+$ by

$$X_a F(x, \theta) = \int_{-\tau(x, \theta)}^{0} \langle F(x + t\theta) \theta, \theta \rangle e^{-\int_t^0 a(x + s\theta) \, ds} \, dt, \quad (x, \theta) \in \Gamma_+.$$

We approach the range characterization through its connection with the transport model as follows: The boundary value problem

$$\theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) = \langle F(x) \theta, \theta \rangle \quad (x, \theta) \in \Omega \times S^1,$$

$$u|_{\Gamma_-} = 0$$

has a unique solution in $\Omega \times S^1$ and

$$u|_{\Gamma_+}(x, \theta) = X_n F(x, \theta), \quad (x, \theta) \in \Gamma_+. \tag{8}$$

The $X$-ray transform of 2-tensors occurs in the linearization of the boundary rigidity problem [32]: For $\epsilon > 0$ small, let

$$g'(x) := I + \epsilon F(x) + o(\epsilon), \quad x \in \Omega,$$

be a family of metrics perturbations from the Euclidean, where $I$ is the identity matrix and $F$ is as in (1). For an arbitrary pair of boundary points $x, y \in \Gamma$ let $d_\epsilon(x, y)$ denote their distance in the metric $g^\epsilon$. The boundary rigidity problem asks for the recovery of the metric $g^\epsilon$ from knowledge of $d_\epsilon(x, y)$ for all $x, y \in \Gamma$. In the linearized case one seeks to recover $F(x)$ from $\frac{d}{d \epsilon}|_{\epsilon=0} d_\epsilon^2(x, y)$. Taking into account the length minimizing property of geodesic one can show that

$$\frac{1}{|x-y|} \left. \frac{d}{d \epsilon} \right|_{\epsilon=0} d_\epsilon^2(x, y) = \int_{-|x-y|}^{0} \langle F(x + t\theta)\theta, \theta \rangle dt = X F(x, \theta),$$

where $\theta := \frac{x - y}{|x-y|} \in S^1$.

2. Preliminaries

In this section we briefly introduce the properties of Bukhgeim’s $A$-analytic maps [7] needed later.

For $z = x_1 + ix_2$, we consider the Cauchy-Riemann operators

$$\overline{\partial} = (\partial_{x_1} + i\partial_{x_2}) / 2, \quad \partial = (\partial_{x_1} - i\partial_{x_2}) / 2. \tag{9}$$
Let $l_{\infty}(l_1)$ be the space of bounded (respectively summable) sequences, $L : l_{\infty} \to l_{\infty}$ be the left shift

$$L(u_{-1}, u_{-2}, \ldots) = (u_{-2}, u_{-3}, u_{-4}, \ldots).$$

**Definition 2.1.** A sequence valued map

$$z \mapsto u(z) := (u_{-1}(z), u_{-2}(z), u_{-3}(z), \ldots)$$

is called $L$-analytic, if $u \in C(\Omega; l_{\infty}) \cap C^1(\Omega; l_{\infty})$ and

$$\partial u(z) + L \partial u(z) = 0, \quad z \in \Omega.$$

For $0 < \alpha < 1$ and $k = 1, 2$, we recall the Banach spaces in [30]:

$$l^1, k_{\infty}(\Gamma) := \left\{ u = (u_{-1}, u_{-2}, \ldots) : \sup_{\zeta \in \Gamma} \sum_{j=1}^{\infty} j^k |u_{-j}(\zeta)| < \infty \right\},$$

$$C^\alpha(\Gamma; l_1) := \left\{ u : \sup_{\zeta \in \Gamma} \|u(\zeta)\|_{l_1} + \sup_{\zeta, \eta \in \Gamma, \zeta \neq \eta} \frac{\|u(\zeta) - u(\eta)\|_{l_1}}{|\zeta - \eta|^\alpha} < \infty \right\}.$$

By replacing $\Gamma$ with $\Omega$ and $l_1$ with $l_{\infty}$ in (12) we similarly define $C^\alpha(\Omega; l_1)$, respectively, $C^\alpha(\Omega; l_{\infty})$.

At the heart of the theory of $A$-analytic maps lies a Cauchy-like integral formula introduced by Bukhgeim in [7]. The explicit variant (13) appeared first in Finch [8]. The formula below is restated in terms of $L$-analytic maps as in [31].

**Theorem 2.1.** [31, Theorem 2.1] For some $g = (g_{-1}, g_{-2}, g_{-3}, \ldots) \in l^1, k_{\infty}(\Gamma) \cap C^\alpha(\Gamma; l_1)$ define the Bukhgeim-Cauchy operator $B$ acting on $g$,

$$\Omega \ni z \mapsto ((Bg)_{-1}(z), (Bg)_{-2}(z), (Bg)_{-3}(z), \ldots),$$

by

$$(Bg)_{-n}(z) := \frac{1}{2\pi i} \int_{\Gamma} g_{-n-j}(\zeta) \frac{(\zeta - z)^j}{(\zeta - z)^{j+1}} d\zeta$$

and

$$-\frac{1}{2\pi i} \int_{\Gamma} g_{-n-j}(\zeta) \frac{(\zeta - z)^{j-1}}{(\zeta - z)^j} d\zeta, \quad n = 1, 2, 3, \ldots$$

Then $Bg \in C^{1, \alpha}(\Omega; l_{\infty}) \cap C(\Omega; l_{\infty})$ and it is also $L$-analytic.

For our purposes further regularity in $Bg$ will be required. Such smoothness is obtained by increasing the assumptions on the rate of decay of the
terms in $g$ as explicit below. For $0 < \alpha < 1$, let us recall the Banach space $Y_{\alpha}$ in [30]:

$$Y_{\alpha} = \left\{ g \in l_{1,2}^1(\Gamma) : \sup_{\xi, \mu \in \Gamma, \xi \neq \mu} \sum_{j=1}^{\infty} \frac{|g_{-j}(\xi) - g_{-j}(\mu)|}{|\xi - \mu|^\alpha} < \infty \right\}.$$  

**Proposition 2.1.** [31, Proposition 2.1] If $g \in Y_{\alpha}, \alpha > 1/2$, then

$$B_{g} \in C^{1,\alpha}(\Omega; l_{1}) \cap C^{\alpha}(\Omega; l_{1}) \cap C^{2}(\Omega; l_{\infty}).$$  

The Hilbert transform associated with boundary of $L$-analytic maps is defined below.

**Definition 2.2.** For $g = \langle g_{-1}, g_{-2}, g_{-3}, \ldots \rangle \in l_{1,1}^1(\Gamma) \cap C^{\alpha}(\Gamma; l_{1})$, we define the Hilbert transform $H_{g}$ componentwise for $n \geq 1$ by

$$(H_{g})_{-n}(\xi) = \frac{1}{\pi} \int_{\Gamma} \frac{g_{-n}(\zeta)}{\zeta - \xi} d\zeta$$

$$+ \frac{1}{\pi} \int_{\Gamma} \left\{ \frac{d\zeta}{\zeta - \xi} - \frac{d\overline{\zeta}}{\zeta - \overline{\xi}} \right\} \sum_{j=1}^{\infty} g_{-n-j}(\zeta) \left( \frac{\overline{\zeta} - \xi}{\zeta - \xi} \right)^{j}, \xi \in \Gamma.$$  

The following result justifies the name of the transform $H$. For its proof we refer to [30, Theorem 3.2].

**Theorem 2.2.** For $0 < \alpha < 1$, let $g \in l_{1,1}^1(\Gamma) \cap C^{\alpha}(\Gamma; l_{1})$. For $g$ to be boundary value of an $L$-analytic function it is necessary and sufficient that

$$(I + iH)g = 0,$$

where $H$ is as in (16).

3. THE NON-ATTENUATED CASE

In this section we assume $a \equiv 0$. We establish necessary and sufficient conditions for a sufficiently smooth function on $\Gamma \times S^1$ to be the X-ray data of some sufficiently smooth real valued symmetric 2-tensor $F$. For $\theta = (\cos \varphi, \sin \varphi) \in S^1$, a calculation shows that

$$(\mathbf{F}(x)\theta, \theta) = f_{0}(x) + \overline{f_{2}(x)}e^{2i\varphi} + f_{2}(x)e^{-2i\varphi},$$

where

$$f_{0}(x) = \frac{f_{11}(x) + f_{22}(x)}{2},$$

and

$$f_{2}(x) = \frac{f_{11}(x) - f_{22}(x)}{4} + i\frac{f_{12}(x)}{2}.$$  

The transport equation in (6) becomes

$$\theta \cdot \nabla u(x, \theta) = f_{0}(x) + \overline{f_{2}(x)}e^{2i\varphi} + f_{2}(x)e^{-2i\varphi}, \quad x \in \Omega.$$
For $z = x_1 + ix_2 \in \Omega$, we consider the Fourier expansions of $u(z, \cdot)$ in the angular variable $\theta = (\cos \varphi, \sin \varphi)$:

$$u(z, \theta) = \sum_{-\infty}^{\infty} u_n(z) e^{i n \varphi}.$$ 

Since $u$ is real valued its Fourier modes occur in conjugates,

$$u_{-n}(z) = \overline{u_n(z)}, \quad n \geq 0, \quad z \in \Omega.$$

With the Cauchy-Riemann operators defined in (9) the advection operator becomes

$$\theta \cdot \nabla = e^{-i \varphi} \partial + e^{i \varphi} \partial.$$

Provided appropriate convergence of the series (given by smoothness in the angular variable) we see that if $u$ solves (20) then its Fourier modes solve the system

(21) \hspace{1cm} \bar{\partial} u_1(z) + \partial u_{-1}(z) = f_0(z),

(22) \hspace{1cm} \bar{\partial} u_{-1}(z) + \partial u_{-3}(z) = f_2(z),

(23) \hspace{1cm} \bar{\partial} u_{2n}(z) + \partial u_{2n-2}(z) = 0, \quad n \leq 0,

(24) \hspace{1cm} \bar{\partial} u_{2n-1}(z) + \partial u_{2n-3}(z) = 0, \quad n \leq -1,

The range characterization is given in terms of the trace

(25) \hspace{1cm} g := u|_{\Gamma \times S^1} = \begin{cases} \chi \mathcal{F}(x, \theta), & (x, \theta) \in \Gamma_+, \\ 0, & (x, \theta) \in \Gamma_- \cup \Gamma_0. \end{cases}

More precisely, in terms of its Fourier modes in the angular variables:

(26) \hspace{1cm} g(\zeta, \theta) = \sum_{-\infty}^{\infty} g_n(\zeta) e^{i n \varphi}, \quad \zeta \in \Gamma.

Since the trace $g$ is also real valued, its Fourier modes will satisfy

(27) \hspace{1cm} g_{-n}(\zeta) = \overline{g_n(\zeta)}, \quad n \geq 0, \quad \zeta \in \Gamma.

From the negative even modes, we built the sequence

(28) \hspace{1cm} g_{\text{even}} := \langle g_0, g_{-2}, g_{-4}, \ldots \rangle.

From the negative odd modes starting from mode $-3$, we built the sequence

(29) \hspace{1cm} g_{\text{odd}} := \langle g_{-3}, g_{-5}, g_{-7}, \ldots \rangle.

Next we characterize the data $g$ in terms of the Hilbert Transform $\mathcal{H}$ in (16). We will construct simultaneously the right hand side of the transport equation (20) and the solution $u$ whose trace matches the boundary data $g$. Construction of $u$ is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation.
Except from negative one mode $u_{-1}$ all non-positive modes are defined by Bukhgeim-Cauchy integral formula in (13) using boundary data. Other then having the trace $g_{-1}$ on the boundary $u_{-1}$ is unconstrained. It is chosen arbitrarily from the class of functions

$$
\Psi_g := \left\{ \psi \in C^1(\Omega; \mathbb{C}) : \psi|_{\Gamma} = g_{-1} \right\}.
$$

**Theorem 3.1** (Range characterization in the non-attenuated case). Let $\alpha > 1/2$.

(i) Let $F \in C^{1,\alpha}_0(\Omega; \mathbb{R}^{2 \times 2})$. For $g := \begin{cases} \chi F(x, \theta), & (x, \theta) \in \Gamma_+, \\ 0, & (x, \theta) \in \Gamma_- \cup \Gamma_0, \end{cases}$ consider the corresponding sequences $g_{even} \in (28)$ and $g_{odd} \in (29)$. Then $g_{even}, g_{odd} \in l^1_\infty(\Gamma) \cap C^\alpha(\Gamma; l^1)$ satisfy

$$
[I + i\mathcal{H}]g_{even} = 0, \tag{31}
$$

$$
[I + i\mathcal{H}]g_{odd} = 0, \tag{32}
$$

where the operator $\mathcal{H}$ is the Hilbert transform in (16).

(ii) Let $g \in C^\alpha(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1))$ be real valued with $g|_{\Gamma_- \cup \Gamma_0} = 0$. If the corresponding sequence $g_{even}, g_{odd} \in Y_\alpha$ satisfies (31) and (32), then there exists a real valued symmetric 2-tensor $F \in C(\Omega; \mathbb{R}^{2 \times 2})$, such that $g|_{\Gamma_+} = X F$. Moreover for each $\psi \in \Psi_g$ in (30), there is a unique real valued symmetric 2-tensor $F_\psi$ such that $g|_{\Gamma_+} = X F_\psi$.

**Proof.** (i) **Necessity**

Let $F \in C^{1,\alpha}_0(\Omega; \mathbb{R}^{2 \times 2})$. Since $F$ is compactly supported inside $\Omega$, for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\Gamma_0$ which yields $g \in C^{1,\alpha}(\Gamma \times S^1)$. Moreover, $g$ is the trace on $\Gamma \times S^1$ of a solution $u \in C^{1,\alpha}(\Omega \times S^1)$ of the transport equation (20). By [30, Proposition 4.1] $g_{even}, g_{odd} \in l^1_\infty(\Gamma) \cap C^\alpha(\Gamma; l^1)$.

If $u$ solves (20) then its Fourier modes satisfy (21), (22), (23) and (24). Since the negative even Fourier modes $u_{2n}$ of $u$ satisfies the system (23) for $n \leq 0$, then

$$
z \mapsto u_{even}(z) := \langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \cdots \rangle
$$

is $L$-analytic in $\Omega$ and the necessity part in Theorem 2.2 yields (31).

The equation (24) for negative odd Fourier modes $u_{2n+1}$ starting from mode $-3$ yield that the sequence valued map

$$
z \mapsto u_{odd}(z) := \langle u_{-3}(z), u_{-5}(z), u_{-7}(z), \cdots \rangle
$$

is $L$-analytic in $\Omega$ and the necessity part in Theorem 2.2 yields (32).

(ii) **Sufficiency**
To prove the sufficiency we will construct a real valued symmetric 2-tensor \( F \) in \( \Omega \) and a real valued function \( u \in C^1(\Omega \times S^1) \cap C(\overline{\Omega} \times S^1) \) such that \( u|_{\Gamma \times S^1} = g \) and \( u \) solves (20) in \( \Omega \). The construction of such \( u \) is in terms of its Fourier modes in the angular variable and it is done in several steps.

**Step 1:** The construction of negative even modes \( u_{2n} \) for \( n \leq 0 \).
Let \( g \in C^\alpha (\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1)) \) be real valued with \( g|_{\Gamma_- \cup \Gamma_0} = 0 \). Let the corresponding sequences \( g^{\text{even}} \) satisfying (31) and \( g^{\text{odd}} \) satisfying (32). By [30, Proposition 4.1(ii)] \( g^{\text{even}}, g^{\text{odd}} \in Y_\alpha \). Use the Bukhgeim-Cauchy Integral formula (13) to construct the negative even Fourier modes:

\[
\langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \ldots \rangle := B g^{\text{even}}(z), \quad z \in \Omega.
\]

By Theorem 2.1, the sequence valued map

\[
z \mapsto \langle u_0(z), u_{-2}(z), u_{-4}(z), \ldots \rangle,
\]

is \( L \)-analytic in \( \Omega \), thus the equations

\[
\overline{\partial} u_{-2k} + \partial u_{-2k-2} = 0,
\]

are satisfied for all \( k \geq 0 \). Moreover, the hypothesis (31) and the sufficiency part of Theorem 2.2 yields that they extend continuously to \( \Gamma \) and

\[
u_{-2k}|_\Gamma = g_{-2k}, \quad k \geq 0.
\]

**Step 2:** The construction of positive even modes \( u_{2n} \) for \( n \geq 1 \).
All of the positive even Fourier modes are constructed by conjugation:

\[
u_{2k} := \overline{u_{-2k}}, \quad k \geq 1.
\]

By conjugating (34) we note that the positive even Fourier modes also satisfy

\[
\overline{\partial} u_{2k+2} + \partial u_{2k} = 0, \quad k \geq 0.
\]

Moreover, they extend continuously to \( \Gamma \) and

\[
u_{2k}|_\Gamma = \overline{u_{-2k}}|_\Gamma = \overline{g_{-2k}} = g_{2k}, \quad k \geq 1.
\]

Thus, as a summary, we have shown that

\[
\overline{\partial} u_{2k} + \partial u_{2k-2} = 0, \quad \forall k \in \mathbb{Z},
\]

\[
u_{2k}|_\Gamma = g_{2k}, \quad \forall k \in \mathbb{Z}.
\]

**Step 3:** The construction of modes \( u_{-1} \) and \( u_1 \).
Let \( \psi \in \Psi_g \) as in (30). We define

\[
u_{-1} := \psi, \quad \text{and} \quad \nu_1 := \overline{\psi}.
\]

Since \( g \) is real valued, we have

\[
u_1|_\Gamma = \overline{g_{-1}} = g_1.
\]
Step 4: The construction of negative odd modes $u_{2n-1}$ for $n \leq -1$.

Use the Bukhgeim-Cauchy Integral formula (13) to construct the other odd negative Fourier modes:

$$\langle u_{-3}(z), u_{-5}(z), \cdots \rangle := \mathcal{B}g^{\text{odd}}(z), \quad z \in \Omega.$$  

(43)

By Theorem 2.1, the sequence valued map

$$z \mapsto \langle u_{-3}(z), u_{-5}(z), u_{-7}(z), \ldots \rangle,$$

(43)

is $L$-analytic in $\Omega$, thus the equations

$$\bar{\partial}u_{2k-1} + \partial u_{2k-3} = 0,$$

(44)

are satisfied for all $k \leq -1$. Moreover, the hypothesis (32) and the sufficiency part of Theorem 2.2 yields that they extend continuously to $\Gamma$ and

$$u_{2k-1}|_{\Gamma} = g_{2k-1}, \quad \forall k \leq -1.$$  

(45)

Step 5: The construction of positive odd modes $u_{2n+1}$ for $n \geq 1$.

All of the positive odd Fourier modes are constructed by conjugation:

$$u_{2k+3} := \overline{u_{-(2k+3)}}, \quad k \geq 0.$$  

(46)

By conjugating (44) we note that the positive odd Fourier modes also satisfy

$$\bar{\partial}u_{2k+3} + \partial u_{2k+1} = 0, \quad \forall k \geq 1.$$  

(47)

Moreover, they extend continuously to $\Gamma$ and

$$u_{2k+3}|_{\Gamma} = \overline{u_{-(2k+3)}}|_{\Gamma} = \overline{g_{-(2k+3)}} = g_{2k+3}, \quad k \geq 0.$$  

(48)

Step 6: The construction of the tensor field $F_\psi$ whose X-ray data is $g$.

We define the 2-tensor field

$$F_\psi := \begin{pmatrix} f_0 + 2 \Re f_2 & 2 \Im f_2 \\ 2 \Im f_2 & f_0 - 2 \Re f_2 \end{pmatrix},$$

(49)

where

$$f_0 = 2 \Re(\partial \psi), \quad \text{and} \quad f_2 = \bar{\partial} \psi + \partial u_{-3}.$$  

(50)

In order to show $g|_{\Gamma_+} = X F_\psi$ with $F_\psi$ as in (49), we define the real valued function $u$ via its Fourier modes

$$u(z, \theta) := u_0(z) + \psi(z) e^{-i\varphi} + \overline{\psi(z)} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n}(z) e^{-in\varphi} + \sum_{n=2}^{\infty} u_n(z) e^{in\varphi},$$

(51)

and check that it has the trace $g$ on $\Gamma$ and satisfies the transport equation (20).

Since $g \in C^\alpha(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1))$, we use [30, Corollary 4.1] and [30, Proposition 4.1 (iii)] to conclude that $u$ defined in (51) belongs to
$C^{1,\alpha}(\Omega \times S^1) \cap C^{\alpha}(\overline{\Omega} \times S^1)$. In particular $u(\cdot, \theta)$ for $\theta = (\cos \varphi, \sin \varphi)$ extends to the boundary and its trace satisfies

$$u(\cdot, \theta)|_{\Gamma} = \left( u_0 + \psi e^{-i\varphi} + \overline{\psi} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n} e^{-i(n-1)\varphi} + \sum_{n=2}^{\infty} u_n e^{i(n+1)\varphi} \right) |_{\Gamma},$$

$$= u_0 + \psi e^{-i\varphi} + \overline{\psi} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n} e^{-i(n-1)\varphi} + \sum_{n=2}^{\infty} u_n e^{i(n+1)\varphi},$$

$$= g_0 + g_{-1} e^{-i\varphi} + g_1 e^{i\varphi} + \sum_{n=2}^{\infty} g_{-n} e^{-i(n-1)\varphi} + \sum_{n=2}^{\infty} g_n e^{i(n+1)\varphi},$$

where in the third equality above we used (40), (45), (48), (42) and definition of $\psi \in \Psi_g$ in (30).

Since $u \in C^{1,\alpha}(\Omega \times S^1) \cap C^{\alpha}(\overline{\Omega} \times S^1)$, the following calculation is also justified:

$$\theta \cdot \nabla u = e^{-i\varphi} \overline{\partial u_0} + e^{i\varphi} \partial u_0 + e^{-2i\varphi} \overline{\partial \psi} + \partial \psi + e^{2i\varphi} \partial \overline{\psi}$$

$$+ \sum_{n=2}^{\infty} \overline{\partial u_{-n}} e^{-i(n+1)\varphi} + \sum_{n=2}^{\infty} \partial u_{-n} e^{-i(n-1)\varphi}$$

$$+ \sum_{n=2}^{\infty} \partial u_n e^{i(n+1)\varphi}.$$

Rearranging the modes in the above equation yields

$$\theta \cdot \nabla u = e^{-2i\varphi}(\overline{\partial \psi} + \partial u_{-3}) + e^{2i\varphi}(\overline{\partial \psi} + \overline{\partial u_3})$$

$$+ e^{-i\varphi}(\overline{\partial u_0} + \overline{\partial u_{-2}} + \partial u_0 + \partial u_{-2})$$

$$+ \sum_{n=1}^{\infty} (\overline{\partial u_{-n}} + \partial u_{n-2}) e^{-i(n+1)\varphi} + \sum_{n=1}^{\infty} (\overline{\partial u_{n+2}} + \partial u_n) e^{i(n+1)\varphi}.$$

Using (39), (44), and (47) simplifies the above equation

$$\theta \cdot \nabla u = e^{-2i\varphi}(\overline{\partial \psi} + \partial u_{-3}) + e^{2i\varphi}(\overline{\partial \psi} + \overline{\partial u_3}) + \overline{\partial \psi} + \partial \psi.$$

Now using (50), we conclude (20).

$$\theta \cdot \nabla u = e^{-2i\varphi} f_2 + e^{2i\varphi} f_2 + f_0 = \langle F_\psi \theta, \theta \rangle.$$

As the source is supported inside, there are no incoming fluxes: hence the trace of a solution $u$ of (20) on $\Gamma_-$ is zero. We give next a range condition
only in terms of \( g \) on \( \Gamma_+ \), where \( g := u_{+ \times S^1} \). More precisely, let \( \tilde{u} \) be the solution of the boundary value problem

\[
\theta \cdot \nabla \tilde{u}(x, \theta) = \langle F(x)\theta, \theta \rangle, \quad x \in \Omega,
\]

(52)

\[
\tilde{u}(z, \theta) = -\frac{1}{2} g_{\mid \Gamma_+}(z, -\theta), \quad (z, \theta) \in \Gamma_-.
\]

Then one can see that

\[
\tilde{u}_{\mid \Gamma_+} = \frac{1}{2} g_{\mid \Gamma_+},
\]

(53)

and therefore \( \tilde{u}_{\mid \Gamma \times S^1} \) is an odd function of \( \theta \). This shows that we can work with the following odd extension:

\[
\tilde{g}(z, \theta) := \frac{g(z, \theta) - g(z, -\theta)}{2}, \quad (z, \theta) \in (\Gamma \times S^1) \setminus \Gamma_0,
\]

(54)

and \( \tilde{g} = 0 \) on \( \Gamma_0 \). Note that \( \tilde{g} \) is the trace of \( \tilde{u} \) on \( \Gamma \times S^1 \).

The range characterization can be given now in terms of the odd Fourier modes of \( \tilde{g} \), namely in terms of

\[
\tilde{g} := \langle \tilde{g}_{-3}, \tilde{g}_{-5}, \tilde{g}_{-7}, \ldots \rangle.
\]

(55)

**Corollary 3.1.** Let \( \alpha > 1/2 \).

(i) Let \( F \in C^{1,\alpha}_0(\Omega; \mathbb{R}^{2 \times 2}) \), \( \tilde{u} \) be the solution of (52) and \( \tilde{g} \) as in (55). Then \( \tilde{g} \in L^{1,1}_\infty(\Gamma) \cap C^{\alpha}(\Gamma; l_1) \) and

\[
[I + i \mathcal{H}] \tilde{g} = 0,
\]

(56)

where the operator \( \mathcal{H} \) is the Hilbert transform in (16).

(ii) Let \( g \in C^{\alpha}(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1)) \) be real valued with \( g_{\mid \Gamma_+ \cup \Gamma_0} = 0 \). Let \( \tilde{g} \) be its odd extension as in (54) and the corresponding \( \tilde{g} \) as in (55). If \( \tilde{g} \) satisfies (56), then there exists a real valued symmetric 2-tensor \( F \in C(\Omega; \mathbb{R}^{2 \times 2}) \), such that \( g_{\mid \Gamma_+} = X F \). Moreover for each \( \psi \in \Psi_{\tilde{g}} \) in (30), there is a unique real valued symmetric 2-tensor \( F_{\psi} \) such that \( g_{\mid \Gamma_+} = X F_{\psi} \).

### 4. The Attenuated Case

In this section we assume an attenuation \( a \in C^{2,\alpha}(\overline{\Omega}), \alpha > 1/2 \) with

\[
\min_{\overline{\Omega}} a > 0.
\]

We establish necessary and sufficient conditions for a sufficiently smooth function \( g \) on \( \Gamma \times S^1 \) to be the attenuated X-ray data, with attenuation \( a \), of some sufficiently smooth real symmetric 2-tensor, i.e. \( g \) is the trace on \( \Gamma \times S^1 \) of some solution \( u \) of

\[
\theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) = \langle F(x)\theta, \theta \rangle, \quad (x, \theta) \in \Gamma \times S^1.
\]

(57)
Different from 1-tensor case in [31] (where there is uniqueness), in the 2-tensor case there is non-uniqueness: see the class of function in (82).

As in [30] we start by the reduction to the non-attenuated case via the special integrating factor $e^{-h}$, where $h$ is explicitly defined in terms of $a$ by

$$h(z, \theta) := Da(z, \theta) - \frac{1}{2} (I - iH) Ra(z \cdot \theta^\perp, \theta),$$

where $\theta^\perp$ is orthogonal to $\theta$, $Da(z, \theta) = \int_0^\infty a(z + t\theta)dt$ is the divergence beam transform of the attenuation $a$, $Ra(s, \theta) = \int_{-\infty}^\infty a(s\theta^\perp + t\theta)dt$ is the Radon transform of the attenuation $a$, and the classical Hilbert transform $Hh(s) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{h(t)}{s-t}dt$ is taken in the first variable and evaluated at $s = z \cdot \theta^\perp$. The function $h$ was first considered in the work of Natterer [21]; see also [8], and [6] for elegant arguments that show how $h$ extends from $S^1$ inside the disk as an analytic map.

The lemma 4.1 and lemma 4.2 below were proven in [31] for a vanishing at the boundary, $a \in C^{1,\alpha}_0(\Omega)$, $\alpha > 1/2$. We explain here why the vanishing assumption is not necessary: we extend $a$ in a neighbourhood $\tilde{\Omega}$ of $\Omega$ with compact support, $\tilde{a} \in C^{1,\alpha}_0(\tilde{\Omega})$. We apply the results [31, Lemma 4.1 and Lemma 4.2] for the extension $\tilde{a}$ and use it on $\Omega$.

**Lemma 4.1.** [31, Lemma 4.1] Assume $a \in C^{p,\alpha}(\Omega)$, $p = 1, 2$, $\alpha > 1/2$, and $h$ defined in (58). Then $h \in C^{p,\alpha}(\Omega \times S^1)$ and the following hold

(i) $h$ satisfies

$$\theta \cdot \nabla h(z, \theta) = -a(z), \; (z, \theta) \in \Omega \times S^1.$$  

(ii) $h$ has vanishing negative Fourier modes yielding the expansions

$$e^{-h(z, \theta)} := \sum_{k=0}^{\infty} \alpha_k(z)e^{ik\varphi}, \quad e^{h(z, \theta)} := \sum_{k=0}^{\infty} \beta_k(z)e^{ik\varphi}, \; (z, \theta) \in \Omega \times S^1,$$

with

(iii)

$$z \mapsto (\alpha_1(z), \alpha_2(z), \alpha_3(z), \ldots, ) \in C^{p,\alpha}(\Omega; l_1) \cap C(\Omega; l_1),$$

$$z \mapsto (\beta_1(z), \beta_2(z), \beta_3(z), \ldots, ) \in C^{p,\alpha}(\Omega; l_1) \cap C(\Omega; l_1).$$
(iv) For any \( z \in \Omega \)

(63) \[ \overline{\partial} \beta_0(z) = 0, \]

(64) \[ \overline{\partial} \beta_1(z) = -a(z)\beta_0(z), \]

(65) \[ \overline{\partial} \beta_{k+2}(z) + \partial \beta_k(z) + a(z)\beta_{k+1}(z) = 0, \quad k \geq 0. \]

(v) For any \( z \in \Omega \)

(66) \[ \overline{\partial} \alpha_0(z) = 0, \]

(67) \[ \overline{\partial} \alpha_1(z) = a(z)\alpha_0(z), \]

(68) \[ \overline{\partial} \alpha_{k+2}(z) + \partial \alpha_k(z) + a(z)\alpha_{k+1}(z) = 0, \quad k \geq 0. \]

(vi) The Fourier modes \( \alpha_k, \beta_k, k \geq 0 \) satisfy

(69) \[ \alpha_0\beta_0 = 1, \quad \sum_{m=0}^{k} \alpha_m\beta_{k-m} = 0, k \geq 1. \]

From (59) it is easy to see that \( u \) solves (57) if and only if \( v := e^{-h}u \) solves

(70) \[ \theta \cdot \nabla v(z, \theta) = \langle F(z)\theta, \theta \rangle e^{-h(z,\theta)}. \]

If \( u(z, \theta) = \sum_{n=-\infty}^{\infty} u_n(z) e^{in\varphi} \) solves (57), then its Fourier modes satisfy

(71) \[ \overline{\partial} u_1(z) + \partial u_{-1}(z) + a(z)u_0(z) = f_0(z), \]

(72) \[ \overline{\partial} u_0(z) + \partial u_{-2}(z) + a(z)u_{-1}(z) = 0, \]

(73) \[ \overline{\partial} u_{-1}(z) + \partial u_{-3}(z) + a(z)u_{-2}(z) = f_2(z), \]

(74) \[ \overline{\partial} u_n(z) + \partial u_{n-2}(z) + a(z)u_{n-1}(z) = 0, \quad n \leq -2, \]

where \( f_0, f_2 \) as defined in (19).

Also, if \( v := e^{-h} = \sum_{n=-\infty}^{\infty} v_n(z) e^{in\varphi} \) solves (70), then its Fourier modes satisfy

(75) \[ \overline{\partial} v_1(z) + \partial v_{-1}(z) = \alpha_0(z)f_0(z) + \alpha_2(z)f_2(z), \]

\[ \overline{\partial} v_0(z) + \partial v_{-2}(z) = \alpha_1(z)f_2(z), \]

\[ \overline{\partial} v_{-1}(z) + \partial v_{-3}(z) = \alpha_0(z)f_2(z), \]

\[ \overline{\partial} v_n(z) + \partial v_{n-2}(z) = 0, \quad n \leq -2, \]

where \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) are the Fourier modes in (60), and \( f_0, f_2 \) as defined in (19).

The following result shows that the equivalence between (74) and (75) is intrinsic to negative Fourier modes only.
Lemma 4.2. [31, Lemma 4.2] Assume $a \in C^{1,\alpha}(\Omega), \alpha > 1/2$.

(i) Let $v = \langle v_{-2}, v_{-3}, \ldots \rangle \in C^1(\Omega, l_1)$ satisfy (75), and $u = \langle u_{-2}, u_{-3}, \ldots \rangle$ be defined componentwise by the convolution

$$u_n := \sum_{j=0}^{\infty} \beta_j v_{n-j}, \quad n \leq -2,$$

where $\beta_j$'s are the Fourier modes in (60). Then $u$ solves (74) in $\Omega$.

(ii) Conversely, let $u = \langle u_{-2}, u_{-3}, \ldots \rangle \in C^1(\Omega, l_1)$ satisfy (74), and $v = \langle v_{-2}, v_{-3}, \ldots \rangle$ be defined componentwise by the convolution

$$v_n := \sum_{j=0}^{\infty} \alpha_j u_{n-j}, \quad n \leq -2,$$

where $\alpha_j$'s are the Fourier modes in (60). Then $v$ solves (75) in $\Omega$.

The operators $\partial, \overline{\partial}$ in (9) can be rewritten in terms of the derivative in tangential direction $\partial_\tau$ and derivative in normal direction $\partial_n$,

$$\partial_n = \cos \eta \partial_{x_1} + \sin \eta \partial_{x_2},$$
$$\partial_\tau = -\sin \eta \partial_{x_1} + \cos \eta \partial_{x_2},$$

where $\eta$ is the angle made by the normal to the boundary with $x_1$ direction. Since the boundary $\Gamma$ is known, $\eta$ is a known function on the boundary. In these coordinates

$$\partial = \frac{e^{-i\eta}}{2} (\partial_n - i \partial_\tau), \quad \overline{\partial} = \frac{e^{i\eta}}{2} (\partial_n + i \partial_\tau).$$

Next we characterize the attenuated X-ray data $g$ in terms of its Fourier modes $g_0, g_{-1}$ and the negative index modes $\gamma_{-2}, \gamma_{-3}, \gamma_{-4}$ of

$$e^{-h(\zeta, \theta)} g(\zeta, \theta) = \sum_{k=-\infty}^{\infty} \gamma_k(\zeta) e^{ik\varphi}, \quad \zeta \in \Gamma.$$

To simplify the statement, let

$$g_h := \langle \gamma_{-2}, \gamma_{-3}, \gamma_{-4}, \ldots \rangle,$$

and from the negative even, respectively, negative odd Fourier modes, we build the sequences

$$g_h^{\text{even}} = \langle \gamma_{-2}, \gamma_{-4}, \ldots \rangle, \quad \text{and} \quad g_h^{\text{odd}} = \langle \gamma_{-3}, \gamma_{-5}, \ldots \rangle.$$

Note that $\gamma_{-1}$ is not included in the $g_h^{\text{odd}}$ definition. As before we construct simultaneously the right hand side of the transport equation (57) together with the solution $u$. Construction of $u$ is via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation. Apart from zeroth mode $u_0$ and negative one mode $u_{-1}$, all Fourier modes are constructed uniquely from the data $g_h^{\text{even}}, g_h^{\text{odd}}$. The
mode $u_0$ will be chosen arbitrarily from the class $\Psi_0^\alpha$ with prescribed trace and gradient on the boundary $\Gamma$ defined as

\begin{align}
\Psi_0^\alpha := \left\{ \psi \in C^2(\Omega; \mathbb{R}) : \psi|_\Gamma = g_0, \quad \partial_n\psi|_\Gamma = -2 \Re e^{-i\eta} \left( \partial \sum_{j=0}^\infty \beta_j (B g_h)_{-2-j} \right|_\Gamma + a|_\Gamma g_{-1} \right\},
\end{align}

where $B$ be the Bukhgeim-Cauchy operator in (13), $\beta_j$’s are the Fourier modes in (60) and $g_h$ in (80). The mode $u_{-1}$ is defined in terms of $u_0$, see (99).

Recall the Hilbert transform $H$ in (16).

**Theorem 4.1** (Range characterization in the attenuated case). Let $a \in C^{2,\alpha}(\Omega)$, $\alpha > 1/2$ with $\min \alpha > 0$.

(i) Let $F \in C^{1,\alpha}_0(\Omega; \mathbb{R}^{2 \times 2})$. For $g := \left\{ X_a F(x, \theta), \quad (x, \theta) \in \Gamma_+, \quad 0, \quad (x, \theta) \in \Gamma_- \cup \Gamma_0 \right\}$, consider the corresponding sequences $g_h^{\text{even}}$, $g_h^{\text{odd}}$ as in (81). Then $g_h^{\text{even}}$, $g_h^{\text{odd}} \in l_1^{1,1}(\Gamma) \cap C^\alpha(\Gamma; l_1)$ satisfy

\begin{align}
[I + iH] g_h^{\text{even}} &= 0, \quad \left[I + iH\right] g_h^{\text{odd}} = 0, \quad \text{and}
\end{align}

\begin{align}
\partial_r g_0 = -2 \Im e^{-i\eta} \left( \partial \sum_{j=0}^\infty \beta_j (B g_h)_{-2-j} \right|_\Gamma + a|_\Gamma g_{-1} \right),
\end{align}

where $H$ is the Hilbert transform in (16), $B$ is the Bukhgeim-Cauchy operator in (13), $\beta_j$’s are the Fourier modes in (60) and $g_h$ in (80).

(ii) Let $g \in C^\alpha(\Gamma; C^{1,\alpha}(S^1)) \cap C(\Gamma; C^{2,\alpha}(S^1))$ be real valued with $g|_{\Gamma_- \cup \Gamma_0} = 0$. If the corresponding sequences $g_h^{\text{even}}$, $g_h^{\text{odd}} \in Y_a$ satisfying (83) and (84) then there exists a symmetric 2-tensor $F \in C(\Omega; \mathbb{R}^{2 \times 2})$, such that $g|_{\Gamma_+} = X_a F$. Moreover for each $\psi \in \Psi_0^\alpha$ in (82), there is a unique real valued symmetric 2-tensor $F_\psi$ such that $g|_{\Gamma_+} = X_a F_\psi$.

**Proof.** (i) Necessity

Let $F \in C^{1,\alpha}_0(\Omega; \mathbb{R}^{2 \times 2})$. Since $F$ is compactly supported inside $\Omega$, for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\Gamma_0$ which yields $g \in C^{1,\alpha}(\Gamma \times S^1)$. Moreover, $g$ is the trace on $\Gamma \times S^1$ of a solution $u \in C^{1,\alpha}(\Omega \times S^1)$. By [30, Proposition 4.1] $g_h^{\text{even}}$, $g_h^{\text{odd}} \in l_1^{1,1}(\Gamma) \cap C^\alpha(\Gamma; l_1)$.

Let $\nu := e^{-i\theta} u_n = \sum_{n=-\infty}^\infty e^{i n \theta}$, then the negative Fourier modes of $\nu$ satisfy (75). In particular its negative odd subsequence $\langle v_{-3}, v_{-5}, \ldots \rangle$ and negative even subsequence $\langle v_{-2}, v_{-4}, \ldots \rangle$ are $L$-analytic with traces $g_h^{\text{odd}}$. 

respectively $g_h^{even}$. The necessity part of Theorem 2.2 yields (83):

$$[I + i\mathcal{H}]g_h^{odd} = 0, \quad [I + i\mathcal{H}]g_h^{even} = 0.$$ 

If $u$ solves (57), then its Fourier modes satisfy (71), (72), (73), and (74). The negative Fourier modes of $u$ and $v$ are related by

$$u_n = \sum_{j=0}^{\infty} \beta_j v_{n-j}, \quad n \leq 0,$$

where $\beta_j$'s are the Fourier modes in (60). The restriction of (72) to the boundary yields

$$\partial u_0|_\Gamma = -\partial u_{-2}|_\Gamma - (au_{-1})|_\Gamma.$$ 

Expressing $\partial$ in the above equation in terms of $\partial_r$ and $\partial_n$ as in (78) yields

$$\frac{e^{in}}{2} (\partial_r + i\partial_n)u_0|_\Gamma = -\partial u_{-2}|_\Gamma - a|_\Gamma g_{-1}.$$ 

Simplifying the above expression and using $\partial_r u_0|_\Gamma = \partial_r g_0$, yields

$$\partial_n u_0|_\Gamma + i\partial_r g_0 = -2e^{-in} (\partial u_{-2}|_\Gamma + a|_\Gamma g_{-1}).$$ 

The imaginary part of the above equation yields (84). This proves part (i) of the theorem.

(ii) **Sufficiency**

To prove the sufficiency we will construct a real valued symmetric 2-tensor $F$ in $\Omega$ and a real valued function $u \in C^1(\Omega \times S^1) \cap C(\overline{\Omega} \times S^1)$ such that $u|_{\Gamma \times S^1} = g$ and $u$ solves (57) in $\Omega$. The construction of such $u$ is in terms of its Fourier modes in the angular variable and it is done in several steps.

**Step 1: The construction of negative modes** $u_n$ for $n \leq -2$.

Let $g \in C^\alpha(\Gamma; C^{1,\alpha}(S^1)) \cap C(\overline{\Gamma}; C^{2,\alpha}(S^1))$ be real valued with $g|_{\Gamma_0 \cup \Gamma_0} = 0$. Let the corresponding sequences $g_h^{even}$, $g_h^{odd}$ as in (81) satisfying (83) and (84). By [30, Proposition 4.1(ii)] and [30, Proposition 5.2(iii)] $g_h^{even}$, $g_h^{odd} \in Y_\alpha$. Use the Bukhgeim-Cauchy Integral formula (13) to define the $\mathcal{L}$-analytic maps

$$v^{even}(z) = \langle v_{-2}(z), v_{-4}(z), \ldots \rangle := B_{g_h^{even}}(z), \quad z \in \Omega,$$

$$v^{odd}(z) = \langle v_{-3}(z), v_{-5}(z), \ldots \rangle := B_{g_h^{odd}}(z), \quad z \in \Omega.$$ 

By intertwining let also define

$$v(z) := \langle v_{-2}(z), v_{-3}(z), \ldots \rangle, \quad z \in \Omega.$$ 

By Proposition 2.1

$$v^{even}, v^{odd}, v \in C^{1,\alpha}(\Omega; l_1) \cap C^\alpha(\overline{\Omega}; l_1) \cap C^2(\Omega; l_{\infty}).$$
Moreover, since \( g_h^{\text{even}} \) and \( g_h^{\text{odd}} \) satisfy the hypothesis (83), by Theorem 2.2 we have

\[
v_{\text{even}} \big|_r = g_h^{\text{even}} \quad \text{and} \quad v_{\text{odd}} \big|_r = g_h^{\text{odd}}.
\]

In particular

\[
v_n \big|_r = \sum_{k=0}^{\infty} (\alpha_k \big|_r) g_{n-k}, \quad n \leq -2.
\]  
(89)

For each \( n \leq -2 \), we use the convolution formula below to construct

\[
u_n := \sum_{j=0}^{\infty} \beta_j v_{n-j}.
\]
(90)

Since \( a \in C^{2,\alpha}(\Omega) \), by (62), the sequence \( z \mapsto \langle \beta_0(z), \beta_1(z), \beta_2(z), \ldots \rangle \) is in \( C^{2,\alpha}(\Omega; l_1) \cap C^{\alpha}(\Omega; l_1) \). Since convolution preserves \( l_1 \), the map is in

\[
z \mapsto \langle u_{-2}(z), u_{-3}(z), \ldots \rangle \in C^{1,\alpha}(\Omega; l_1) \cap C^{\alpha}(\Omega; l_1).
\]  
(91)

Moreover, since \( v \in C^2(\Omega; l_\infty) \) as in (88), we also conclude from convolution that

\[
z \mapsto \langle u_{-2}(z), u_{-3}(z), \ldots \rangle \in C^2(\Omega; l_\infty).
\]  
(92)

The property (91) justifies the calculation of traces \( u_n \big|_r \) for each \( n \leq -2 \):

\[
u_n \big|_r = \sum_{j=0}^{\infty} \beta_j \big|_r (v_{n-j} \big|_r).
\]

Using (89) in the above equation gives

\[
u_n \big|_r = \sum_{j=0}^{\infty} \beta_j \big|_r \sum_{k=0}^{\infty} \alpha_k \big|_r g_{n-j-k}.
\]

A change of index \( m = j + k \), simplifies the above equation

\[
u_n \big|_r = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \alpha_k \beta_{m-k} g_{n-m},
\]

\[= \alpha_0 \beta_0 g_n + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \alpha_k \beta_{m-k} g_{n-m}.
\]

Using Lemma 4.1 (vi) yields

\[
u_n \big|_r = g_n, \quad n \leq -2.
\]  
(93)

From the Lemma 4.2, the constructed \( u_n \) in (90) satisfy

\[
\bar{\partial} u_n + \partial u_{n-2} + au_{n-1} = 0, \quad n \leq -2.
\]  
(94)

**Step 2:** The construction of positive modes \( u_n \) for \( n \geq 2 \).
All of the positive Fourier modes are constructed by conjugation:
\( u_n := \overline{u_{-n}}, \quad n \geq 2. \)  
(95)

Moreover using (93), the traces \( u_n|_\Gamma \) for each \( n \geq 2 \):
\( u_n|_\Gamma = \overline{u_{-n}}|_\Gamma = \overline{g_{-n}} = g_n, \quad n \geq 2. \)  
(96)

By conjugating (94) we note that the positive Fourier modes also satisfy
\( \overline{\partial u_{n+2}} + \partial u_n + au_{n+1} = 0, \quad n \geq 2. \)  
(97)

**Step 3: The construction of modes \( u_0, u_{-1} \) and \( u_1. \)**

Let \( \psi \in \Psi^a_g \) as in (82) and define
\( u_0 := \psi, \)  
(98)

and
\( u_{-1} := \frac{-\overline{\partial} \psi - \partial u_2}{a}, \quad u_1 := \overline{u_{-1}}. \)  
(99)

By the construction \( u_0 \in C^2(\Omega; l_\infty) \) and \( u_{-1} \in C^1(\Omega; l_\infty) \), and
\( \overline{\partial} u_0 + \partial u_{-2} + au_{-1} = 0 \)  
(100)

is satisfied. Furthermore, by conjugating (100) yields
\( \partial u_0 + \partial u_2 + au_1 = 0. \)  
(101)

Since \( \psi \in \Psi^a_g \), the trace of \( u_0 \) satisfies
\( u_0|_\Gamma = g_0. \)  
(102)

We check next that the trace of \( u_{-1} \) is \( g_{-1} : \)
\( u_{-1}|_\Gamma = \left. \frac{-\overline{\partial} \psi - \partial u_2}{a} \right|_\Gamma \)
\( = -\frac{1}{a} \left|_\Gamma \frac{e^{in}}{2} (\partial_n + i\partial_r) \psi|_\Gamma - \frac{1}{a} \right|_\Gamma \partial u_{-2}|_\Gamma \)
\( = -\frac{1}{2a} \left|_\Gamma e^{in} \left\{ \partial_n \psi|_\Gamma + i\partial_r \psi|_\Gamma + 2e^{-in} \partial u_{-2}|_\Gamma \right\} \right|_\Gamma \)
\( = g_{-1}, \)  
(103)

where the last equality uses (84) and the condition in class (82).

**Step 4: The construction of the tensor field \( F_\psi \) whose attenuated X-ray data is \( g. \)**

We define the 2-tensor
\( F_\psi := \begin{pmatrix} f_0 + 2 \Re f_2 & 2 \Im f_2 \\ 2 \Im f_2 & f_0 - 2 \Re f_2 \end{pmatrix}, \)  
(104)
where

\begin{align}
  f_0 &= -2 \Re e \left( \frac{\partial \psi + \partial u_{-2}}{a} \right) + a \psi, \quad \text{and} \\
  f_2 &= -\partial \left( \frac{\partial \psi + \partial u_{-2}}{a} \right) + \partial u_{-3} + au_{-2}. 
\end{align}

Note that \( f_2 \) is well defined as \( u_{-2} \in C^2(\Omega; l_{\infty}) \) from (92).

In order to show \( g|_{\Gamma_\ell} = X_n F \psi \) with \( F \psi \) as in (104), we define the real valued function \( u \) via its Fourier modes

\begin{equation}
  u(z, \theta) := u_0(z) + u_{-1} e^{-i\varphi} + \overline{u_{-1}(z)} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n}(z) e^{-in\varphi} + \sum_{n=2}^{\infty} u_n(z) e^{in\varphi}.
\end{equation}

We check below that \( u \) is well defined, has the trace \( g \) on \( \Gamma \) and satisfies the transport equation (57).

For convenience consider the intertwining sequence

\[ u(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), u_{-3}(z), \ldots \rangle, \quad z \in \Omega. \]

Since \( u \in C^{1,\alpha}(\Omega; l_1) \cap C^\alpha(\overline{\Omega}; l_1) \), by [30, Proposition 4.1 (iii)] we conclude that \( u \) is well defined by (107) and as a function in \( C^{1,\alpha}(\Omega \times S^1) \cap C^\alpha(\overline{\Omega} \times S^1) \). In particular \( u(\cdot, \theta) \) for \( \theta = (\cos \varphi, \sin \varphi) \) extends to the boundary and its trace satisfies

\[
  u(\cdot, \theta)|_{\Gamma} = \left( u_0 + u_{-1} e^{-i\varphi} + \overline{u_{-1}} e^{i\varphi} + \sum_{n=2}^{\infty} u_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} u_n e^{in\varphi} \right)|_{\Gamma} \\
  = u_0|_{\Gamma} + u_{-1}|_{\Gamma} e^{-i\varphi} + \overline{u_{-1}}|_{\Gamma} e^{i\varphi} + \sum_{n=2}^{\infty} (u_{-n}|_{\Gamma}) e^{-in\varphi} + \sum_{n=2}^{\infty} (u_n|_{\Gamma}) e^{in\varphi} \\
  = g_0 + g_{-1} e^{-i\varphi} + g_1 e^{i\varphi} + \sum_{n=2}^{\infty} g_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} g_n e^{in\varphi} \\
  = g(\cdot, \theta),
\]

where is the third equality we have used (93), (96), (102), and (103).
Since \( u \in C^{1,\alpha}(\Omega \times \mathbb{S}^1) \cap C^{\alpha}(\overline{\Omega} \times \mathbb{S}^1) \), the following calculation is also justified:

\[
\theta \cdot \nabla u + au = e^{-i\varphi} \partial u_0 + e^{i\varphi} \partial u_0 + e^{-2i\varphi} \partial u_{-1} + \partial u_1 + \partial u_{-1} + e^{2i\varphi} \partial u_1
\]

\[
+ \sum_{n=2}^{\infty} \partial u_{-n} e^{-i(n+1)\varphi} + \sum_{n=2}^{\infty} \partial u_{-n} e^{-i(n-1)\varphi}
\]

\[
+ \sum_{n=2}^{\infty} \partial u_n e^{i(n-1)\varphi} + \sum_{n=2}^{\infty} \partial u_n e^{i(n+1)\varphi}
\]

\[
+ au_0 + au_{-1} e^{-i\varphi} + au_1 e^{i\varphi} + \sum_{n=2}^{\infty} au_{-n} e^{-in\varphi} + \sum_{n=2}^{\infty} au_n e^{in\varphi}.
\]

Rearranging the modes in the above equation yields

\[
\theta \cdot \nabla u + au = e^{-2i\varphi} (\partial u_{-1} + \partial u_{-3} + au_{-2}) + e^{2i\varphi} (\partial u_{1} + \partial u_{3} + au_{2})
\]

\[
+ e^{-i\varphi} (\overline{\partial u}_{0} + \partial u_{-2} + au_{-1}) + e^{i\varphi} (\overline{\partial u}_{0} + \partial u_{2} + au_{1})
\]

\[
+ \partial u_{1} + \partial u_{-1} + au_{0} + \sum_{n=2}^{\infty} (\overline{\partial u}_{n+2} + \partial u_{n} + au_{n+1}) e^{i(n+1)\varphi}
\]

\[
+ \sum_{n=2}^{\infty} (\overline{\partial u}_{-n} + \overline{\partial u}_{-n-2} + au_{-n-1}) e^{-i(n+1)\varphi}.
\]

Using (94), (97), (100) and (101) simplifies the above equation

\[
\theta \cdot \nabla u + au = e^{-2i\varphi} (\overline{\partial u}_{-1} + \overline{\partial u}_{-3} + au_{-2}) + e^{2i\varphi} (\partial u_{1} + \partial u_{3} + au_{2})
\]

\[
+ \overline{\partial u}_{1} + \partial u_{-1} + au_{0}.
\]

Now using (105) and (106), we conclude (57)

\[
\theta \cdot \nabla u + au = e^{-2i\varphi} f_2 + e^{2i\varphi} \overline{f}_2 + f_0 = \langle F_{\psi} \theta, \theta \rangle.
\]

\[\square\]

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References


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